Acoustics Beyond the Wave Equation

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1 Navier-Stokes Equation

Acoustics concerns itself with the study of small oscillations in fluids (liquids and gases). The traditional study of acoustics concerns itself with the linearized equations of fluid mechanics, however, the topic of this lecture concerns itself with fluctuations that violate the assumptions of linearity. The fundamental equations of Nonlinear Acoustics are those of fluid dynamics, a mathematical description of which begins with continuity equations. As these equations have been treated several times in the past, they will be covered briefly.

1.1 Continuity Equation (of Mass)

The total mass flowing out of a volume V per unit time is

$$\oint_{\partial V} \rho \mathbf{v} \cdot d\mathbf{s}. \tag{1}$$

The rate of change of the mass of fluid in V is

$$-\frac{\partial}{\partial t} \int \rho \, dV \tag{2}$$

These two expressions must be equal,

$$\frac{\partial}{\partial t} \int \rho \, dV = \oint \rho \mathbf{v} \cdot d\mathbf{s}. \tag{3}$$

Green's theorem may be used to convert the surface integral into a volume integral.

$$\oint \rho \mathbf{v} \cdot d\mathbf{s} = \int \nabla \cdot (\rho \mathbf{v}) \; d\mathbf{s} \tag{4}$$

Therefore,

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0 \tag{5}$$

Since this equation is valid for any volume, the integrand must vanish.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \tag{6}$$

Alternately,

Theorem 1 *Equation of Continuity*

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \tag{7}$$

1.2 Euler's Equation

The total force acting on a volume of fluid is

$$-\oint_{\partial V} p \, d\mathbf{s}, \tag{8}$$

which, after application of Green's theorem, becomes

$$-\oint_{\partial V} p \, d\mathbf{s} = -\int_V \nabla p \, dV. \tag{9}$$

The equation of motion of a volume element in the fluid is

$$\rho \frac{dv}{dt} = -\nabla p. \tag{10}$$

This equation is the fluid equivalent of Newton's Third Law. Before it may be used in fluid mechanics, it must be transformed from a Lagrangian (moving with fluid) coordinate system to an Eulerian (stationary) system. The change in velocity if a volume element has two parts: the change in velocity at that point in space, and the change due to any gradient in the velocity field.

$$dv = \left(\frac{\partial \mathbf{v}}{\partial t} \right) dt + (d\mathbf{r} \cdot \nabla) \, \mathbf{v} \tag{11}$$

Dividing by dt give an equation that relates the Lagrangian coordinate velocities to the Eulerian coordinate velocities.

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \, \mathbf{v} \tag{12}$$

The equation of motion thus becomes

Theorem 2 *Euler's Equation*

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \, \mathbf{v} \right] = -\nabla p. \tag{13}$$

1.3 Momentum Equation

Euler's equation does not take into account any dissipation that may occur due to viscosity or thermal conduction. In order to generalize the treatment to encompass these effects, it is necessary to consider momentum as well. For the moment, dissipative effects are still neglected. Tensor notation is used for simplicity.

The rate of change of momentum of a fluid volume is

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i \tag{14}$$

The time derivative of density is given by the equation of continuity,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho v_k)}{\partial x_k}, \tag{15}$$

which in conjunction with Euler's equation,

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial \rho}{\partial x_i}, \tag{16}$$

results in

$$\begin{aligned} \frac{\partial}{\partial t} &= -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial (\rho v_k)}{\partial x_k} \\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k). \end{aligned} \tag{17}$$

By writing

$$\frac{\partial p}{\partial x_i} = \delta_{ik} \frac{\partial p}{\partial x_k}, \tag{18}$$

the result is the momentum flux equation.

Theorem 3 *Euler's Equation in terms of Momentum Flux*

$$\begin{aligned} \frac{\partial}{\partial t}(\rho v_i) &= -\frac{\partial \Pi_{ik}}{\partial x_k} \\ \Pi_{ik} &= p\delta_{ik} + \rho v_i v_k \end{aligned} \tag{19}$$

1.4 Navier-Stokes Equation

In order to include viscosity, the momentum flux tensor is modified to account for viscous stresses.

$$\begin{aligned} \Pi_{ik} &= p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} \\ &= -\sigma_{ik} + \rho v_i v_k \end{aligned} \tag{20}$$

The tensor

$$\sigma_{ik} = -p\delta_i k + \sigma'_{ik} \tag{21}$$

is the stress tensor, while σ'_{ik} is the viscous stress tensor.

The most general rank-two tensor incorporating viscosity must satisfy certain conditions to be physically realistic. It must vanish when the velocity is constant, and it must vanish when the fluid is in uniform rotation. It may be shown that the most general linear function incorporating derivatives $\partial v_i / \partial x_k$ is

$$\boxed{\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l},} \tag{22}$$

where the first and second viscosity coefficients are both positive

$$\eta > 0, \zeta > 0. \tag{23}$$

Viscosity may be included in the equations of motion by adding $\partial \sigma'_{ik} / \partial x_k$ to the right side of Euler's equation

$$\rho \left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} \right). \tag{24}$$

The result is

$$\rho \left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k} \left[\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l} \right) \right] + \frac{\partial}{\partial x_i} \left(\zeta \frac{\partial v_l}{\partial x_l} \right). \tag{25}$$

If the viscosity coefficients may be considered constant, the result is the so-called *Navier-Stokes Equation*

