Toward Relative Mass With a Pulley-Based Differential Transmission

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***Abstract*—In the well-known electromechanical analogy that converts between electrical and mechanical system representations, mass is the dual of a grounded capacitor. Consequently, any electrical circuit that contains ungrounded capacitors, such as a filter, does not have a mechanical equivalent. A new mechanical system element representing a pulley-based differential transmission is proposed, which, when connected to a mass, is shown to simulate a capacitor in the general case. This new differential mass model provides an additional conceptual framework to model complex mechanical systems such as robotic manipulators.**

***Index Terms*—Dual, electromechanical system analogy, grounded capacitor, relative mass, system model.**

I. Introduction

EQUIVALENT circuit modeling in mechanical, electrical, and electromechanical circuits has its origins in the Maxwell model of solids and the development of the concept of impedance and its generalization, reactance. The idea of decomposing driving-point impedances into terms that represent simple electrical elements began with Foster [1] with the concept that the poles and zeros of a reactance function determine its frequency response. In 1931, Brune [2] described the conditions for network synthesis, given a positive real matrix rational impedance or admittance function. The application of network synthesis, in the context of circuit simulation (both electrical and mechanical), was introduced by Paynter [3] with the introduction of bond graphs and the concept of effort and flow variables implemented in a graphical setting. Bond graph theory and application have continued to develop, with the classical systems dynamics text by Karnopp *et al.* [4], the book by Wellstead [5], and enhanced application and theoretical developments [6], [7]. With the advent of computer technology, in particular, object-oriented modeling, simulation languages have been developed and applied to the bond graph methodology [8]–[10]. The application of engineering electromechanical system simulations is numerous, and it includes magnetic circuits [11], mechatronics, and electromechanical transducers [12], [13]. Extensions to descriptions of classical engineering systems via the graphical representation of the underlying differential equations include the comparison of different methodologies to model multibody systems [14], synthesis of active elements for mechanical systems [15] and microelectromechanical systems [16], and, most recently, the application of differential geometry and Hamiltonian dynamics to the creation of a power-conservative geometric structure [17]. The efficacy of the application of comprehensive engineering system modeling technology cannot be overlooked in the pedagogical context, given the increased assimilation of material that is required in current engineering curricula [18], [19].

A proposal to derive a relative mass model and apply it to the modeling of robot manipulators was originally presented by Stocco and Yedlin [20]. A more complete development of the idea is presented here by starting with the often neglected reference terminal of the mass symbol and interpreting the problem as a need for a mechanical isolation transformer. It is then shown that the proposed model enables one to implement a fundamental system, such as a mechanical bandpass filter, which would not otherwise be possible. Finally, more complete robot examples are presented here that include fewer simplifications than in the previous work. Note from the examples presented that the proposed relative mass model provides no new capabilities that do not already exist with electrical circuit models or bond graphs. This proposal merely gives a similar level of freedom to those with a personal preference for mechanical system models.

Section II of this paper describes conventional electromechanical analogies and points out a shortcoming in the mass model. In Section III, a new pulley-based differential transmission model is proposed that can be added to the mass model to overcome its shortcoming. Section IV shows how the differential mass can be used to represent mass properties of robot manipulators with off-diagonal terms in their mass matrices, while Section V presents concluding remarks and suggests other applications of this study.

II. Motivation

The ability to define an electromechanical equivalent circuit has important applications in electrical circuit analysis, mechanical system analysis, and electromechanical system design. It can be used to represent a hybrid electromechanical system as a pure electrical circuit that can then be simulated by a circuit analysis tool such as SPICE. It is also a potent teaching tool since it spans two seemingly dissimilar areas of study with a common set of fundamentals. The background behind the technique can be found in a large number of textbooks (see [21] for example) on system modeling and control.

The idea stems from the parallelism in the differential equations that describe electrical and mechanical systems, each of which involve an across variable, a through variable, and an

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impedance or admittance variable. In electrical circuits, voltage $E(s)$ is the across variable and current $I(s)$ is the through variable. In mechanical systems, there are two schools of thought. One approach treats velocity $V(s)$ as the across variable and force $F(s)$ as the through variable (i.e., flow [22]). This results in the correspondence between resistance R and damping B, inductance L and stiffness K, and capacitance C and mass M, as given in (1)–(3) below. Here, these equations are referred to as the mass/capacitor (M/C) analogy and are given by

$$E(s) = I(s)R, \qquad V(s) = F(s)\frac{1}{B} \tag{1}$$

$$E(s) = I(s)sL, \qquad V(s) = F(s)\frac{s}{K} \tag{2}$$

$$E(s) = I(s)\frac{1}{sC}, \qquad V(s) = F(s)\frac{1}{sM}. \tag{3}$$

An alternate approach treats force as the across variable and velocity as the through variable. This results in the correspondence between resistance and damping, inductance and mass, and capacitance and stiffness, as given in (4)–(6) below. Here, these equations are referred to as the mass/inductor (M/L) analogy and are given by

$$E(s) = I(s)R, \qquad F(s) = V(s)B \tag{4}$$

$$E(s) = I(s)sL, \qquad F(s) = V(s)sM \tag{5}$$

$$E(s) = I(s)\frac{1}{sC}, \qquad F(s) = V(s)\frac{K}{s}. \tag{6}$$

This mathematical similarity stems from the fact that each element is an impedance to the transmission of energy, be it electrical or mechanical, with either a proportional (1) and (4), integral (3) and (6), or differential (2) and (5) relationship between the across and through variables. As such, the product of the across and through variables corresponds to the rate of energy being dissipated, where an imaginary value denotes energy that is stored and returned without loss, which is given in (7), shown below, for an electrical system and in (8), also shown below, for a mechanical system. Note that this property holds for both the M/C and M/L analogies:

$$\text{units}(EI) = \text{V}\cdot\text{A} = \frac{\text{J}\cdot\text{C}}{\text{C}\cdot\text{s}} = \frac{\text{J}}{\text{s}} \tag{7}$$

$$\text{units}(VF) = \frac{\text{m}}{\text{s}}\cdot\text{N} = \frac{\text{m}}{\text{s}}\cdot\frac{\text{kg}\cdot\text{m}}{\text{s}^2} = \frac{\text{kg}\cdot\text{m}^2}{\text{s}^3} = \frac{\text{J}}{\text{s}}. \tag{8}$$

The two analogies are summarized by the s-domain representations of Ohm's Law, as shown in Table I. Note that, in practice, velocities sum in series, while forces sum in parallel. For example, two velocity sources may not be placed in parallel since an object cannot simultaneously move at two different velocities with respect to a common reference. By a similar reasoning, two force sources may not be placed in series with one another. Nevertheless, the M/L analogy equates forces to voltages that do sum in series and velocities to currents that do sum in parallel. In the M/C analogy, on the other hand, all across and impedance variables (electrical and mechanical) sum in series, while all through and admittance variables sum in parallel.

