

An Isolated Mass Model for Intuitive Electro-Mechanical Analogies

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Abstract—In the well-known electro-mechanical analogy used to transform between electrical and mechanical systems, mass is replaced by a grounded capacitor. This is sufficient for converting a mechanical system into its electrical equivalent, but does not allow any electrical system to be modeled by a mechanical equivalent. A two-terminal isolated mass element, previously proposed by the author, simulates a capacitor in the general case and can be used to build a topologically identical mechanical equivalent of any electric circuit. It is the missing link of perhaps the most powerful technique available for building student intuition on the analogous behaviour of electrical and mechanical systems.

I. INTRODUCTION, GOALS & OBJECTIVES

Converting between an electrical and mechanical system is rarely a necessary step when analyzing an electrical or mechanical system. The behaviour of either is entirely governed by the differential equations describing its internal states and can always be solved directly using mathematical methods. The incentive for performing a transformation is merely to present a particular problem in a domain which is more familiar to the analyst. This is particularly useful when students are first exposed to electric circuits since they have a lifetime of experience with mechanical interactions which may be observed with the naked eye, but little appreciation for electrical interactions which cannot be observed without test equipment. Any child who has ever played with a Slinky™ knows how a spring works but they are unlikely to have any appreciation for inductance.

Since the purpose of electro-mechanical transformation is to provide insight into an unfamiliar domain, any dissimilarity between the two domains is a source of confusion that is greatly self defeating. The greatest such source is the fundamental dissimilarity between a mass and a capacitor which have different numbers of terminals and are only analogous under certain conditions. A new model involving a mass and pulley system, previously developed by the author, mimics both the

topology and the behaviour of a capacitor in the general sense. When used in conjunction with the conventional spring and damper models, a topologically identical mechanical model may be developed for any electric circuit. This modified analogy is easier to apply and more intuitive to students, particularly when they are first introduced to it.

A proposal to derive a relative mass model was originally presented in [1],[2]. A more complete development of the idea is presented in [3] where the often neglected reference terminal of the mass model is freed of its implied ground connection by a mechanical isolation transformer. It is shown that the proposed model enables one to implement a mechanical band-pass filter, which would otherwise, not be possible. Finally, examples are presented which demonstrate how the technique may be applied to higher order mechanical models such as robot systems.

Section II of this paper describes conventional electro-mechanical analogies and points out the shortcomings of the mass model. Section III describes the isolated mass model while Section IV shows how it may be used as a teaching tool to develop greater intuition in students. Section V presents concluding remarks and identifies the potential impact.

II. CONVENTIONAL MODELS

The ability to define an electro-mechanical equivalent circuit is a potent teaching tool that spans two seemingly dissimilar areas of study with a common set of fundamentals. The background behind the technique can be found in a large number of text books ([4] for example) on system modelling and control.

The idea stems from the duality of the differential equations that describe electrical and mechanical systems, each of which involve an across variable, a through variable and an impedance or admittance variable. In electrical circuits, voltage $E(s)$ is the across variable and current $I(s)$ is the through variable. In mechanical systems, it is convenient to treat velocity $V(s)$ as the across variable and force $F(s)$ as the through variable. This results in the correspondence between resistance R and damping B , inductance L and stiffness K , and capacitance C and mass M shown in (1-3).

This mathematical similarity demonstrates that each element is an impedance to the transmission of energy, be it electrical or mechanical, with either a proportional (1), integral (3) or

$$E(s) = I(s)R \quad V(s) = F(s)\frac{1}{B} \quad (1)$$

$$E(s) = I(s)sL \quad V(s) = F(s)\frac{s}{K} \quad (2)$$

$$E(s) = I(s)\frac{1}{sC} \quad V(s) = F(s)\frac{1}{sM} \quad (3)$$

differential (2) relationship between the across and through variables. As such, the product of the across and through variables corresponds to the rate of energy, or power being dissipated where an imaginary value denotes energy that is stored and returned without loss. This is shown in (4) for an electrical system and in (5) for a mechanical system.

$$\text{units}(EI) = VA = \frac{JC}{Cs} = \frac{J}{s} \quad (4)$$

$$\text{units}(VF) = \frac{m}{s}N = \frac{mKgm}{s^2} = \frac{Kgm^2}{s^3} = \frac{J}{s} \quad (5)$$

Deriving an analogous system involves replacing each component in the original system with its equivalent in the alternate domain. This ideally requires topological consistency between components that are to be substituted for one another. Resistors, inductors and capacitors all share the following three fundamental traits.

