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STANDARD TESTING METHODS FOR EXTENSIONAL AND BENDING ELECTROACTIVE POLYMER ACTUATORS

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ABSTRACT

Certain electroactive polymer actuators produce mechanical strains in excess of 1% under the application of a voltage. These actuator materials include conducting polymers, ionomeric polymers or IPMCs, carbon nanotube and polymer-nanotube composites, ferroelectric polymers and dielectric elastomer materials. The ability to generate mechanical strain in excess of 1% under voltage application differentiates electroactive polymers from other types of active materials, such as piezoelectric polymers and ceramics, which are generally limited to strains less than 1%. To date there has been no development of standardized testing methods for this class of electroactive materials. The development of standardized testing methods would create consistent metrics on which to compare actuator materials. In this work we present a set of standard testing methods for two classes of electroactive polymer actuators. Extensional actuators are defined as materials whose dominant response is expansion or contraction upon application of an electric field. Bender actuators are defined as those whose dominant response is a bending deflection upon the application of a voltage. The standardized testing method for each of these actuators contains detailed schematics of test fixtures for testing of mechanical, electrical, and electromechanical properties. The test method for each actuator type specifies the environmental conditions that must be controlled and reported for each test condition. Furthermore, methods of obtaining data and processing the results are discussed to create a consistent database of material properties and actuator performance metrics. The key properties tested in this method are the stress and strain induced by the applied voltage, material properties, and properties associated with the polymer composition. In addition, we propose the development of a standardized nomenclature and parameter definition that will facilitate analysis of published data and allow direct comparison of data between various research groups and materials developers.

INTRODUCTION

In the past ten years there have been significant advances in the development of electroactive polymer materials that exhibit mechanical response upon stimulation by an applied voltage. This class of materials includes materials such as conducting polymers [3,4,8], carbon nanotubes [2], ionomeric polymer metal-composites [7], nanotube-polymer composites, ferroelectric polymers [9], liquid crystal elastomers and dielectric elastomers [6]. The physical mechanisms that govern the behavior of these materials varies widely, and to some extent there is still debate regarding the physical mechanisms that produce electromechanical response. In general, though, these materials can be separated into *ionic materials*, or those that require charge and mass transport for electromechanical coupling, and *electronic materials*, or those whose electromechanical coupling is associated with polarization-based mechanisms or electrostatic pressure such as Maxwell stress. For a comprehensive overview of the different material types, see Bar-Cohen [1], and a recent review of performance parameters of

Table 1: Nomenclature for measured geometric, material, and stimulus parameters.

Property	Variable	Preferred Units	Acceptable Units
Geometry			
Total Length	L_t	m	mm, cm? cm, non-SI?
Free Length	L_f	m	mm, cm
width	w	m	mm, cm
Thickness (or height)	h	m	mm, μm
Material			
Density (wet & dry)	ρ	kg/m^3	g/cm^3
Young's Modulus	Y	MPa	
Conductivity	σ	S/cm	1/Ω-cm
Applied Stimulus			
Electric potential	v	V	mV, μV
Current	i	A	mA, μA
Current Density	j	A/m^2	mA/cm^2
Charge	q	C	
Charge Density*	ρ_c	C/m^3	
Scan Rate, Ramp Rate		mV/sec	V/sec
Reference Electrode		vs. SCE	vs. Ag/AgCl

*In conducting polymers and carbon nanotubes the charge density is of particular importanve since strain is a function of charge. It is also often preferable to denote charge per carbon or per monomer, but this is not always simple to determine.

different material types has been presented by Madden, et al [5].

Although the physical mechanisms governing behavior differ widely, electroactive polymer actuators can generally be regarded as extensional actuators or bending actuators. Extensional actuators are those that expand or contract upon application of an electric potential. Extensional materials are often configured such that the expansion and contraction occurs primarily in a single direction. Examples include conducting polymer fibers and dielectric elastomer actuators. Bending actuators are those whose dominant motion is a bending deformation upon application of an electric field. Certain ionic materials are naturally bender actuators, e.g., ionic polymer-metal composites, while materials that exhibit extensional expansion and contraction can be configured as bending actuators by combining them with inactive substrates or a counteracting electrode. One example of a bending actuator fabricated from extensional-actuation materials is a bending bimorph constructed from conducting polymers in which a thin layer of the conducting polymer polypyrrole is coated with a metal backing [8,4].