TABLE I
ACROSS AND THROUGH VARIABLES

	Electrical Circuit	*Mechanical System*	
		M/C Analogy	*M/L Analogy*
Across	E(s)	V(s)	F(s)
Through	I(s)	F(s)	V(s)
Ohm's Law	E(s)=I(s)Z(s)	V(s)=F(s)Z(s)	F(s)=V(s)Z(s)

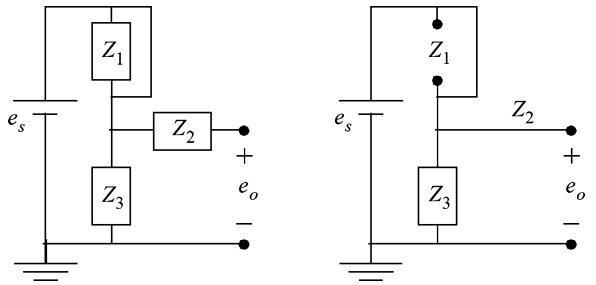

Fig. 1. Equivalent open- and short-circuit models.

Since this is considerably more intuitive, the M/C analogy is used throughout this paper unless otherwise specified.

To derive an analogous system involves replacing each component in the original system with its equivalent in the alternate domain. This, of course, requires topological consistency between components that are to be substituted for one another. Resistors, inductors, and capacitors all share the following three fundamental traits.

1) They have exactly two terminals that can be connected to any node in a circuit.
2) They are symmetrical about their two terminals (i.e., flipping a device over does not affect its response).
3) They obey Ohm's law in the s-domain.

Since voltage is a relative measurement, Ohm's law is better represented by (9), where $E_1(s)$ and $E_2(s)$ are the two terminal voltages of an element with impedance $Z(s)$. It follows that the traits listed earlier are not independent of one another, since Ohm's law (trait 3) can only be defined for a two-terminal device (trait 1) and holds for both positive and negative voltages and currents (trait 2), which can be given by

$$E_2(s) - E_1(s) = I(s)Z(s). \tag{9}$$

Ohm's law also implies that any impedance whose terminals are shorted together will have zero voltage drop across it and, therefore, zero current through it. Since the impedance passes no current, it simulates an open circuit (see Z_1 in Fig. 1). Similarly, any impedance with either of its terminals left floating will have zero current through it and, therefore, zero voltage drop across it. Since the impedance experiences no voltage drop, it simulates a short circuit (see Z_2 in Fig. 1). Note that replacing Z_1 in Fig. 1 with a short circuit would allow current to flow and would contradict Ohm's law. Similarly, replacing Z_2 with an open circuit would allow the positive terminal of e_o to float (i.e., have an arbitrary voltage) and result in a potentially nonzero voltage drop across Z_2.

Electrical Symbol	Mechanical Symbol	Relationship
Current Source: $I(s)$	Force Source: $F(s)$	$F(s) = I(s)$
Voltage Source: $E(s)$	Velocity Source: $V(s)$	$V(s) = E(s)$
Resistor: R	Damper: B	$B = 1/R$
Inductor: L	Spring: K	$K = 1/L$
Capacitor: C	Mass: M	$M = C$

Fig. 2. Electromechanical equivalents.

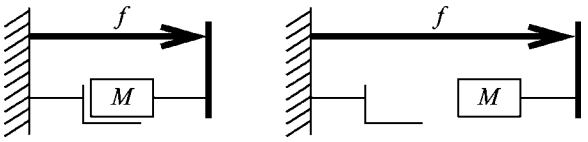

Fig. 3. Implied behavior of mass symbol during translation.

Voltage and current sources share the following two traits.

1) They have exactly two terminals.
2) They are directional with respect to their two terminals (i.e., flipping a device over changes its sign).

Unlike passive components, there are restrictions on how sources may be connected. For example, connecting two dissimilar voltage sources in parallel or two dissimilar current sources in series will result in an unsolvable circuit. By the M/C analogy, the commonly accepted electromechanical equivalents are shown in Fig. 2. Additional symbols exist for angular motion, but the linear motion symbols shown above are more commonly used.

All components in Fig. 2 have two terminals and obey Ohm's law. Note, however, that the mass symbol contains a reference terminal, which is very often neglected. This reference terminal was originally proposed to maintain consistency between components (i.e., a mass symbol can extend similar to a spring or damper, as illustrated in Fig. 3) and to indicate that the energy stored in a mass corresponds to its velocity with respect to an unambiguous reference.

The reason to neglect the reference terminal is because, for any object that exists on earth, the reference is always earth. Consider the example of a payload M_p traveling on a slippery train car M_t, as shown in Fig. 4. It is a simple matter to change the reference point of the spring and damper so that they store and/or dissipate energy when the payload moves with respect to the ground [see Fig. 4(a)] or when the payload moves with respect to the train car [see Fig. 4(b)]. However, the energy stored in the payload always depends on its velocity with respect to the ground. It is independent of the velocity of the train car, regardless of how the system is configured.

Since the reference terminal of the mass is always connected to ground, it is commonly omitted. This implicit connection impacts the symmetry of the mass symbol. Consider the mechanical system shown in Fig. 5(a) that involves a force applied to a mass. Flipping the mass over (i.e., interchanging its terminals) implicitly moves the earth ground from the left to the right [see Fig. 5(b)]. Theoretically, the ground could be left in its original position, as shown in Fig. 5(c), to imply that the mass is taken to be the reference position (n_1) and that the force is effectively pushing the earth (n_2) away from the reference. Although unconventional, this is the mechanical representation of a floating ground, which is common in electric circuits and is theoretically acceptable.

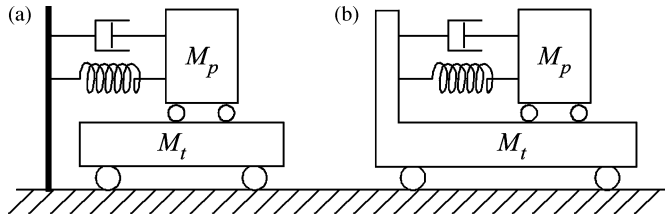

Fig. 4. Train car and payload example.