Theorem 4 *Navier-Stokes Equation*

$$\boxed{\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{1}{3}\eta \right) \nabla\nabla \cdot v} \tag{26}$$

1.5 Approximations in Sound Propagation

Acoustics concerns itself with small fluctuations in pressure and density about an equilibrium state.

$$p = p_0 + p', \; \rho = \rho_0 + \rho' \tag{27}$$

Neglecting quantities of second order (terms with products of small fluctuations), the continuity equation becomes,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla v = 0, \tag{28}$$

and Euler's equation,

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right] = -\nabla p, \tag{29}$$

becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\frac{1}{\rho_0} \right) \nabla p' = 0. \tag{30}$$

The fluctuations in pressure and density may be approximated by a linear relationship for isentropic flow,

$$p' = \left(\frac{\partial p}{\partial \rho_0} \right)_s \rho', \tag{31}$$

resulting in an equation in terms of only p' and $\mathbf{v}$.

$$\frac{\partial p'}{\partial t} + \rho_0 \left(\frac{\partial p}{\partial \rho_0} \right)_s \nabla \mathbf{v} = 0 \tag{32}$$

The two variables may be related using a scalar potential $v = \nabla \phi$ so that

$$p' = -\rho_0 \frac{\partial \phi}{\partial t}. \tag{33}$$

Substitution into the previous equation results in the scalar wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0; \tag{34}$$

where

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho} \right)_s}. \tag{35}$$

2 Absorption in Fluids

There are many mechanisms for the absorption of sound including thermal conduction, viscosity, structural changes in molecules, cavitation, and scattering. The particular mechanism that will be examined here is second viscosity, which can qualitatively explain the bulk of the frequency effects encountered in absorption. In particular, chemical relaxation processes, such as the presence of salt in sea water, will be considered.

Consider some physical quantity ξ with an equilibrium value ξ_0 in a mixture. To first order, small changes perturbations in ξ relax to equilibrium values with some time constant τ.

$$\frac{\partial \xi}{\partial t} = -\frac{\xi - \xi_0}{\tau} \tag{36}$$

A small periodic compression and expansion is applied to the fluid with a time variation $e^{-i\omega t}$. Thus, the equilibrium value of the parameter also changes as a function of time,

$$\xi_0 = \xi_{00} + \xi_0'. \tag{37}$$

The value of ξ will have some time-varying form,

$$\xi = \xi_{00} + \xi'. \tag{38}$$

In terms of frequency, the relaxation equation is written

$$-i\omega\xi' = -\frac{\xi' - \xi_0'}{\tau}, \tag{39}$$

so that ξ' is also a periodic function of time, related to ξ_0' by

$$\xi' = \frac{\xi_0'}{1 - i\omega\tau}. \tag{40}$$

The state of the fluid is a function of the pressure, density, entropy, and ξ, so the isentropic rate of change of pressure with density is given by

$$\frac{\partial p}{\partial \rho} = \left(\frac{\partial p}{\partial \rho}\right)_\xi + \left(\frac{\partial p}{\partial \xi}\right)_\rho \frac{\partial \xi}{\partial \rho}. \tag{41}$$

The derivative of ξ with respect to density is just

$$\frac{\partial \xi}{\partial \rho} = \frac{\partial \xi'}{\partial \rho} = \frac{1}{1 - i\omega\tau}\frac{\partial \xi_0'}{\partial \rho} = \frac{1}{1 - i\omega\tau}\frac{\partial \xi_0}{\partial \rho}, \tag{42}$$

so that

$$\frac{\partial p}{\partial \rho} = \frac{1}{1 - i\omega\tau}\left[\left(\frac{\partial p}{\partial \rho}\right)_\xi + \left(\frac{\partial p}{\partial \xi}\right)_\rho \frac{\partial \xi_0}{\partial \rho} - i\omega\tau\left(\frac{\partial p}{\partial \rho}\right)_\xi\right]. \tag{43}$$

This rate may be compared to that for a very slow change where ξ is effectively at equilibrium.

$$\frac{\partial p}{\partial \rho} = \frac{1}{1 - i\omega\tau}\left[\left(\frac{\partial p}{\partial \rho}\right)_{eq} - i\omega\tau\left(\frac{\partial p}{\partial \rho}\right)_\xi\right]. \tag{44}$$

If the pressure is perturbed from thermodynamic equilibrium by an adiabatic change in density, the over-pressure, or excess pressure above its equilibrium value, is given by

$$p - p_0 = \left[\left(\frac{\partial p}{\partial \rho}\right) - \left(\frac{\partial p}{\partial \rho}\right)_{eq}\right]\delta\rho = \frac{i\omega\tau}{1 - i\omega\tau}\left[\left(\frac{\partial p}{\partial \rho}\right)_{eq} - \left(\frac{\partial p}{\partial \rho}\right)_\xi\right]\delta\rho \tag{45}$$

To find the density changes due to motion in the fluids, we use the equation of continuity in terms of total time derivatives,

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{46}$$

which in terms of frequency becomes

$$i\omega\delta\rho + \rho\nabla\cdot\mathbf{v} = 0. \tag{47}$$

The change in density is just

$$\delta\rho = \frac{\rho}{i\omega}\nabla\cdot\mathbf{v}. \tag{48}$$

Substituting the change in density into the over-pressure formula,

$$p - p_0 = \frac{\tau\rho}{1 - i\omega\tau}\left(c_0^2 - c_\infty^2\right)\nabla\cdot\mathbf{v} \tag{49}$$

where

$$c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{eq},\ c_\infty^2 = \left(\frac{\partial p}{\partial \rho}\right)_\xi. \tag{50}$$

The meanings of these quantities will soon become apparent.