Rapid advances in material fabrication have given rise to various methods of testing electroactive polymers. In most cases the groups involved in the fabrication of the material have developed test protocols and test equipment that enable the measurement of pertinent performance parameters. The most common performance metrics for extensional actuators are stress and strain induced by the electric potential or current in isotonic (constant force) and isometric (constant length) conditions, and for bending actuators the most common metric is radius of curvature. These parameters are most often measured as a function of time to a specified input waveform. Additionally, mechanical properties such as elastic modulus and electrical properties such as conductivity are generally measured as a function of composition or environmental conditions.

The goal of this paper is to present a standard set of testing procedures for extensional and bending actuators stimulated by voltage. We believe that the development of a testing standard will facilitate advances in the field by presenting a consistent set of experimental protocols across a range of material types. Not only will this facilitate direct comparisons between materials synthesized by different groups, it will also facilitate direct comparisons in the performance of different types of electroactive polymers. Researchers will benefit from this development because it will enable a common foundation from which to compare experimental data,

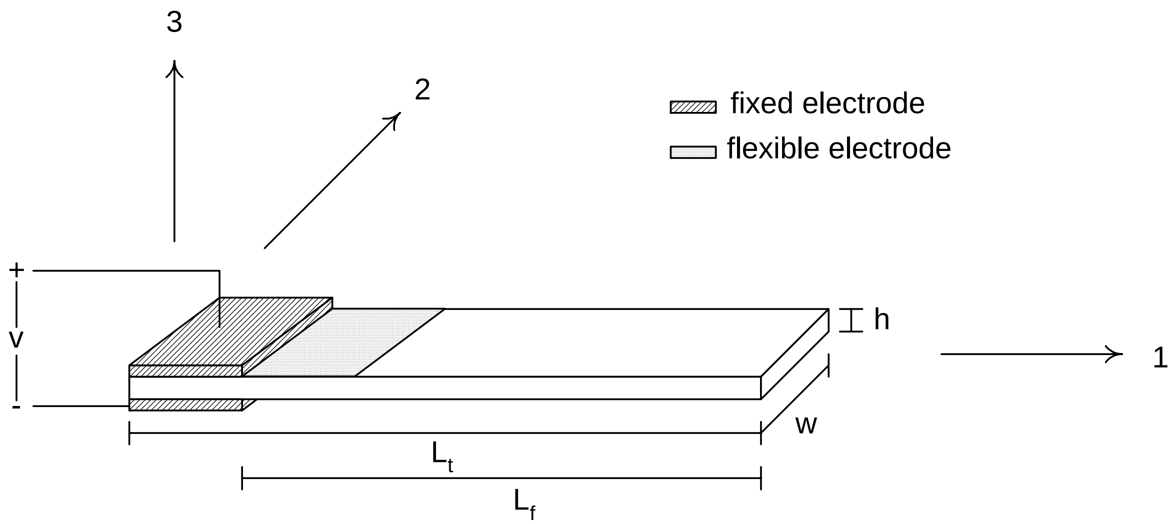

Figure 1: Test geometry.

while end users will benefit by a consistent comparison of performance metrics across a range of material types.

The paper is organized in the following manner. Section 2 describes the nomenclature and basic operating characteristics of bending and extensional actuators. The two sections that follow describe the testing methods associated with each actuator type. The final section of the paper presents representative experimental results using the standardized testing protocols.

NOMENCLATURE AND OPERATING CHARACTERISTICS

For the purposes of standardizing testing methodologies we will define two types of electroactive polymer actuators that exhibit electromechanical coupling. The first type of actuator is defined as an *extensional actuator* because the dominant motion produced by electrical stimulus is an extension or contraction along a single dimension. It should be noted that so-called extensional actuators might exhibit expansion and contraction in multiple dimensions (due to Poisson effects or nature of the mechanism) but for the purposes of the present discussion we assume that the predominant motion of interest is in a single dimension. The second actuator type that we will define is a *bending actuator*. A bending actuator is so defined because the predominant mechanical motion is caused by differential expansion and contraction through the thickness of the material (or laminate of several materials) caused by the application of a voltage. Differential expansion and contraction produces a bending moment about the thickness resulting in a change in the radius of curvature.