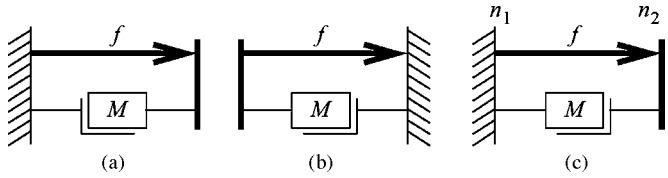

Fig. 5. Force/mass example.

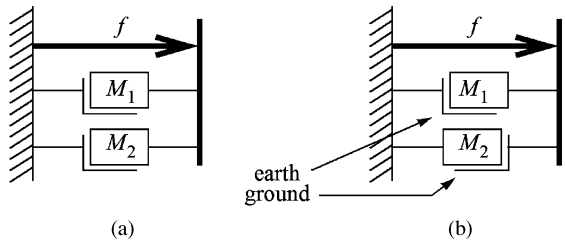

Fig. 6. Parallel mass example.

However, if a second mass is introduced, as shown in Fig. 6(a), flipping one of the masses over [see M_2 in Fig. 6(b)] creates a conflict. In Fig. 6(b), M_1 is connected to the reference of M_2 and *vice versa*. In order for the force f to translate M_1 with respect to its own reference, it must also translate the references of M_1 and M_2 with respect to one another. In other words, the earth must be pushed away from itself. This is similar to the conflict when two electrical sources are connected incorrectly and result in a system with no physical meaning. As a minimum requirement, all masses in a mechanical system should share a common reference node since they represent a common entity.

Consequently, a standard rule-of-thumb when converting a mechanical system to an analogous electrical circuit is to start by replacing all masses by grounded capacitors, as shown in Fig. 7. This fixes the reference voltage of the capacitor to zero and guarantees correspondence between the following two equations:

$$E_2(s) - E_1(s) = E_2(s) - 0 = E_2(s) = I(s)\frac{1}{sC} \quad (10)$$

$$V_2(s) - V_1(s) = V_2(s) - 0 = V_2(s) = F(s)\frac{1}{sM}. \quad (11)$$

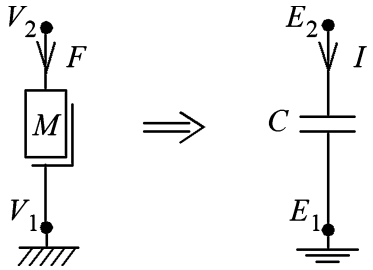

Fig. 7. Capacitor/mass equivalent models.

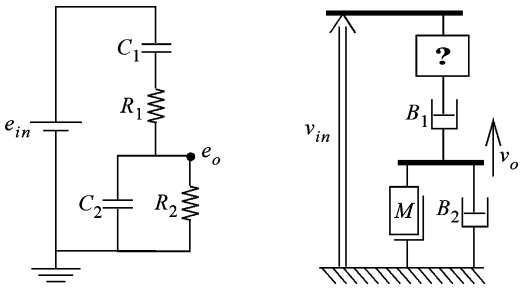

Fig. 8. Bandpass filter example.

Since no connectivity restrictions exist for capacitors, a capacitor can always be used to simulate a mass but a mass cannot always be used to simulate a capacitor. Consider, for example, the bandpass filter in Fig. 8. Here, R_1, R_2, and C_2 are replaced by B_1, B_2, and M, respectively, in the equivalent mechanical system, but there is no component that can be used in place of C_1, since it does not share a node with C_2 that can be used to represent earth ground.

Note that the bandpass filter shown in Fig. 8 can be represented in the mechanical domain if the M/C analogy is abandoned in favor of the M/L analogy that replaces capacitors by springs instead of masses. However, topological consistency is lost since serial connections become parallel connections and *vice versa*, and even this ceases to be an option if inductors are added to the circuit shown in Fig. 8.

III. Pulley-Based Differential Transmission Model

It would be useful to have a mechanical component that simulates a capacitor, in general, so that any electrical circuit, even one that contains capacitors with no common nodes, can be represented by an equivalent mechanical system. It should have two symmetric terminals, should obey Ohm's law, and should be able to accommodate a nonzero velocity at both terminals simultaneously. In other words, it should have no connectivity constraints.

In circuit design, the usual way to eliminate a ground reference is by adding an isolation transformer. For example, the grounded capacitor shown in Fig. 9 is made to act like an ungrounded capacitor by connecting it to an isolation transformer. Similarly, the implicit ground reference of a mass could be eliminated by connecting it to the mechanical equivalent of an isolation transformer.

A transformer scales up voltage and scales down current by its winding ratio similar to a mechanical gearbox that scales up velocity and scales down force by its tooth count ratio. With

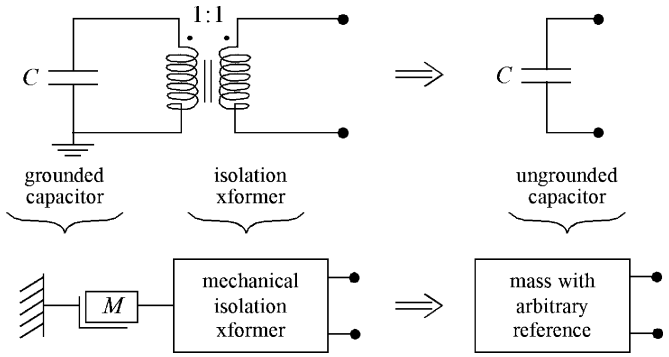

Fig. 9. Achieving relative mass with a mechanical isolation transformer.

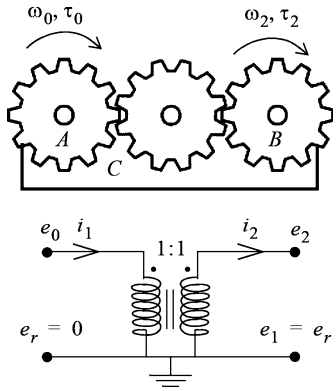

Fig. 10. Transformer and gearbox analogy.

unity winding and tooth ratios, the models shown in Fig. 10 and described by (12)–(15) are equivalent

$$\omega_2 = \omega_0 \tag{12}$$

$$\tau_2 = \tau_0 \tag{13}$$

$$e_2 = e_0 \tag{14}$$

$$i_2 = i_1. \tag{15}$$

Note that the mechanical system shown in Fig. 10 represents an ideal 1:1 transmission system with no friction, backlash, or contact dynamics whatsoever. A gearbox is used in this example to indicate that there is no slippage, but two identical pulleys joined by an ideal timing belt would be valid as well.