The stress tensor σ_{ik} may be modified by the following viscous term

$$-(p - p_0)\delta_{ik} = \frac{\tau\rho}{1 - i\omega\tau}\left(c_\infty^2 - c_0^2\right)\delta_{ik}\frac{\partial v_l}{x_l}. \tag{51}$$

By comparing this modification with the general form for the viscous stress tensor, we find that the relaxation process is equivalent to the presence of a second viscosity

$$\boxed{\zeta = \frac{\tau\rho}{1 - i\omega\tau}\left(c_\infty^2 - c_0^2\right)} \tag{52}$$

The next goal is to find out how the presence of a relaxation process affects the propagation of sound in a fluid. The basic equations for sound propagation may be kept if certain generalizations may be made. The wave number and frequency are still related by

$$k = \frac{\omega}{c},\ c = \sqrt{\frac{\partial p}{\partial \rho}}, \tag{53}$$

where $\partial p/\partial \rho$ has been calculated above. The quantity c becomes complex, resulting in a complex wave number,

$$k = \omega\sqrt{\frac{1 - i\omega\tau}{c_0^2 - c_\infty^2 i\omega\tau}} \tag{54}$$

The low frequency limit is

$$k = \frac{\omega}{c_0} + \frac{i\omega^2\tau}{2c_o^3}\left(c_\infty^2 - c_0^2\right),\ \omega\tau \ll 1, \tag{55}$$

while the high frequency limit is

$$k = \frac{\omega}{c_\infty} + i\frac{c_\infty^2 - c_0^2}{2\tau c_\infty^3},\ \omega\tau \gg 1. \tag{56}$$

By separating the real and imaginary parts of k,

$$k = k_1 + ik_2 \tag{57}$$

and taking their ratio k_2/k_1, we find the absorption per unit wavelength. Note that in both limiting cases, the absorption is small. The absorption per wavelength is maximum at an intermediate frequency

$$\omega^* = \frac{1}{\tau}\sqrt{\frac{c_0}{c_\infty}}. \tag{58}$$

Since the second viscosity, ζ is greater than zero, it may be seen that

$$c_\infty > c_0. \tag{59}$$

If multiple relaxation processes are present, quantities may be defined that are independent of one another such that

$$\frac{\partial \xi_n}{\partial t} = -\frac{\xi_n - \xi_n 0}{\tau_n}. \tag{60}$$

By proceeding as before,

$$c^2 = c_\infty^2 + \sum_n \frac{a_n}{1 - i\omega\tau_n} \tag{61}$$

where

$$c_\infty^2 = \left(\frac{\partial p}{\partial \rho}\right)_\xi, \; a_n = \left(\frac{\partial p}{\partial \xi_n}\right)\left(\frac{\partial \xi_n}{\partial \rho}\right)_{eq}. \tag{62}$$

Figure 1 shows the frequency dependent absorption in seawater due to the presence of boric acid and magnesium sulfate. The data was parametrically fit to a plot in Kinsler et. al.

3 A Nonlinear Wave Equation

3.1 Absorption of Sound

At high frequencies, the absorption of sound due to viscosity and thermal conductivity is proportional to the square of frequency.

$$\begin{aligned} \gamma &= \frac{\omega^2}{2\rho c^3}\left[\left(\frac{4}{3}\eta + \zeta\right) + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right] \\ &\equiv a\omega^2 \end{aligned} \tag{63}$$

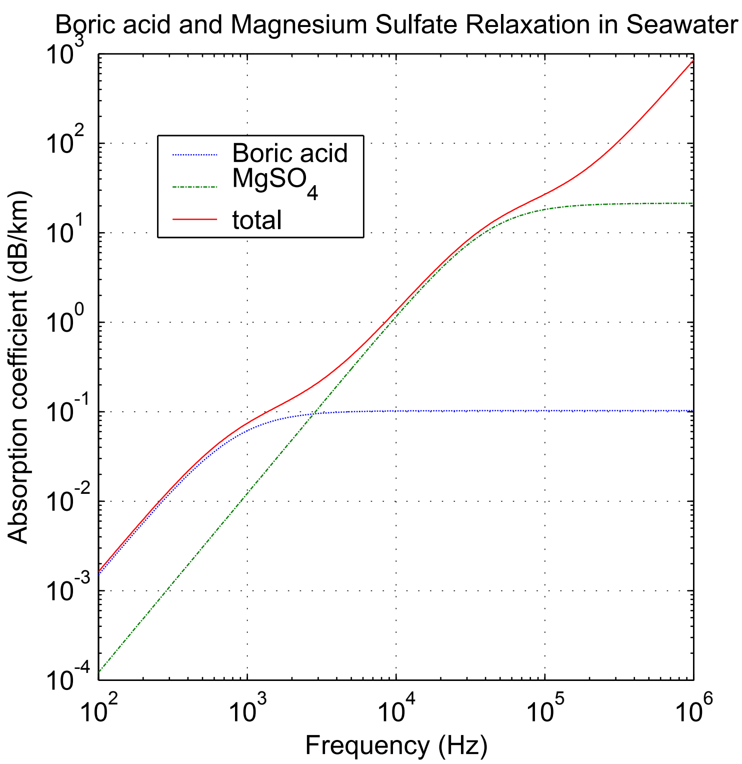

Figure 1: Sound absorption in seawater

Our goal is to include this absorption into the equation for a wave moving in the negative x-direction,

$$
\begin{aligned}
p' &= p'(x+ct) \\
\frac{\partial p'}{\partial x} &= \frac{1}{c}\frac{\partial p'}{\partial t}.
\end{aligned} \tag{64}
$$