The nomenclature that we will utilize for this paper is shown in Figure 1. For both types of actuators we define a coordinate system in which the '3' direction is defined as the direction of the gradient of the electric potential caused by the application of an electric potential. In extensional actuators the '1' direction is defined as the direction of the dominant elongation or contraction. The '2' direction is defined with an axis that is orthogonal to the other two directions in which the '3' direction follows the right-hand rule. In certain circumstances the applied field can be in the same direction as the direction of actuation, or the total potential is in fact more relevant. The unit normal vectors in the three directions are defined as x_i, i = 1, 2, 3. The deflection in the three directions are denoted $u_i(\mathbf{x})$, i = 1, 2, 3. Nomenclature for the actuator geometry is shown in Figure 1 and listed in Table 1. In this work we will use SI units to define all variables. Acceptable SI units for the actuator properties are also listed in Table 1.

Table 2: Test Measurement Parameters

Measurement	Variable	Method	Comments
Geometry			
Thickness	h	Micrometer, profilometer measurements at a minimum of three locations along the length of the polymer. Specified width is the mean of the measurements.	Note measurement accuracy. Uncertainty should be stated, particularly for thickness. In laminates the thickness of each layer needs to be measured as well.
Width	w	Micrometer measurements at a minimum of three locations along the polymer. Specified width is the mean of the measurements.	Note measurement accuracy
Diameter (circular cross-section)	d_c	Micrometer measurements at a minimum of three locations along the polymer. Specified diameter is the mean of the measurements.	
Total Length	L_t	Ruler or micrometer	Note accuracy
Free Length	L_f	Ruler or micrometer	Note accuracy
Length to measurement point	L_m	Ruler or micrometer	Note accuracy
Environment			
Humidity	R_H	Hygrometer	Note whether controlled or uncontrolled
Temperature	T	Thermometer	Note whether controlled or uncontrolled
Polymer Composition			
Base material			
Electrode material			
Mobile ion			

It is assumed that the external stimulus for the actuator is an applied voltage v or an applied current, i. The current density is defined as the induced or applied current divided by the total electrode area, A_e,

$$j = \frac{i}{A_e} \qquad (1)$$

The total electrode area is defined as the area over which the applied electric potential is constant. The electrode can be separated into a fixed electrode and a flexible electrode. The fixed electrode is a region of high electrical conductivity over which the displacement of the actuator in the dominant direction is equal to zero. The flexible electrode is a region of high electrical conductivity which does not restrain the motion of the actuator.

ACTUATOR TESTING METHODS

The standardized test fixture for bending actuators is a clamp with a fixed electrode. The test sample will be placed in the clamp so that the free length of the sample, L_f, is free to move in the '3' direction. The environmental

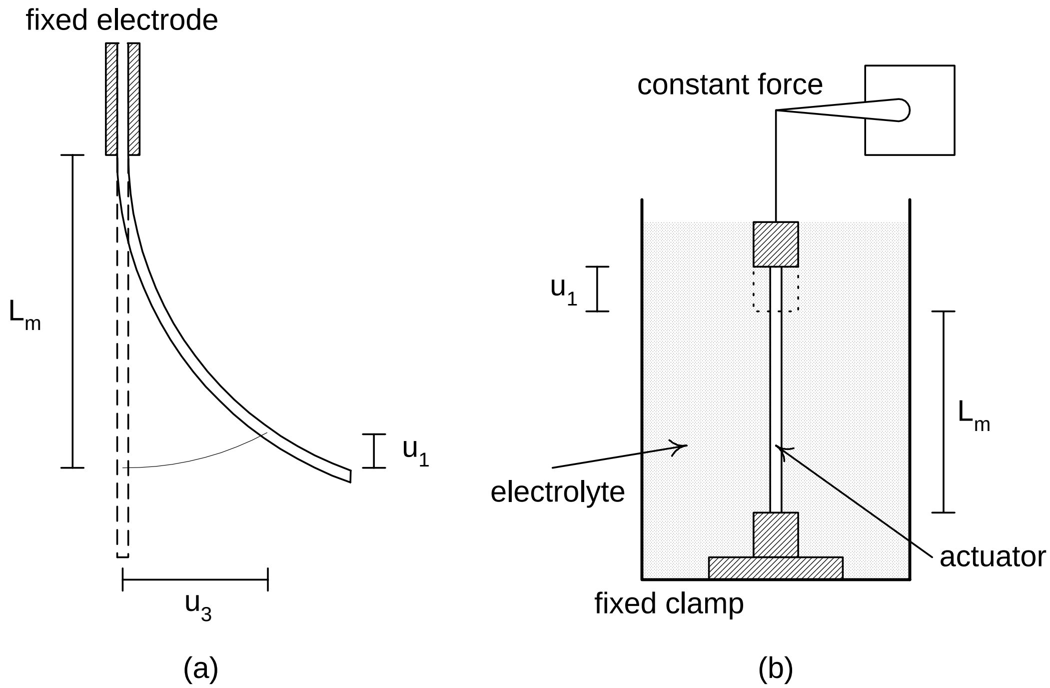

Figure 2: (a) Free deflection test configuration for bending actuators; (b) isotonic measurement of deflection for extensional actuators.

conditions and material parameters that are to be recorded are listed in Table 2.