Note, however, that the transformer in shown Fig. 10 is not an isolation transformer, since its input and output ports share a common reference node ($e_1 = e_r$). The same can be said for the gearbox since both gears (A and B) are fixed to a common reference body C. This mechanical constraint is removed by making gear B a planetary gear by mounting it on a rotating member E, which rotates at a velocity ω_1 about a common axis with gear D, as shown in Fig. 11. Combining (16) and (17) results in (18)

$$\omega_1 - \omega_4 = \omega_2 - \omega_1 \tag{16}$$

$$\omega_4 = -\omega_0 \tag{17}$$

$$\omega_0 = \omega_2 - 2\omega_1 \tag{18}$$

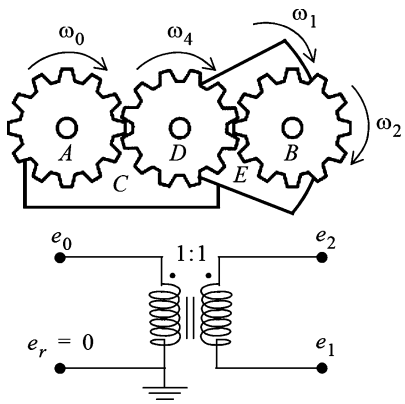

Fig. 11. Electromechanical differential transmissions.

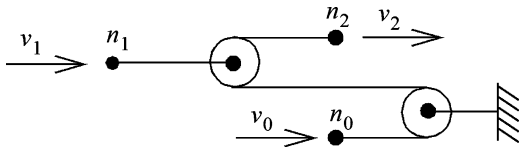

Fig. 12. Linear motion differential transmission.

which is similar to the equation for an isolation transformer (20)

$$e_0 - e_r = e_2 - e_1 \tag{19}$$

$$e_0 = e_2 - e_1; \quad e_r = 0 \tag{20}$$

except for the factor of 2, which multiplies the reference velocity ω_1.

To correspond with the more commonly used linear motion symbols shown in Fig. 2, a linear version of the isolation transformer is proposed, as shown in Fig. 12, which has a similar velocity relationship (22), shown below, as the planetary gearbox shown in Fig. 11 [see (18)]:

$$v_1 = \frac{1}{2}(v_2 - v_0) \tag{21}$$

$$v_0 = v_2 - 2v_1. \tag{22}$$

Similar to the planetary gearbox, the linear transmission has the same equation as the electrical isolation transformer, except for the factor of 2 applied to velocity of node n_1. Note, however, that setting v_0 to 0 in (22) (i.e., connecting node n_0 to ground) results in the following equation:

$$v_2 = 2v_1. \tag{23}$$

Therefore, an additional pulley can be added to the system, as shown in Fig. 13, to scale down the velocity of node n_1 by a factor of 2, thereby resulting in a velocity equation (26), shown below, that exactly mirrors that of the electrical isolation transformer (20). The resulting system is a linear differential transmission where node n_0 is translated at a velocity equal to the difference between the velocities of nodes n_1 and n_2 and is given by

$$v_0 = v_2 - 2v_3 \tag{24}$$

$$v_1 = 2v_3 \tag{25}$$

$$v_0 = v_2 - v_1. \tag{26}$$

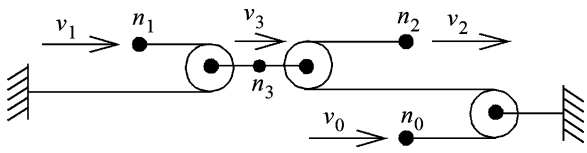

Fig. 13. Unity-gain linear motion differential transmission.

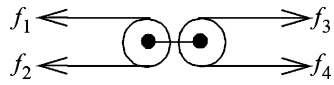

Fig. 14. Free-body diagram of center pulleys and stiff member.

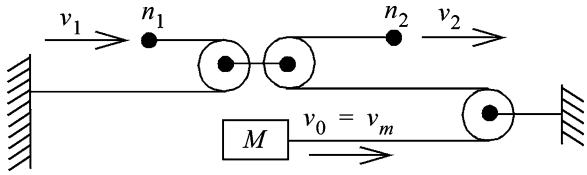

Fig. 15. Differential mass model.

Since the proposed transmission system shown in Fig. 13 is intended to model an ideal electric isolation transformer, therefore, all cables and pulleys have zero mass and friction, and the cable is infinitely stiff and long. There is zero gravity and the ideal cables do not buckle under a compressive load; therefore, the system always holds its shape, regardless of whether cable forces are compressive or tensile. This idealized model is not unlike an ideal spring that is massless, linear, and neither bottoms out nor reaches full extension or an ideal damper that is massless, infinitely long, and friction-free. In addition, it is later shown that the ideal cables in the proposed model may be replaced by ones that need not resist buckling.

Since the connection at node n_3 is a stiff member, a free-body diagram can be drawn for the member and pulleys (see Fig. 14). Since the pulleys have neither mass nor friction, therefore, no reaction forces can occur, and both the pulleys and cables are in a constant state of equilibrium. Equation (27), shown below, follows from the fact that a cable in equilibrium has a common tension throughout, and (28), also shown below, follows from the absence of reaction forces. Combining (27) and (28) results in (29), shown below, which states that the tension in both cables is equal:

$$f_1 = f_2, \qquad f_3 = f_4 \tag{27}$$

$$f_1 + f_2 = f_3 + f_4 \tag{28}$$

$$f_1 = f_2 = f_3 = f_4. \tag{29}$$

Connecting a mass to the proposed differential transmission results in the system shown in Fig. 15, where equal velocities at nodes n_1 and n_2 will not cause any translation of the mass, but only a translation of the two massless pulleys. A difference in the two velocities, however, will cause the mass to be translated by the difference.

Although the system shown in Fig. 15 uses "unrealistic" cables that can be compressed without buckling, this system is,

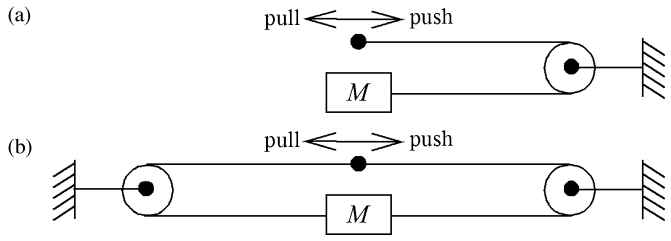

Fig. 16. Pull–pull transmission system.

