Homework 2 -- Due: see website (Released: February 8)

1. Problems with trees (10+10+10 points)

(a) Develop an algorithm to determine the second smallest element in a binary search tree. (You may assume that all keys are integers.) What is the computational complexity of your algorithm?

(b) An extended binary tree is a binary tree with the restriction that each node has either no children or two children. Any node with no children (leaf) is called an external node and a node with two children is called an internal node.

If T is an extended binary tree with *n* internal nodes, then it has *n+1* external nodes. The internal path length of an extended binary tree T, denoted IPL(T), is the sum of the levels of all the internal nodes. (The root is at level 0.) The external path length of T, EPL(T), is the sum of the levels of all the external nodes.

Prove, using structural induction, that for any extended binary tree with n internal nodes, EPL(T)-IPL(T) = 2n:

(c) A ternary tree is a tree that allows each node to have up to three children (left, middle and right). One or more of these children may be null (NIL) pointers. Prove, using structural induction, that in a non-empty ternary tree, the number of null pointers is one more than twice the number of nodes in the tree.

2. Functions and order estimates [10+10+10 points]

(a) Determine if the following functions from R to R are injective, surjective, and/or bijective.

i. $f(x) = x^3+1$

ii. $f(x) = x/(x^2+1)$

iii. $f(x) = x^2/(x^2+1)$

(b) Assume that the set N = {0,1,2…} and assume that the set Z = {…, -2, -1, 0, 1, 2, …}. Define a bijection between N and Z. Prove that the function you define is bijective

(c) Give the best (tightest) possible big-O estimates for the following functions.

i. $f(n) = n^2(500\log n+n^2)+n(n^2-14)$

ii. $f(n) = (n^3+4n)(n^2-5)$

iii. $f(n) = (n!+3n^3)(2^n-2n)n$

3.) [20 points] Find an explicit formula for f(n) where f(1)=1 and f(n)=f(n-1)+2n -1. Prove your result using induction.

4. **A grading job [12 points]**

Assign a grade of A (correct) or F (failure) to each of the following proofs. If you give a F, please explain exactly what is wrong with the structure or the reasoning in the proof. You should justify all your answers (remember, saying that the claim is false is not a justification).

(a) For all positive integers n, $\sum_{i=1}^{n} i = (n-1)(n+2)/2$.

Proof. The proof will be by induction.
Base case: The claim is valid for $n=1$.
Induction step: Assume that $\sum_{i=1}^{k} i = (k-1)(k+2)/2$. Then $\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$. By the inductive hypothesis, $\sum_{i=1}^{k+1} i = (k-1)(k+2)/2 + (k+1)$. Collecting terms and simplifying, the righthand side becomes $k(k+3)/2$, which can be written as $[(k+1)-1][(k+1)+2]/2$. Thus

$$\sum_{i=1}^{k+1} i = [(k+1)-1][(k+1)+2]/2,$$

which completes the induction step.

□

(b) For every $n \in N$, $n^2 + n$ is odd.

Proof. The proof will be by induction.
Base case: The natural number 1 is odd.
Induction step: Suppose $k \in N$ and $k^2 + k$ is odd. Then, $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + (2k+2)$ is the sum of an odd and an even integer. Therefore, $(k+1)^2 + (k+1)$ is odd. By the Principle of Mathematical Induction, the property that $n^2 + n$ is odd is true for all natural numbers n.

□

(c) For all $x, n \in N$, if $nx = 0$ and $n > 0$, then $x = 0$.

Proof. The proof will be by induction.
Base case: If $n = 1$, then the equation $nx = 0$ implies $x = 0$, since $nx = 1 \cdot x = x$ in this case.
Induction step: Fix $k > 0$, and assume that $kx = 0$ implies $x = 0$. Suppose that $(k+1)x = 0$. Note that $(k+1)x = kx + x$, hence we can conclude that $kx + x = 0$, or in other words, $kx = -x$. Now there are two cases:

i. $x = 0$. In this case, $kx = -x = -0 = 0$, so $kx = 0$. Consequently, the inductive hypothesis tells us that $x = 0$.

ii. $x > 0$. In this case, $-x < 0$ (since $x > 0$). At the same time, $kx \geq 0$ (since $k, x > 0$). But this is impossible, since we know $kx = -x$. We have a contradiction, and therefore Case 2 cannot happen.

In either case, we can conclude that $x = 0$. This completes the proof of the induction step.

□

(d) For all $x, y, n \in \{0, 1, 2, \ldots, \}$, if $\max(x, y) = n$, then $x = y$.

Proof. The proof will be by induction.

Base case: Suppose that $n = 0$. If $\max(x, y) = 0$ and $x, y \in \{0, 1, 2, \ldots\}$, then $x = 0$ and $y = 0$, hence $x = y$.

Induction step: Assume that, whenever we have $\max(x, y) = k$, then $x = y$ must follow. Next suppose x, y are such that $\max(x, y) = k+1$. Then it follows that $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. In this case, we have $x = y$, completing the induction step.

$\square$

5. **Forming project groups [8 points]**

You are teaching a computer engineering course and would like to divide the students into groups for projects such that each group has 4 or 5 students. Using induction, show that it is always possible to divide a class with 12 or more students into reasonable groups (with 4 or 5 students per group). What is the problem when there are less than 12 students in the class?

6. **Some results concerning sets [20 points]**

(a) Is the following statement true for any three sets A, B, C? Prove your answer.

$$|A \cup B \cup C| = |A - B - C| + |B - A - C| + |C - A - B|.$$

(b) Give an example where the following "theorem" fails: *For sets A, B, C and D, let $L ::= (A \cup C) \times (B \cup D)$, $R ::= (A \times B) \cup (C \times D)$. Then $L = R$.*

(c) Identify the mistake in this proof of the "theorem" stated above.

Proof. Since L and R are both sets of pairs, it is sufficient to prove that $(x, y) \in L \leftrightarrow (x, y) \in R$ for all x, y. The proof will be a chain of iff (if and only if) implications:

$$\begin{aligned} (x, y) \in L, &\quad \text{iff} \\ (x \in A \cup C) \wedge (y \in B \cup D), &\quad \text{iff} \\ (x \in A \vee x \in C) \wedge (y \in B \vee y \in D), &\quad \text{iff} \\ (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D), &\quad \text{iff} \\ ((x, y) \in A \times B) \vee ((x, y) \in C \times D), &\quad \text{iff} \\ (x, y) \in (A \times B) \cup (C \times D) = R. & \end{aligned}$$

$\square$

(d) Correct the above proof to show that $R \subseteq L$.

(e) The distributive law for sets informs us that union distributes over intersection:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Use this to prove the generalization of the distributive law for $n \geq 3$ sets:

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap (A \cup B_3) \cap \cdots \cap (A \cup B_n).$$

Additional suggested problems (not graded):

- Define a bijection between N and NxN.
- Sets (page 130 in textbook): 29, 42
- Functions (page 146): 16, 20,
- Complexity of algorithms (page 199): 7, 8, 26; (page 259): 9-18
- Induction (pg 330): 9, 34, 43
- Trees (pg 709): 8, 13-18