The standardized test fixture for an extensional actuator is a fixed clamp on one end of the actuator and a clamp on the opposite end that attaches to an arm that can be set to constant position or constant force. Generally the actuator fixture is placed within a liquid electrolyte. The attachment from the movable clamp to the arm is a thin thread.

Free Deflection / Isotonic Measurement

The free deflection test for a bending actuator consists of applying a voltage to the actuator and measuring the deflection of the measurement point located at a distance L_m from the fixed end. Two types of tests can be performed: time domain or frequency domain. In the time domain test the voltage is applied and the displacement of the measurement point is measured as a function of time. In the frequency domain test a broadband excitation signal is used to excite the response over a specified bandwidth and the displacement is measured at the measurement location. Fourier analysis is used to convert the time domain signals into a frequency response function (FRF) between the measured displacement and the input voltage. The response must of course be linear (i.e. obey superposition and scaling) in the actuation range tested in order for frequency methods to be valid.

For the free displacement tests we assume that the bender actuator exhibits a constant curvature along the length of the cantilever. This is the case when the resistance per unit length of the flexible electrode is small enough such that the electric potential is approximately constant along the length of the cantilever and the geometry is uniform. Deformed shapes that exhibit decreasing change in curvature with length generally indicate an electrode with poor conductivity along the length. Curvature as a function of length can be measured and analyzed by digitizing the images and using analysis tools. Twisting actuators indicate non-uniformities. The displacement at the measurement point is defined by the displacement in the ‘3’ direction, u_3, and the displacement in the ‘1’ direction, u_1, as shown in Figure 2a. The displacement can be measured using a number of techniques including digital video, laser vibrometer, or an optical displacement probe. In all cases it is critical for the measurement to be a non-contact technique to eliminate mechanical loading of the sample being tested.

Isotonic measurement for an extensional actuator is performed by applying the voltage and measuring the deflection of the sample at the movable end. Generally the isotonic measurement is performed with a slight

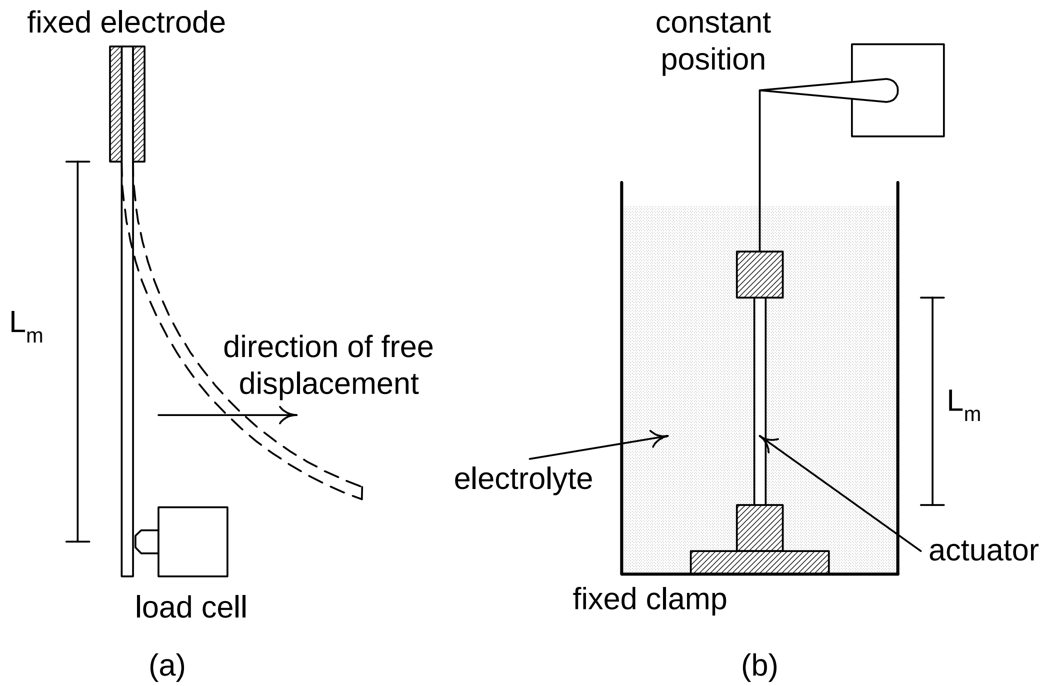

Figure 3: (a) Blocked force test configuration for a bending actuator; (b) isometric measurement for extensional actuators.

pretension to eliminate buckling in the sample. The pretension should be recorded for each test. The deflection u_1 at the movable end is measured with a non-contact technique such as video imagining, laser vibrometry, or an optical probe.