The true wave complex wave number is

$$
k = \frac{\omega}{c} + ia\omega^2, \tag{65}
$$

so to capture this absorption, the wave equation may be modified as follows

$$
\begin{aligned}
p' &\sim e^{ikx-\omega t} \\
\frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} &= ac^3\frac{\partial^2 p'}{\partial x^2}.
\end{aligned} \tag{66}
$$

This may be checked in the frequency domain. Note that this is one of many possible modifications to account for the dispersion. Another frequent one involves the second time-derivative of p.

3.2 Nonlinearity in Sound

Through a short derivation, it may be shown that acoustic nonlinearities may be accounted for by the addition of another term

$$
\frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} - \alpha_p p'\frac{\partial p'}{\partial x} = ac^3\frac{\partial^2 p'}{\partial x^2}. \tag{67}
$$

The derivation is included here for completeness. The exact equations for one-dimensional gas flow without dissipation are

$$
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x},\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{68}
$$

These terms may be expanded to second order to capture small nonlinear effects,

$$
p = p_0 + p',\ \rho = \rho_0 + \frac{p'}{c^2} + \frac{1}{2}p'^2\left(\frac{\partial^2 \rho}{\partial p^2}\right)_s. \tag{69}
$$

Note that

$$
\frac{\partial}{\partial t} = c\frac{\partial}{\partial x},\ v = -\frac{p'}{c\rho_0}, \tag{70}
$$

so that the equations for one-dimensional flow become

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0, \tag{71}$$

and

$$\frac{\partial v}{\partial x} + \frac{1}{\rho c^2}\frac{\partial p'}{\partial t} = c\rho\left(\frac{\partial^2 V}{\partial p^2}\right)_s p'\frac{\partial p'}{\partial x}, \tag{72}$$

where $V = 1/\rho$ and the following relation has been used:

$$\left(\frac{\partial^2 \rho}{\partial p^2}\right)_s = \frac{2}{\rho c^4} - \rho^2\left(\frac{deV}{\partial p^2}\right)_s. \tag{73}$$

Differentiating (71) by x and (72) by t, followed by subtraction gives

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)p' = c^2\rho^2\left(\frac{deV}{\partial p^2}\right)_s \frac{\partial}{\partial x}\left(p'\frac{\partial p'}{\partial x}\right). \tag{74}$$

To the same level of accuracy,

$$\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t} \simeq 2\frac{\partial}{\partial x}, \tag{75}$$

hence:

Theorem 5 *Nonlinear Wave Equation*

$$\begin{aligned} \frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} - \alpha_p p'\frac{\partial p'}{\partial x} &= ac^3\frac{\partial^2 p'}{\partial x^2} \\ \alpha_p &= \frac{1}{2}\left(\frac{c^3}{V^2}\right)\left(\frac{\partial^2 V}{\partial p^2}\right)_s \end{aligned} \tag{76}$$

A traveling wave solution to this equation is

$$p = 1/2(p_1 + p_2) + 1/2(p_2 - p_1)\tanh\frac{(p_2 - p_1)(x + v_1 t)}{4ac^3/\alpha_p}. \tag{77}$$

It turns out that if the thickness of the shock based on this result is actually calculated, it is on the order of the mean free path of the gas molecules. The equations of fluid mechanics are not sufficient to investigate the internal structure of shocks.