Fig. 17. Block diagrams of differential mass with explicit and implicit ground references.

in fact, physically realizable. Consider the pulley system shown in Fig. 16(a) that cannot be pushed without buckling. Adding a return path to the cable [see Fig. 16(b)] keeps the cable in tension at all times, thereby allowing one to "push" on the cable without buckling. Pull–pull transmission systems, such as capstan drives, commonly apply this technique to exploit the benefits of cable-based actuation. Although return paths could be added to Fig. 15, they would not help to provide intuition; therefore, it is preferred that return paths be assumed.

Differentiating (26) results in (30) and (31), shown below, which is transformed to the s-domain in (32), also shown below. Note that in (32), F is the tensile force, which is equal in both cables:

$$\dot{v}_0(t) = \dot{v}_m(t) = a_m(t) = \frac{f}{M} \tag{30}$$

$$\dot{v}_2 - \dot{v}_1 = \frac{f}{M} \tag{31}$$

$$V_2(s) - V_1(s) = F(s)\frac{1}{sM}. \tag{32}$$

Equation (32) is simply Ohm's law for a mass (11) with the $V_1(s) = 0$ constraint removed. This result exactly matches the equation for a "not necessarily grounded" capacitor (10).

The block diagrams shown in Fig. 17 may be used to represent the differential mass element that may or may not include an explicit ground reference, which is implied when an explicit ground reference is absent. Unlike the mass shown in Fig. 2, the two terminals of the differential mass (n_1 and n_2) can be connected to any node in a mechanical circuit.

The instantaneous power in the differential mass $P_d(t)$ (33), shown below, is computed by substituting (30) into the equation for mechanical power. This corresponds to the instantaneous power in a capacitor $P_c(t)$, shown below in (34). From (32), the complex impedance $Z_d(s)$ of the differential mass can be obtained directly (35), shown below, and is given by

$$P_d(t) = \frac{\text{work}}{\text{unit time}} = f(t)v(t) = Ma_m(t)v_m(t)$$
$$= M\frac{f(t)}{M}(v_2(t) - v_1(t)) = f(t)(v_2(t) - v_1(t)) \tag{33}$$

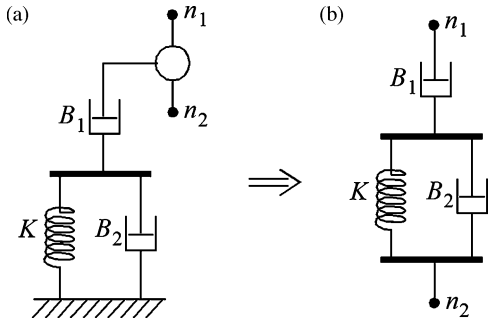

Fig. 18. Isolated BK network.

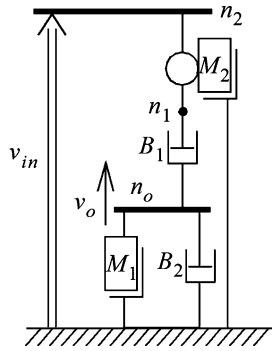

Fig. 19. Mechanical model of a bandpass filter.

$$P_c(t) = i(t)(e_2(t) - e_1(t)) \tag{34}$$

$$Z_d(s) = \frac{1}{Ms}. \tag{35}$$

The proposed differential mass is thereby shown to contain two symmetric, interchangeable terminals, to obey Ohm's law, and to satisfy the same differential equation as a mass but with an arbitrary reference velocity, thereby making it a general mechanical equivalent of a capacitor.

Of course, an isolation transformer may be used to remove the ground reference from any electrical network. In other words, the proposed pulley model shown in Fig. 13 could be used to remove the ground reference from any mechanical network and not just a mass. For example, adding a pulley system to the grounded BK (spring/damper) network shown in Fig. 18(a) results in the equivalent of the ungrounded BK network shown in Fig. 18(b).

The proposed differential mass element can be applied to the bandpass filter example shown in Fig. 8. Substituting a differential mass for the unknown element results shown in Fig. 19, which may be analyzed similar to an electric circuit. At very low frequencies [see (36) below], the impedances of both the conventional and differential masses approach infinity (37), shown below, and due to the finite impedance of damper B_2, all of the input velocity is "dropped" across the differential mass M_2 [see (38), shown below]. In other words, the masses simulate mechanical open circuits similar to the capacitors in Fig. 8.

At very high frequencies [see (39), shown below], the impedances of both the conventional and differential masses approach zero (40), shown below, and all of the input velocity is

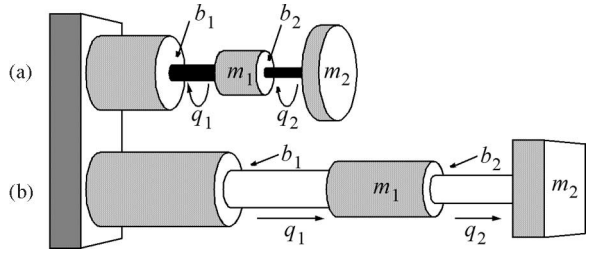

Fig. 20. Redundant rotary and prismatic actuator.

"dropped" across damper B_1 [see (41), shown below]. In other words, the masses simulate mechanical short circuits similar to the capacitors in Fig. 8. At all other frequencies, the output velocity v_o is nonzero and finite, and a mechanical bandpass filter is realized:

$$v_2 = v_{in}(\omega = 0) \tag{36}$$

$$Z_{M_2}(s) = \frac{1}{M_2(j0)} = \infty \tag{37}$$

$$v_2 - v_1 = v_{in} \tag{38}$$

$$v_2 = v_{in}(\omega = \infty) \tag{39}$$

$$Z_{M_1}(s) = Z_{M_2}(s) = \frac{1}{M(j\infty)} = 0 \tag{40}$$

$$v_1 - v_0 = v_{in}. \tag{41}$$

IV. Application to a Robot Mass Matrix

A. Serial 1-DOF Robots

Due to the constraints associated with conventional mass, there are mechanical systems which cannot be described by a mechanical system diagram. Elaborate transmission systems, such as robotic manipulators, may contain mass elements that are only present when relative motion occurs between individual motion stages. Systems such as these can only be modeled using electric circuits since capacitors can be used to model this property but conventional masses cannot.