Blocked Force Test, Isometric Measurement and Stiffness Test

The force produced by a bending cantilever sample when the displacement of the measurement point is constrained to be zero is called the blocked force. The test fixture for measuring blocked force consists of placing a cantilevered sample into a fixed electrode. A load cell that measures the appropriate force range is placed at a distance L_m from the fixed electrode as shown in

Figure 3a. The load cell is placed in a manner so that it constrains the motion of the sample for the polarity of the applied voltage. Due to the requirement of measuring the low-frequency response, the load cell must be capable of DC measurement.

This test setup is appropriate for measuring force applied to the load cell by the test sample, whereas a slight modification to the test procedure must be made to measure bidirectional force produced by the test sample. Bidirectional force can be measured by preloading the sample using the load cell such that a constant force is applied to the sample and measured by the load cell. Preloading the sample will allow the measurement of an alternating force that is no greater than the DC bias. This is common in the measurement of force produce by AC voltage applied to the polymer.

It is important to know the stiffness of the structure, and in the case of a laminate, of each layer of the laminate. On a device level, the stiffness is critical in evaluating resistance to perturbations produced in the environment. This stiffness can change as a function of state. The stiffness of each layer is important in evaluating strain and stress generated in active layers, which can then be used to generalize the properties of the active material (e.g. determine stresses produced in addition to the strain generated). Stiffness of a beam is determined by measuring force on the load cell as a function of deflection. These measurements must often be made as a function of voltage. The stiffness of individual layers can also be determined by performing linear stress – strain tests and relating the measured modulus to the stiffness. Generally in EAP materials there is some time dependent response in both the active and passive behaviors. In principle once the active and passive properties are known the two can be superimposed, but this is not necessarily true.

Isometric measurements for extensional actuators are performed by controlling the arm in the test fixture to be constant force. The prescribed force is chosen to make the deflection of the movable clamp negligible compared to the deflection obtained in the isotonic measurement. We define the isometric force as the force measured when the deflection u_1 is at least two orders of magnitude smaller than the isotonic measurement. The force produced by the actuator is measured as a function of the applied voltage in either the time or frequency domain.

Current Measurement and Electrical Impedance Test

The current induced in the polymer due to an applied voltage can be measured with the fixture and circuits shown in Figure 4. As in all current measurements it is important to make sure that the impedance of the measurement circuitry, leads and contacts is small compared to that of the device under test. Current measurements are performed in both the time and frequency domain. Time domain measurements of the current response are most useful in extracting low-frequency material behavior. The frequency response of the voltage referenced to the current is called the electrical impedance of the polymer sample. The electrical impedance is most useful for extracting broadband information regarding the sample characteristics.

Time domain measurements of the current response can be classified according to the applied voltage (or current) waveform. Step voltage excitation will generally produce a sharp rise in the current followed by an exponential-type decay. Sine wave excitation will produce a transient response followed by a steady-state response. For linear systems the relative amplitude and phase shift between the input voltage and output current can be measured and related to the frequency response of the material. Deviations from linearity are measured as distortion in the current waveform to the sinusoidal input, or lack of proportional scaling of the output as the input amplitude is changed. Triangular voltage waveforms are useful for analyzing capacitive and electrochemical behavior.