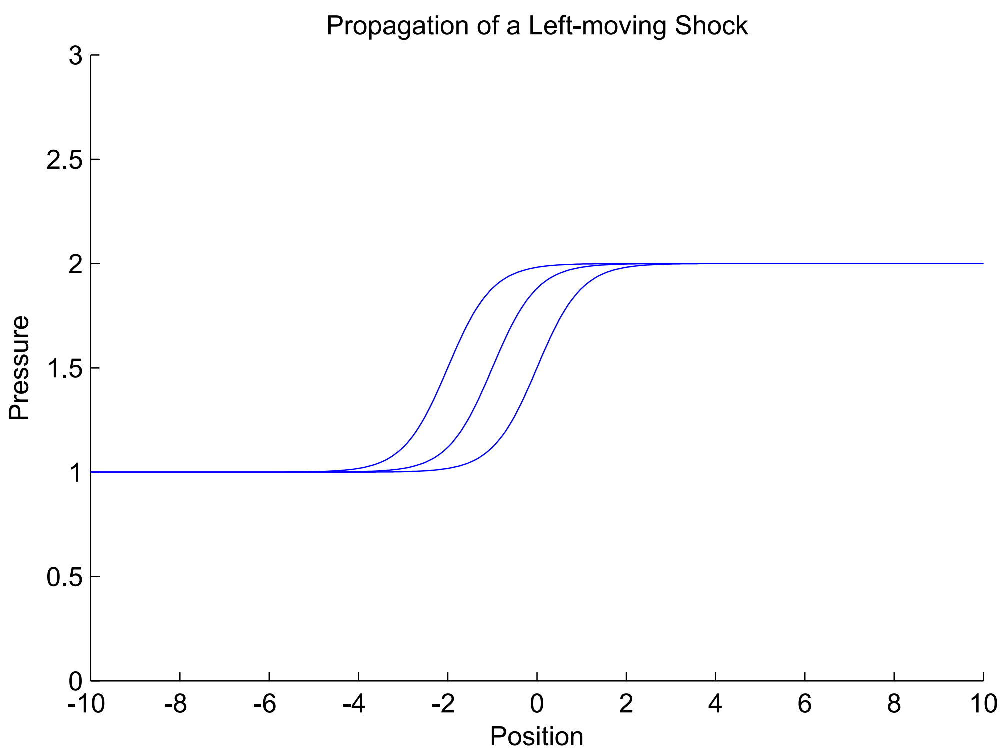

Figure 2: An example of a left-moving shock solution.

4 The Parametric Array

4.1 What is a parametric array?

The *Parametric End Fire Array* is a term coined by Westervelt because of its resemblance to the corresponding sonar array in underwater acoustics. Westervelt noted that the nonlinear interactions between two intense beams act as a distribution of sources.

In general, if a slight nonlinearity is introduced into a linear system containing two different frequencies, there will be radiation at those frequencies, as well as the sum and difference of those frequencies.

For example, we may have a system that performs the following:

$$\begin{aligned} p(\omega_1) + p(\omega_2) \longrightarrow & p(\omega_1) + p(\omega_2) \\ & + \epsilon p(\omega_1 + \omega_2) + \epsilon p(\omega_1 - \omega_2) \end{aligned} \tag{78}$$

Here is the key aspect of a parametric array: in air, or in water there is absorption. If the absorption coefficient is assumed to be a constant, α, the decay will be proportional to the number of wavelengths distance through which the signal has passed. If ω_1 and ω_2 are high, but close together, their signals will decay very rapidly, but their difference frequency $\omega_1 - \omega_2$ will propagate far.

4.2 Greens Functions and Retarded Potentials

Before asking how does a nonlinear source radiate, we must first understand how any source radiates. The linear wave equation may be written as

$$\Box^2 p = \nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{79}$$

The box notation is used to draw the similarities between the wave equation in higher dimensions and laplace's equation. This is particularly useful in fields like special relativity where time is treated on an equal footing to space variables.

This similarity becomes particularly useful in the presence of sources. From basic electromagnetics (for the electrical engineers), or elasticity (for the mechanical engineers), we know that a source can be represented by a delta function.

$$\triangle u(r) = -\delta(r' - r) \tag{80}$$

The solution to this problem, or rather, the Green in three-dimensions is

$$u(r) = G(r, r') = \frac{1}{4\pi\,|r' - r|} \tag{81}$$

This allows the general solution to an arbitrary source to be solved by convolution

$$\begin{aligned} \triangle u(r) &= -f(r) = -\int f(r')\delta(r' - r)\, d^3r' \\ u(r) &= \int f(r')G(r, r')\, d^3r' = \frac{1}{4\pi}\int f(r')\frac{1}{|r' - r|}\, d^3r' \end{aligned} \tag{82}$$

If the time variable is trivially included, the equation becomes

$$\triangle u(r, t) = -\delta(r' - r) \tag{83}$$

The green function may be defined as

$$G(r, t; r', t') = \frac{\delta(t' - t)}{4\pi\,|r' - r|} \tag{84}$$

so that with a distributed source,

$$\begin{aligned} \triangle u(r, t) &= -f(r, t) \\ &= -\iint f(r', t')\delta(r' - r)\delta(t' - t)\, d^3r'\, dt' \\ u(r, t) &= \iint f(r', t')G(r, t; r', t')\, d^3r'\, dt' \\ &= \frac{1}{4\pi}\iint f(r', t')\frac{\delta(t' - t)}{|r' - r|}\, d^3r'\, dt' \\ &= \frac{1}{4\pi}\int f(r', t)\frac{1}{|r' - r|}\, d^3r' \end{aligned} \tag{85}$$

which is just the formula obtained previously.

Similarly, there is a Green's function for the wave equation. The full derivation is peripheral to our goals, so I will just point out a few details about its structure. As we have seen, information propagates through the wave equation at a speed c. The Green's function will be identical to that for laplace's equation, except that the integration will not be performed at a fixed time through space, but rather at a *retarded* time, i.e. the pressure at a particular time is influenced by the pressure through space at times delayed by the travel time for a signal from the space to the point in question.

$$G(r,t;r',t') = \frac{\delta\left(t' - \left[t - \frac{|r'-r|}{c}\right]\right)}{4\pi\,|r'-r|} \tag{86}$$

Note that this is only one of several possible Green functions. For example, the wave equation remains the same if time is reversed, resulting in an *advanced* Green function. The pressure for the retarded, or causal, Green function is given by

$$\begin{aligned} \Box^2 p(r,t) &= -f(r,t) \\ p(r,t) &= \frac{1}{4\pi}\int f(r', t-R/c)\frac{1}{|r'-r|}\,d^3r' \\ R &= |r'-r| \end{aligned} \tag{87}$$

In acoustics, the conventional notation for an equation with a simple source strength density is slightly different,

$$\Box^2 p = -\rho_0 \frac{\partial q}{\partial t}, \tag{88}$$

with the result that the frequency domain representation for the pressure field is given by

$$\boxed{p(r) = -\frac{i\omega\rho}{4\pi}\int \frac{q e^{ik|r-r'|}}{|r-r'|}\,d^3r'} \tag{89}$$

4.3 The Lighthill Equation

The next step is to determine how the nonlinear properties of sound may be incorporated into an appropriate source term within the standard wave equation. To do this, the same procedure is used as when deriving the wave equation.