Let us consider the simplified dynamics of a 2-degree-of-freedom (2-DOF) robot [see (42) and (43), shown below], where M is the mass matrix, B is the damping matrix, F is a vector of joint forces/torques, R is a vector of joint rates, and s is the Laplace operator. Gravitational and Coriolis effects are assumed to be negligible for the purpose of this example. If the damping in the system is dominated by the actuator damping coefficients (typical for a serial manipulator), B is a diagonal matrix (43). M, on the other hand, represents the effective mass perceived by each joint and may not be diagonal or otherwise easily simplified and is given by

$$F = BR + MsR \tag{42}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + Ms \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{43}$$

For simple kinematic arrangements, such as a redundant 1-DOF rotary [see Fig. 20(a)] or prismatic [see Fig. 20(b)] actuator with a distal actuator mass/moment m_1 and load mass/moment m_2, the mass matrix M is given in (44), shown below. The system may be modeled by the mechanical system diagram or its electrical equivalent, as shown in Fig. 21,

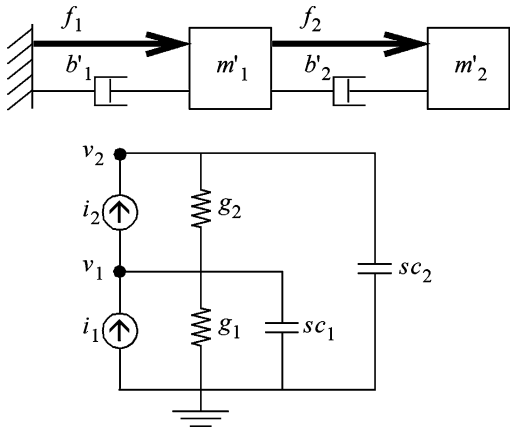

Fig. 21. System models of redundant actuators.

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_2 \end{bmatrix}. \tag{44}$$

Performing nodal analysis on the circuit shown in Fig. 21 results in (45), shown below, by inspection. Note, however, that (45) contains the term $i_1 - i_2$ as well as v_2, which corresponds to the endpoint velocity in the mechanical system, or, in other words, the sum of the joint rates $r_1 + r_2$. To obtain a correspondence between electrical and mechanical component values, the dynamic equation (43) is rearranged in (46), shown below, where the associated damping B' and mass M' matrices are given in (47) and (48), also shown below. From (47), the admittances g_1 and g_2 and capacitances c_1 and c_2 correspond to the equivalent damping and mass values b'_1 and b'_2 and m'_1 and m'_2, respectively [see (49), shown below]:

$$\begin{bmatrix} i_1 - i_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_1 + g_2 & -g_2 \\ -g_2 & g_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} s \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} f_1 - f_2 \\ f_2 \end{bmatrix} = B' \begin{bmatrix} r_1 \\ r_1 + r_2 \end{bmatrix} + M's \begin{bmatrix} r_1 \\ r_1 + r_2 \end{bmatrix} \tag{46}$$

$$B' = \begin{bmatrix} b'_1 + b'_2 & -b'_2 \\ -b'_2 & b'_2 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 \end{bmatrix} \tag{47}$$

$$M' = \begin{bmatrix} m'_1 & 0 \\ 0 & m'_2 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \tag{48}$$

$$\begin{bmatrix} b'_1 \\ b'_2 \\ m'_1 \\ m'_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ m_1 + m_2 \\ m_2 \end{bmatrix}. \tag{49}$$

In this example, it is possible to model the system using conventional masses but only because the manipulator has a single1 DOF; therefore, M' is diagonal, and there is no cross-coupling

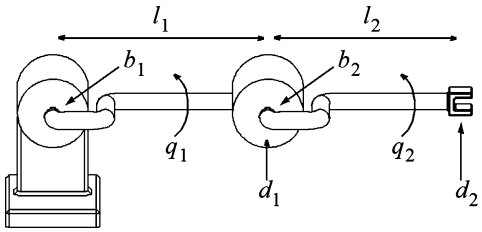

Fig. 22. 2-DOF serial robot.

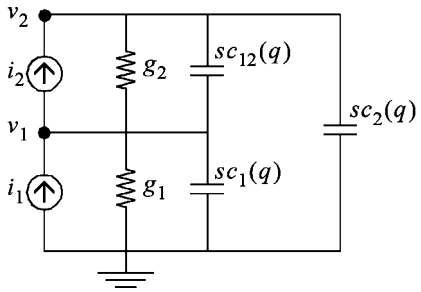

Fig. 23. Electrical model of 2-DOF serial robot.

between actuators. In general, however, effective mass is not always decoupled and the off-diagonal elements of M' may be nonzero. When this is the case, conventional masses cannot be used to model the effective mass of the system since they cannot model the off-diagonal terms that describe inertial effects resulting from relative motion of the actuators.

B. Serial 2-DOF Robots

Consider the 2-DOF serial robot shown in Fig. 22. The mass matrix for this mechanism is approximated in [23] by two point masses d_1 and d_2, which are positioned at the distal actuator and end-effector, as indicated shortly. The resulting mass matrix (50), shown below, has the terms given in (51)–(53), also shown below, where q_1 and q_2 are the joint angles, and l_1 and l_2 are the link lengths. Similar to the previous example, actuator damping coefficients b_1 and b_2 are taken to dominate the total system damping

$$M(q) = \begin{bmatrix} m_1(q) & m_3(q) \\ m_3(q) & m_2(q) \end{bmatrix} \tag{50}$$

$$m_1 = l_2^2 d_2 + 2l_1 l_2 d_2 \cos(q_2) + l_1^2(d_1 + d_2) \tag{51}$$

$$m_2 = l_2^2 d_2 \tag{52}$$

$$m_3 = l_2^2 d_2 + l_1 l_2 d_2 \cos(q_2). \tag{53}$$

The equivalent circuit of this system is shown in Fig. 23. It is similar to Fig. 21, except that the capacitor values are configuration-dependent, and a third capacitor c_{12} is included to model the coupled mass terms that are present. Performing nodal analysis results in (54), shown below, and the corresponding M' matrix in (55), also shown below, that can be rearranged to solve for the mechanical model parameters in terms of the physical mass values in (56), also shown below. Note that B' is the same

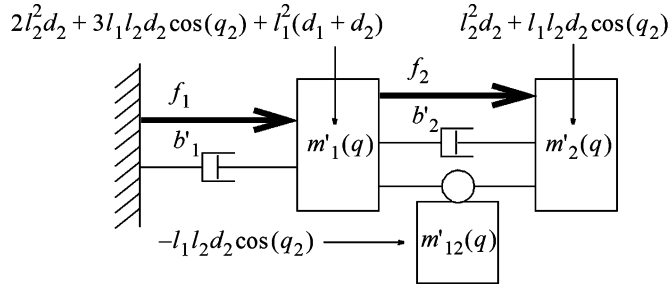

Fig. 24. Mechanical model of 2-DOF serial robot.