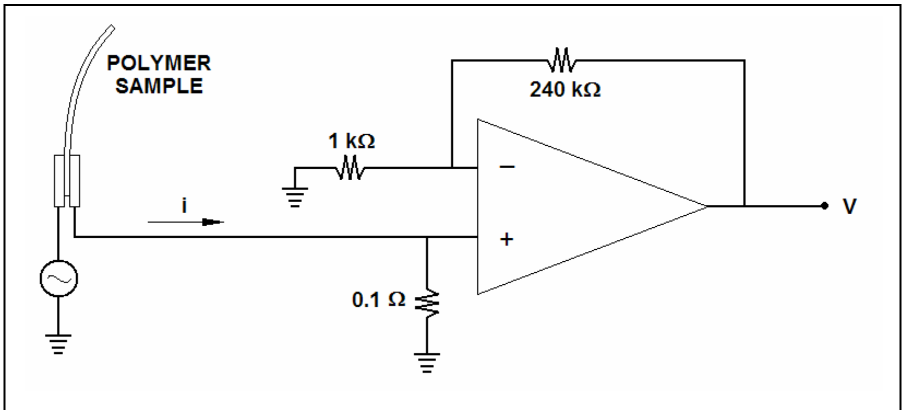

Figure 4: Schematic of the electrical impedance and electrical current measurement for a bender actuator.

Electrical impedance of the polymer sample is measured in the frequency domain. The frequency response of voltage referenced to current is measured over a specified range. The complex-valued impedance is denoted

$$Z(j\omega) = \frac{V(j\omega)}{I(j\omega)} \tag{2}$$

The units of impedance are Ohms. The internal resistance, capacitance, and a range of rate limiting factors can sometimes be identified and extracted by measuring the impedance and fitting or comparing the response of an equivalent circuit model. Impedance, frequency response or dynamic signal analyzers provide the instrumentation and often associated software for frequency response measurement.

DATA PRESENTATION

Time domain data is naturally presented by plotting the output response on the vertical axis versus time on the horizontal axis. The voltage excitation waveform should also be plotted either on the same graph (using right and left vertical axes) or on a graph below or above the graph of the physical variable of interest to help establish the relationship between the controlled and measured quantities.

Frequency domain data is presented by plotting the magnitude and phase of the frequency response as a function of frequency. The magnitude or phase should be on the vertical axis and frequency on the horizontal axis. The magnitude and phase should be plotted above one another with a common frequency axis. The units of the magnitude and the units of frequency should be identified clearly on the axis. Frequency units of Hz are preferred. Bode plots (log-log scale) can be useful if results are over a large range of frequencies and magnitudes. Sometimes Nyquist plots (real vs. imaginary impedance) are useful, especially in extracting electrochemical information. In Nyquist plots the frequency range and increment should be indicated.

DATA ANALYSIS

The measured data from the free deflection, blocked force, stiffness and electrical tests are analyzed to determine intrinsic properties of the bender or extensional actuator. The intrinsic properties are separated into a set of static properties and dynamic properties.

Static Analysis of Bending Actuators

In what follows we assume that the actuator is symmetric about its central axis, that its mechanical response is linear and time independent, and that the elastic properties are not functions of actuation state. These assumptions are not always valid and must be verified []. Referring to Figure 2, when $u_1/L_m << 1$ and $u_3/L_m << 1$ and the curvature of the beam is constant along the length, the strain at the outer fibers of the bender actuator is determined from the expression

$$\varepsilon_1(h/2) = \frac{u_3(L_m)h}{L_m^2} \qquad (3)$$

The derivation for equation (3) is included in Appendix A. If the assumption regarding relative amplitudes of the displacements u_1 and u_3 is not valid, then the strain at the outer fibers is determined from the expression,

$$\varepsilon_1 = \frac{hu_3}{(L_m - u_1)^2 + u_3^2} \qquad (4)$$

This derivation is also included in Appendix A.

The stress generated at the outer fibers of the actuator is

$$\sigma_1(h/2) = Y\varepsilon_1(h/2) = Y\frac{u_3(L_m)h}{L_m^2} \qquad (5)$$

where Y is the modulus of elasticity or Young's modulus. Here we assume that the bending actuator is characterized by one modulus, but the situation can be significantly more complicated for multilayer structures. The Young's modulus for a beam of uniform stiffness (typically one material) can be determined from the force vs. displacement curve using the relationship.

The *intrinsic volumetric energy density* is defined as one-half the product of the stress and strain at the outer fibers,

$$E_v = \frac{1}{2}\varepsilon_1(h/2)\sigma_1(h/2) = \frac{1}{2}Y\varepsilon_1^2(h/2) = \frac{1}{2}\left(\frac{u_3(L_m)h}{L_m^2}\right)^2 \qquad (6)$$

The intrinsic volumetric energy density represents the maximum strain energy stored in the actuator due to the applied electrical input. None of this work is done on the external world, but instead is elastic energy within the structure. As in any spring or capacitor like mechanism, mechanical impedance matching must be done in order to extract this energy externally as work, so that generally less than half can actually be output. The units of energy density are J/m^3. The *actuator volumetric energy density* is defined as

$$E_a = \frac{1}{2}\frac{f_{bl}(L_m)u_3(L_m)}{whL_f} \qquad (7)$$

and represents the maximum output energy per unit volume of the actuator. The units of actuator volumetric energy density are also J/m^3. The intrinsic volumetric energy density must be greater than the actuator volumetric energy density and generally these two quantities are not the same.