Begin with the continuity of mass and momentum-flux equations.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \tag{90}$$

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k} \tag{91}$$

$$\Pi_{ik} = p_{ik} + \rho v_i v_k \tag{92}$$

Note that a pressure stress tensor has been used for convenience,

$$p_{ik} = -\sigma_{ik} \tag{93}$$

which will make the following analysis a little simpler.

To make the equations more like the standard wave equation, a new tensor will be defined which represents the difference between the *true* momentum-flux, and the *acoustic* momentum flux.

$$\boxed{T_{ik} = \Pi_{ik} - \Pi_{ik}^0 = p_{ik} - \rho c^2 \delta_{ik} + \rho v_i v_k} \tag{94}$$

With this definition, the momentum-flux equation becomes

$$\frac{\partial}{\partial t}(\rho v_i) + c^2 \frac{\partial \rho}{\partial x_i} = -\frac{\partial T_{ik}}{\partial x_k} \tag{95}$$

Differentiating by x_i and substituting the mass continuity equation results in the Lighthill equation.

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \nabla^2 \rho = \frac{\partial^2 T_{ik}}{\partial x_i \partial x_k} \tag{96}$$

$$\boxed{c^2 \Box^2 \rho = -T_{ik,ik}} \tag{97}$$

4.4 Westervelt's Approximation for Collinear Beams

Westervelt's contribution was to find the corresponding equation for p, or rather p'. The ' symbol is implicitly present in every equation. His approximation keeps terms up to second order:

$$\begin{aligned} \Box^2 p &\equiv \nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \\ &= \frac{\partial^2}{\partial t^2}\left(\rho - \frac{p}{c^2}\right) - \frac{\partial^2}{\partial x_i x_j}(\rho v_i v_j) \end{aligned} \tag{98}$$

The first term on the right may be evaluating by using the equation of state for the gas. Note that the ' symbol is shown for clarity.

$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho=\rho_0} \rho' + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_{s,\rho=\rho_0} \rho'^2 + \dots \tag{99}$$

Acousticians often introduce nonlinearity parameters for this equation.

$$\begin{aligned} p' \equiv p - p_0 &= A\frac{\rho'}{\rho} + \frac{1}{2}B\left(\frac{\rho'}{\rho}\right)^2 \\ A = c^2\rho_0 \ , \ B &= \left(\frac{\partial^2 p}{\partial \rho^2}\right)_{s,\rho=\rho_0} \rho^2 \end{aligned} \tag{100}$$

The B parameter is a key measure of the nonlinearity in the problem. With these definitions, the first term on the right becomes

$$\begin{aligned} \rho - \frac{p}{c^2} &\simeq -\frac{1}{2c^6}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_{\rho=\rho_0} p^2 \\ = -\frac{1}{2c^4}\frac{B}{A} \end{aligned} \tag{101}$$

The second term on the right may also be simplified. Since the only pressure field of interest is the scattered wave, the contribution due to hydrostatic pressure, p_0, and the contribution due to the two collinear beams, p_1 and p_2 may be ignored.

$$p = p_0 + p_1 + p_2 + p_s \tag{102}$$

The scattered sound field due to the two incident beams is therefore

$$\Box^2 p_s = -\frac{1}{2c^4}\frac{B}{A}\frac{\partial^2 p_i}{\partial t^2} - \rho_0 \nabla^2 v_i^2 \tag{103}$$

where the subscript i refers to the total primary beam. By substituting the linear relation for v_i,

$$\begin{aligned} \nabla^2 v_i &\simeq \frac{1}{\rho_0 c^2}\nabla^2 p_i^2 \\ &= \frac{1}{\rho_0 c^2}\Box^2 p_i^2 + \frac{1}{\rho^2 c^4}\frac{\partial^2 p_i^2}{\partial t^2} \\ &\simeq \frac{1}{\rho^2 c^4}\frac{\partial^2 p_i^2}{\partial t^2} \end{aligned} \tag{104}$$

The Westervelt equation is then

$$\Box^2 p = -\rho_0 \frac{\partial q}{\partial t}$$
$$q = \frac{1}{\rho_0^2 c^4}\left(1 + \frac{B}{2A}\right)\frac{\partial}{\partial t} p_i^2 \tag{105}$$

Where q is the simple source strength density resulting from the primary waves p_i.

4.5 Including absorption

Consider two primary beams,

$$\begin{aligned} P_1 &= P_{10} e^{-\alpha_1 x} cos(\omega_1 t - k_1 x) \\ P_2 &= P_{20} e^{-\alpha_2 x} cos(\omega_2 t - k_2 x) \\ \omega_s &= \omega_1 - \omega_2, \end{aligned} \tag{106}$$

The radiation pattern is

$$p_s(r) = -\frac{i\omega_s \rho_0}{4\pi} \int \frac{q e^{ik_s|r-r'|}}{|r-r'|} dV \tag{107}$$

Since the sources are effectively distributed at the origin, the substitution $dV = Sdx$ may be made where S denotes the cross-sectional area of the array. The far field radiation pattern given by

$$p_s(r) = -\frac{i\omega_s \rho_0 S}{4\pi} \int_0^\infty \frac{q e^{ik_s|r-r'|}}{|r-r'|} dx \tag{108}$$

As with the standard dipole approximation, in the far-field, in the numerator, the approximation

$$|r - r'| \approx r - x' \cos(\theta) \tag{109}$$

may be made, while in the denominator, we may just keep $|r - r'| \approx r$ term. This integral is just one of constants and exponentials, but because of all the variable substitutions, I will simply state the result. The radiated intensity from two primary beams of equal strength at a distance R and angle θ from the source.