diagonal matrix as in (47):

$$\begin{bmatrix} i_1 - i_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_1 + g_2 & -g_2 \\ -g_2 & g_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_{12} & -c_{12} \\ -c_{12} & c_2 + c_{12} \end{bmatrix} s \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{54}$$

$$M'(q) = \begin{bmatrix} m'_1 + m'_{12} & -m'_{12} \\ -m'_{12} & m'_2 + m'_{12} \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_3 - m_2 \\ m_3 - m_2 & m_2 \end{bmatrix} \tag{55}$$

$$\begin{bmatrix} m'_1 \\ m'_2 \\ m'_{12} \end{bmatrix} = \begin{bmatrix} m_1 + m_3 \\ m_3 \\ m_2 - m_3 \end{bmatrix}. \tag{56}$$

Note from (55) that M' is only diagonal when $m'_{12} = 0$, or, in other words, when $m_2 = m_3$. From (52) and (53), this is merely the special case when $q_2 = \pm\pi/2$. Therefore, it is not possible to model this system using only masses due to their implicit ground reference, as described in Section II. The off-diagonal terms can, however, be modeled using the differential mass proposed in Section III. It results in a mechanical system model that is topologically identical to the equivalent circuit shown in Fig. 23, where each grounded capacitor (c_1 and c_2) is replaced by a regular mass, and each ungrounded capacitor (c_{12}) is replaced by a differential mass that can accommodate a nonzero reference velocity. The resulting mechanical system is shown in Fig. 24.

Although m'_{12} has a negative value when $-\pi/2 < q_2 < \pi/2$, the net mass perceived by each actuator is always positive because M is positive definite. When m'_{12} is negative, it simply means that the motion of actuator 1 reduces the net mass perceived by actuator 2, but the net mass perceived by actuator 2 is always greater than 0.

C. Parallel 2-DOF Robots

The same technique can be applied to parallel manipulators, such as the 2-DOF planar manipulator shown in Fig. 25. In a parallel manipulator, each actuator is referenced to ground, but there remains a coupling between the effective mass perceived by each actuator which, like a serial manipulator, is configuration-dependent. This coupling is modeled by c_{12} and m'_{12} in the equivalent electrical and mechanical models shown in Fig. 25.

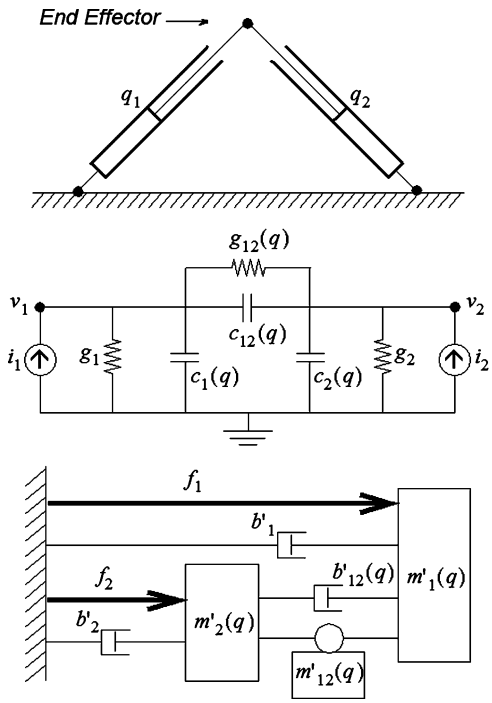

Fig. 25. 2-DOF parallel robot with electrical and mechanical equivalents.

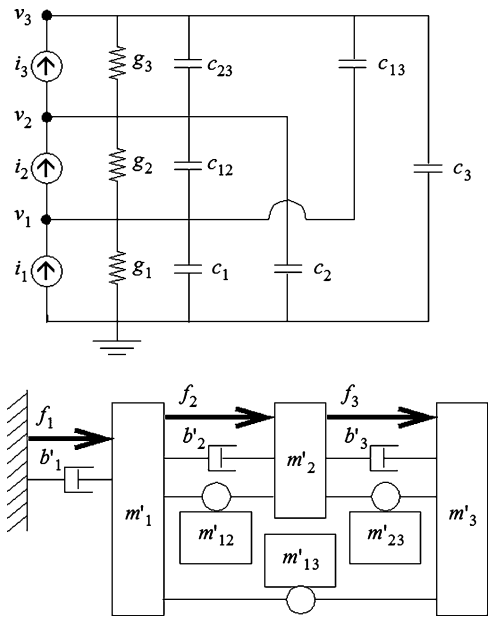

Fig. 26. Model of a 3-DOF serial robot.

Due to the existence of passive joints (represented by dark circles), parallel manipulators also have coupled, configuration-dependent damping terms that cannot necessarily be neglected. These are represented by g_{12} and b'_{12}, which are shown in Fig. 25.

Performing nodal analysis on the circuit shown in Fig. 25 results in (57), shown below, by inspection. For a parallel robot, currents and voltages correspond directly to joint forces and joint rates, and therefore, $B' = B$, and $M' = M$. For a mass matrix of the form given in (50), the elements of the M' matrix are given in (59), shown below. For a damping matrix of the same form, the parameter values are computed similarly:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} g_1 + g_{12} & -g_{12} \\ -g_{12} & g_2 + g_{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_{12} & -c_{12} \\ -c_{12} & c_2 + c_{12} \end{bmatrix} s \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{57}$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} b'_1 + b'_{12} & -b'_{12} \\ -b'_{12} & b'_2 + b'_{12} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} m'_1 + m'_{12} & -m'_{12} \\ -m'_{12} & m'_2 + m'_{12} \end{bmatrix} s \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} m'_1 \\ m'_2 \\ m'_{12} \end{bmatrix} = \begin{bmatrix} m_1 + m_3 \\ m_2 + m_3 \\ -m_3 \end{bmatrix}. \tag{59}$$

D. Multiple DOF Robots

This technique is easily extended to devices with any number of DOFs n. With serial manipulators, the compliance and damping are often dominated by the actuators, and therefore, the damping B and stiffness K matrices are diagonal [see (60) and (61), shown below]. With parallel manipulators, the B and K matrices typically contain off-diagonal terms, but they are easily modeled using conventional elements since springs and dampers have two terminals and no connectivity constraints:

$$B = \operatorname{diag}\left(\begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}\right) \tag{60}$$

$$K = \operatorname{diag}\left(\begin{bmatrix} 1/k_1 & 1/k_2 & \dots & 1/k_n \end{bmatrix}\right). \tag{61}$$