The mechanical coupling coefficient, k^2 is the ratio of intrinsic volumetric energy density to input electrical energy. It represents the proportion of electrical energy converted to mechanical energy per cycle. It is not the same as efficiency however. For example, some of the mechanical energy is not converted to external work, but

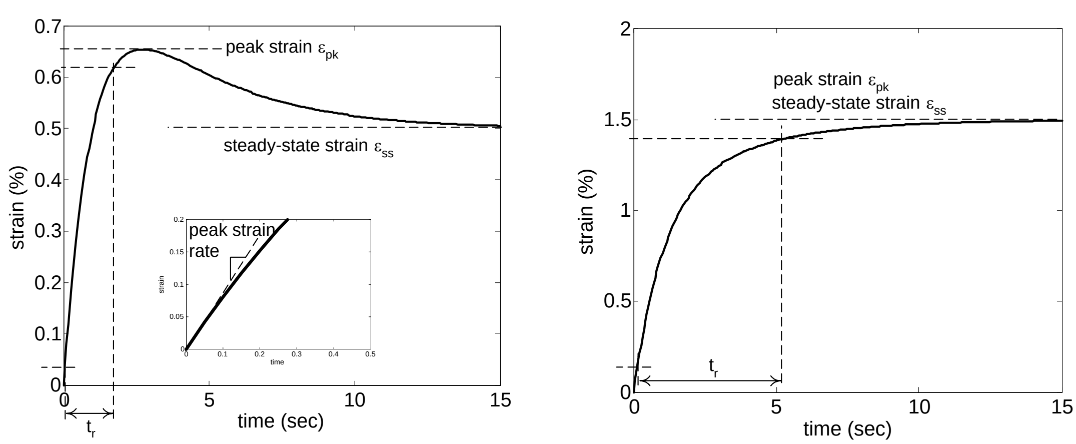

Figure 5: Dynamic strain parameters for bender actuators that exhibit back relaxation (left) and forward relaxation (right).

rather is stored as elastic energy in the actuator structure itself, reducing efficiency. Often however the elastic energy can be converted back to electrical potential energy [e.g. Pelrine and Kornbluh] in many cases, restoring efficiency. It is important to report actuator efficiency, which is given by the ability to convert mechanical work into potential energy (generation) is an important property of any actuator, and can be used in transduction and feedback. Measurement of generation properties is performed using force and/or displacement as inputs.

Static Analysis of Extensional Actuators

The static analysis for extensional actuators assumes that the material is undergoing uniaxial strain and that the stress applied to the material is uniform on the cross-section of the sample. Under these assumptions, the strain in the `1' direction is

$$\varepsilon_1 = \frac{u_1(L_m)}{L_m} \tag{8}$$

The strain is constant along the length under the assumption of uniaxial strain. The free strain is approximated by the isotonic strain at very low levels of stress, and is denoted $\varepsilon_{\mathrm{IT}}$. The stress is computed from the force measurement by the expression

$$\sigma_1 = \frac{f}{A_c}, \tag{9}$$

where A_c is the cross-sectional area of the actuator. The isometric strain is denoted σ_{IM}. The actuator volumetric energy density is computed from the expression

$$E_a = \frac{1}{2}\sigma_{\mathrm{IM}}\varepsilon_{\mathrm{IT}}. \tag{10}$$

Assuming that the elastic modulus Y of the material is constant the volumetric energy density can be written as

$$E_a = \frac{1}{2}Y\varepsilon_{\mathrm{IT}}^2. \tag{11}$$

Dynamic Analysis

The conversions from displacement to strain are also applied to the results of the dynamic free deflection tests. The steady-state strain, peak strain, peak strain rate, and rise time of the bender actuator are determined from the plot of the strain response to a step change in the applied potential. The parameters are defined as:

The steady-state strain ε_{ss} is defined as the strain of the actuator as t approaches ∞.

The peak strain $\varepsilon_{\mathrm{pk}}$ is defined as the maximum absolute value of the strain over the time interval measured.