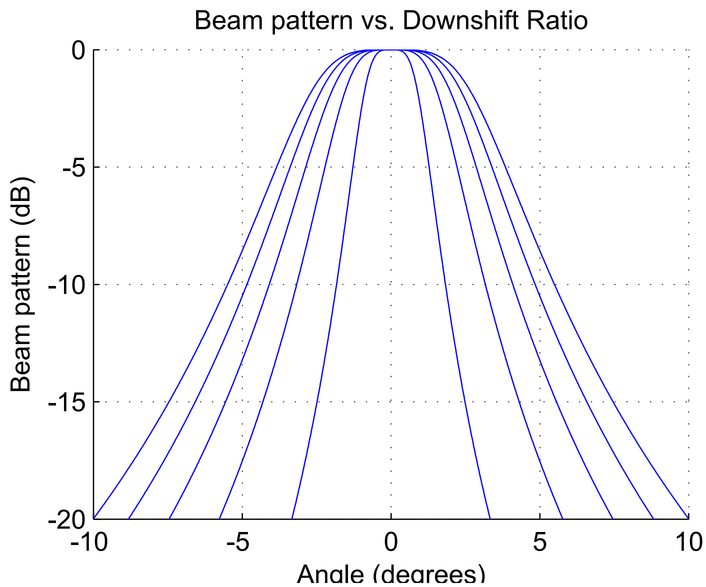

Figure 3: Beam pattern for downshift ratios of 10,30, 50, 70, 90 with a 1MHz source.

$$I_s = \frac{|P|^2}{2\rho_0 c} = \frac{\omega_s^4 P_0^4 S^2 \left(1+\frac{B}{2A}\right)}{2(8\pi)^2 \rho_0^3 c^9 R^2} \frac{1}{\alpha^2 + k_s^2 \sin^4(\theta/2)} \tag{110}$$

The theoretical beam half-width is

$$\theta_{1/2} \simeq 2 \cdot 3^{1/4} \left(\frac{\alpha}{k_s}\right)^{1/2} \tag{111}$$

Which is about twice as wide as the experimental values. In air, with two primary sources of 13 MHz and 14 MHz, the measured beam half-width is approximately 2°. Even better results may be achieved underwater. Note that with a single primary beam, the array becomes an extremely directional receiver for any sound traveling in the direction of the receiver, behaving as a parametric amplifier. This technology is useful for constructing precision depth-sounders.

5 Shock Waves - Optional material

5.1 Characteristics of Burger's Equation

Burgers' Equation

$$u_t + uu_x = \epsilon u_{xx} \tag{112}$$

Primitive form

$$u_t + uu_x = 0 \tag{113}$$

Conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{114}$$

Integral form

$$\int_{x_L}^{x_R} \left[\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right] dt = 0 \tag{115}$$

$$\frac{d}{dt}\int_{x_L}^{x_R} u\, dt = -\left(\frac{u_R^2}{2} - \frac{u_L^2}{2}\right) \tag{116}$$

Arbitrary path $x(t)$ in $x - t$ plane:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dx}{dt} \tag{117}$$

$$\frac{dx}{dt} = u \Longrightarrow \frac{du}{dt} = 0 \Longrightarrow u = u_0(constant) \tag{118}$$

These lines are characteristics

5.2 Shock Formation

Consider the initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{119}$$

For $x < t$

$$\frac{dx}{dt} = 1 \rightarrow x = t + x_0 \rightarrow u(x,t) = 1. \tag{120}$$

For $t < x < 1$

$$\frac{dx}{dt} = 1 - x_0 \rightarrow x = (1 - x_0)t + x_0 \rightarrow u(x,t) = \frac{1-x}{1-t}. \tag{121}$$

For $x < t$

$$\frac{dx}{dt} = 0 \rightarrow x = x_0 \rightarrow u(x,t) = 0. \tag{122}$$

5.3 Shock Path

From the integral form,

$$\frac{d}{dt}\int_{x_L}^{x_R} u\,dt = -\left(\frac{u_R^2}{2} - \frac{u_L^2}{2}\right), \tag{123}$$

and choosing x_L and x_R to be very close to the discontinuity, we find the shock speed s to be given by the *Rankine-Hugoniot* jump condition:

$$-(u_R - u_L)s = -\left(\frac{u_R^2}{2} - \frac{u_L^2}{2}\right), \tag{124}$$

or in a more simplified form,

$$s = \frac{u_L + u_R}{2} \tag{125}$$

5.4 Non-uniqueness

Consider Burgers' equation with a rarefaction,

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{126}$$

One possible solution is a shock with speed

$$s = \frac{u_L + u_R}{2} = 0, \tag{127}$$

while another is a rarefaction wave or expansion fan,

$$u(x,0) = \begin{cases} -1 & \text{if } x < -t \\ x/t & \text{if } -t < x < t \\ 1 & \text{if } x > t \end{cases} \tag{128}$$

Which one is the correct solution? Are there other solutions? Since both solve Burgers' equation, further physical arguments are required. It turns out that the only solution satisfying a thermodynamic condition on entropy is the expansion fan. It may be shown that the solution to Burgers' equation with viscosity reduces to this solution as the viscosity approaches zero. Try verifying this yourself after working Problem 2.