To account for inertial cross-coupling, the model must contain a capacitor and/or a differential mass between every pair of actuators. For example, the electric circuit model and corresponding mechanical system model of a serial 3-DOF manipulator are shown in Fig. 26. The capacitance matrix C, which results from the nodal analysis (62), shown below, of the circuit shown in Fig. 26, is shown below in (63):

$$\begin{bmatrix} i_1 - i_2 \\ i_2 - i_3 \\ i_3 \end{bmatrix} = G(q) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + C(q)s \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{62}$$

$$C(q) = \begin{bmatrix} c_1 + c_{12} + c_{13} & -c_{12} & -c_{13} \\ -c_{12} & c_2 + c_{12} + c_{23} & -c_{23} \\ -c_{13} & -c_{23} & c_3 + c_{23} + c_{13} \end{bmatrix}. \tag{63}$$

Similar to previous examples, the 3×3 mass matrix M' (65), shown at the bottom of the next page, is rearranged into the form shown in (64), shown below, to parallel the current/voltage relationship of (62). For a mass matrix M of the form shown in

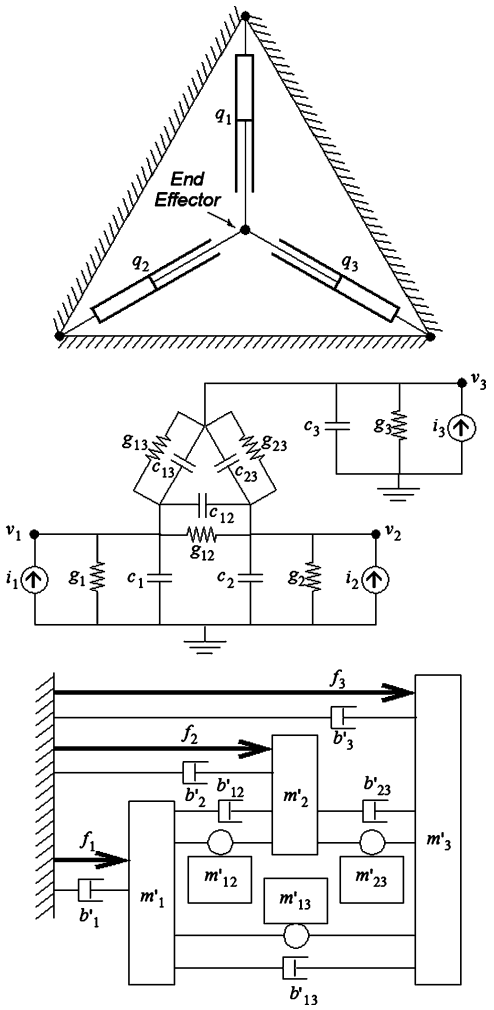

Fig. 27. 3-DOF parallel robot with electrical and mechanical equivalents.

(66), shown below, the entries of the M' matrix are solved in (67), also shown below.

Similarly, for a parallel 3-DOF robot, the electric circuit model and corresponding mechanical system model are shown in Fig. 27. For a mass matrix of the form shown in (66), the elements of M' are given in (68), shown below, and just like with the parallel 2-DOF example, a similar result is easily obtained for the damping matrix:

$$\begin{bmatrix} f_1 - f_2 \\ f_2 - f_3 \\ f_3 \end{bmatrix} = B' \begin{bmatrix} r_1 \\ r_1 + r_2 \\ r_1 + r_2 + r_3 \end{bmatrix} + M's \begin{bmatrix} r_1 \\ r_1 + r_2 \\ r_1 + r_2 + r_3 \end{bmatrix} \tag{64}$$

$$M'(q) = \begin{bmatrix} m'_1 + m'_{12} + m'_{13} & -m'_{12} & -m'_{13} \\ -m'_{12} & m'_2 + m'_{12} + m'_{23} & -m'_{23} \\ -m'_{13} & -m'_{23} & m'_3 + m'_{13} + m'_{23} \end{bmatrix} \tag{65}$$

$$M(q) = \begin{bmatrix} m_1(q) & m_4(q) & m_5(q) \\ m_4(q) & m_2(q) & m_6(q) \\ m_5(q) & m_6(q) & m_3(q) \end{bmatrix} \tag{66}$$

$$\begin{bmatrix} m'_1 \\ m'_2 \\ m'_3 \\ m'_{12} \\ m'_{23} \\ m'_{13} \end{bmatrix} = \begin{bmatrix} m_1 - m_4 \\ m_4 - m_5 \\ m_5 \\ m_2 + m_5 - m_4 - m_6 \\ m_3 - m_6 \\ m_6 - m_5 \end{bmatrix} \tag{67}$$

$$\begin{bmatrix} m'_1 \\ m'_2 \\ m'_3 \\ m'_{12} \\ m'_{23} \\ m'_{13} \end{bmatrix} = \begin{bmatrix} m_1 + m_4 + m_6 \\ m_2 + m_4 + m_6 \\ m_3 + m_5 + m_6 \\ -m_4 \\ -m_5 \\ -m_6 \end{bmatrix} \tag{68}$$

V. Conclusion

A pulley-based differential transmission model is proposed as a mechanical analog of an isolation transformer that has a number of practical applications. First, when combined with a mass, it results in a system, which is completely analogous to a capacitor without any of the connectivity constraints of the mass on its own. This allows any electrical circuit to be converted into a mechanical dual and enables mechanical modeling of complex mechanical systems with coupled mass terms, such as robot manipulators. Second, it can be added to any mechanical network to provide a floating reference when required.

Finally, this proposal has a further application in MEMS filter design [24]. Due to limitations in CMOS technology associated with implementing inductors, higher quality factors may be possible by using mechanically oscillating structures. The challenge that arises is that many favorable filter circuits contain ungrounded capacitors, such as Chebyshev bandpass filters, and implementing these filters using MEMS technology requires the mechanical equivalent of an ungrounded capacitor. As shown here, a MEMS implementation of a differential transmission coupled with a mass would provide just that and enable any electrical filter to be implemented as a micro electromechanical system.

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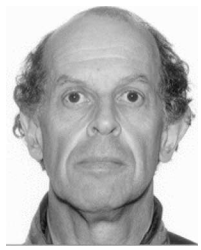

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