The peak strain rate is defined as the maximum value of the derivative of the strain as a function of time.

The rise time of the actuator t_r is defined as the time required for the strain to change from 5% of it is peak value to 95% of its peak value.

The time constant of the response τ is useful in cases where the system time dependence is essentially first order, and represents the time to reach 1-1/e of the steady state response, or has a 45^o phase difference from the steady state.

Certain classes of ionic actuators exhibit relaxation properties at long time scales. For certain materials the relaxation is in the direction that is opposite to the initial motion, while for other materials the relaxation is in the same direction. These two types of relaxation behaviors will be denoted *back relaxation* and *forward relaxation*, respectively. Representative examples of these two types of relaxation behavior and the corresponding dynamic strain parameters are shown in Figure 5.

SUMMARY

Methods for testing the performance characteristics of extensional and bending electroactive polymer actuators were presented. A consistent nomenclature and set of test procedures was defined to facilitate comparison of test results between different research groups and to facilitate direct comparison of different material types.

The final paper will include additional sections on dynamic analysis of actuator response.

ACKNOWLEDGEMENTS

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REFERENCES

1. Bar-Cohen, Editor, 2004, *Electroactive Polymer Actuators and Devices as Artificial Muscles*, SPIE Press, Bellingham, WA.

2. Baughman, R., et al, 1999, "Carbon nanotube actuators," *Science*, vol. 284, pp. 1340-1344.

3. Baughman, R., 1996, “Conducting polymer artificial muscles,” *Synthetic Metals*, vol. 78, pp. 339-353.

4. Madden, J., Cush, R., Kanigan, T., Hunter, I. W., 2000, “Fast contracting polypyrrole actuators,” *Synthetic Metals*, vol. 113, pp. 185-192.

5. Madden, et a., 2004, “Artificial muscle technology: physical principles and naval prospects,” *IEEE Journal of Oceanic Engineering*, vol. 29, pp. 706-728.

6. Pelrine, R., Kornbluh, R., Pei, Q., Joseph, J., 2000, “High-speed electrically actuated elastomers with strain greater than 100%,” *Science*, vol. 287, pp. 836-838.

7. Shahinpoor, M., Bar-Cohen, Y., Simpson., J., Smith, J., 1998, “Ionic polymer-metal composites as biomimetic sensors, actuators, and artificial muscles: a review,” *Smart Materials and Structures*, vol. 7, pp. R15-R30.

8. Smela, E., Inganäs, O., Lundström, I., 1995, “Controlled folding of micrometer-sized structures,” *Science*, vol. 268, pp. 1735-1738.

9. Zhang, Q.M., Bharti, V., Zhao, X., 1998, “Giant electrostriction and relaxor ferroelectric behavior in electron-irradiated poly(vinylidene fluoride-trifluoroethylene) copolymer,” *Science*, vol. 280, pp. 2101-2104.

APPENDIX A

Equations (3) and (4) are derived by assuming that the deflected shape of the bending actuator has a constant curvature. As shown in Figure 2 the deflected position at the measurement point is denoted u_1 and u_3. The deflected shape of the beam is represented as a circle with center at a distance ρ in the ‘2’ direction,

$$(L_m - u_1)^2 + (u_3 - \rho)^2 = \rho^2 \qquad \text{(A1)}$$

and the strain is related to the radius of curvature ρ at the outer fibers of the beam from the expression

$$\varepsilon_1 = \frac{h}{2\rho}\,. \qquad \text{(A2)}$$

Solving equation (A1) for the radius of curvature

$$\rho = \frac{(L_m - u_1)^2 + u_3^2}{2u_3} \qquad \text{(A3)}$$

And substituting the result into equation (A2) yields

$$\varepsilon_1 = \frac{hu_3}{(L_m - u_1)^2 + u_3^2} \qquad \text{(A4)}$$

Equation (A4) is used to compute the strain when measuring the deflection of the measurement point in both the ‘1’ and ‘2’ directions.

In the case in which the displacement is small compared to the length, then equation (A4) is rewritten

$$\varepsilon_1 = \frac{hu_3}{L_m^2\left[(1 - u_1/L_m)^2 + u_3^2/L_m^2\right]} \qquad \text{(A5)}$$

If $u_1/L_m << 1$ and $u_3/L_m << 1$ then equation (A5) reduces to

$$\varepsilon_1 = \frac{hu_3}{L_m^2} \qquad \text{(A6)}$$

which is listed in equation (3) as the expression to use in the case of small displacements.