6 References

- Fluid Mechanics -Landau and Lifshitz
  - Basic acoustics
  - Absorption and second viscosity
  - Nonlinear waves and shocks
- Nonlinear Acoustics -Beyer
  - Parametric arrays
- Fundamentals of Acoustics -Kinsler et al.
  - Parameters for graphs
- MIT course notes - Numerical Methods for PDE's
  - Additional materials for shocks

7 Problems

Problem 1 (Dispersion due to relaxation processes)

The derivative if p with respect to ρ was derived for a single relaxation processes to be

$$\frac{\partial p}{\partial \rho} = \frac{1}{1 - i\omega\tau}\left[c_0^2 - i\omega\tau c_\infty^2\right]. \tag{129}$$

Show that for

$$k = \frac{\omega}{c}, \tag{130}$$

the dispersion formula

$$k = \omega\sqrt{\frac{1 - i\omega\tau}{c_0^2 - c_\infty^2 i\omega\tau}} \tag{131}$$

is obtained. Show that the low and high frequency limits are

$$k = \frac{\omega}{c_0} + \frac{i\omega^2\tau}{2c_o^3}\left(c_\infty^2 - c_0^2\right), \; \omega\tau \ll 1, \tag{132}$$

$$k = \frac{\omega}{c_\infty} + i\frac{c_\infty^2 - c_0^2}{2\tau c_\infty^3}, \; \omega\tau \gg 1. \tag{133}$$

Qualitatively sketch a log-log plot of absorption in seawater as a function of frequency. Include two relaxation processes and the effects of second viscosity.

Problem 2 (Velocity of a shock)

The nonlinear wave equation for sound was shown to be

$$\frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} - \alpha_p p'\frac{\partial p'}{\partial x} = ac^3\frac{\partial^2 p'}{\partial x^2}. \tag{134}$$

Find the velocity of a traveling wave solution of the form

$$p'(x,t) = p'(x + v_1 t) \tag{135}$$

where the limiting pressures as $x \to \pm\infty$ are p_2 and p_1 by the following steps. Set p' to be the change in pressure from the front of the shock, p_1, i.e. $p = p_1 + p'$.

- Substitute the desired form for p' into the nonlinear equation.
- Change all derivatives to derivatives in $\xi = x + v_1 t$.
- Integrate the equation with respect to ξ.
- Note that $\partial p'/\partial\xi$ is zero at $\pm\infty$. Use this and the limiting pressures to find the velocity.

The resulting velocity,

$$v_1 = c + \frac{1}{2}\alpha_p(p_2 - p_1), \tag{136}$$

increases with the pressure difference, a fact which is crucial to shock formation. If one proceeded with the solution, the result would be

$$p = \frac{1}{2}(p_1 + p_2) + 1/2(p_2 - p_1)\tanh\frac{(p_2 - p_1)(x + v_1 t)}{4ac^3/\alpha_p}. \tag{137}$$

Sketch this solution for an appropriate p_1 and p_2 at $t = 0$ and indicate which way the wave is traveling. It turns out that the shock thickness is on the order of the mean free path of the molecules; therefore, fluid mechanics is not a valid approximation for the fine structure of the shock.

Problem 3 (Radiation pattern of a parametric array)

The sound produced by a source volume distribution is given by:

$$\Box^2 p = -\rho_0 \frac{\partial q}{\partial t}, \tag{138}$$

$$p(r) = -\frac{i\omega\rho}{4\pi} \int \frac{q e^{ik|r-r'|}}{|r-r'|}\, d^3r' \tag{139}$$

Westervelt's equation gives the source density q.

$$q = \frac{1}{\rho_0^2 c^4} \left(1 + \frac{B}{2A}\right) \frac{\partial}{\partial t} P_i^2 \tag{140}$$

Two beams of equal strength are used, and the effect of absorption results in the following.

$$\begin{aligned} P_1 &= P_0 e^{-\alpha x} cos(\omega_1 t - k_1 x) \\ P_2 &= P_0 e^{-\alpha x} cos(\omega_2 t - k_2 x) \\ \omega_s &= \omega_1 - \omega_2,\ k_s = k_1 - k_2, \end{aligned} \tag{141}$$

Given that the quadratic mixing produces a source proportional to the mixed frequency, and ignoring time dependence,

$$q \sim e^{-2\alpha x' + ik_s x'}, \tag{142}$$

show that the radiated field pattern in the far-field is given by

$$I_s = \frac{|p_s|^2}{2\rho_0 c} = \frac{const.}{r^2} \times H(\theta)^2 \tag{143}$$

where $H(\theta)$, the beam pattern, is given by

$$\frac{\alpha/k_s}{\sqrt{(\alpha/k_s)^2 + \sin^4(\theta/2)}}. \tag{144}$$

Do not bother to keep track of constants along the way. Note that, in the far-field, an approximation of $|r - r'| = r - x' cos(\theta)$, and $|r - r'| = r$ is necessary. Sketch this solution as a function of frequency for a half-angle of 2-degrees and mention two benefits of these arrays over other transducers.