1. They have exactly 2 terminals which can be connected to any node in a circuit.
2. They are symmetrical about their 2 terminals (i.e. flipping a device over does not affect its response).
3. They obey Ohm's Law in the s-domain.

Voltage and current sources share the following two traits:

1. They have exactly 2 terminals.
2. They are directional with respect to their 2 terminals (i.e. flipping a device over changes its sign).

Unlike passive components, sources have some connectivity constraints. Connecting dissimilar voltage sources in parallel violates KVL while connecting dissimilar current sources in series violates KCL with either condition resulting in an unsolvable circuit.

According to equations (1-3), the electro-mechanical equivalents are as shown in Figure 1. Of course, similar equivalents may be defined for angular motion, but only linear motion symbols are used in this paper.

All components in Figure 1 have two terminals and obey Ohm's Law. However, the reference terminal of the mass symbol is commonly neglected because it is implicitly connected to ground. This implicit connection exists because a non-deformable mass stores energy in the form of inertia which corresponds to its velocity with respect to the earth.

Consequently, the first step in the process for converting a mechanical system into its analogous electrical equivalent is to replace all masses by grounded capacitors as shown in Figure 2. This fixes the reference voltage of the capacitor to zero and guarantees correspondence between (6) and (7).



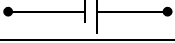
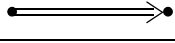

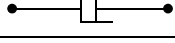



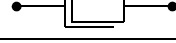
Electrical Symbol	Mechanical Symbol	Relationship
Current Source: $I(s)$ 	Force Source: $F(s)$ 	$F(s) = I(s)$
Voltage Source: $E(s)$ 	Velocity Source: $V(s)$ 	$V(s) = E(s)$
Resistor: R 	Damper: B 	$B = 1/R$
Inductor: L 	Spring: K 	$K = 1/L$
Capacitor: C 	Mass: M 	$M = C$

Fig. 1. Electro-mechanical equivalents.

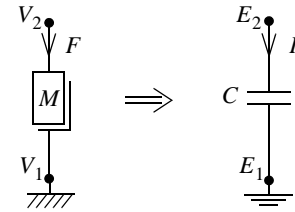


Fig. 2. Capacitor / mass equivalent models.

$$E_2(s) - E_1(s) = E_2(s) - 0 = E_2(s) = I(s)\frac{1}{sC} \quad (6)$$

$$V_2(s) - V_1(s) = V_2(s) - 0 = V_2(s) = F(s)\frac{1}{sM} \quad (7)$$

Since a capacitor has no implicit connections, a capacitor can always be used to simulate a mass but a mass cannot always be used to simulate a capacitor. Consider, for example, the band-pass filter in Figure 3. R_1 , R_2 and C_2 are replaced by B_1 , B_2 and M respectively in the equivalent mechanical system. But there is no mechanical component that can be used to represent C_1 because there is no way to connect the ground symbol to C_1 without changing the circuit. In other words, C_1 and C_2 do not share a common node.

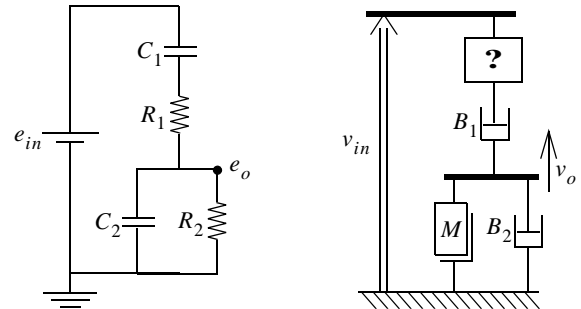


Fig. 3. Band-pass filter example.

III. ISOLATED MASS MODEL

The technique would be much more intuitive if it included a mechanical component which simulated a capacitor in general so that even an electrical circuit with multiple capacitors with no common nodes, could be represented by an equivalent mechanical system. The component should have two symmetric terminals, obey Ohm's Law, and be able to simultaneously accommodate a non-zero velocity at either terminal. In other words, it should contain no implicit connections.

The standard method for removing an unwanted ground from an electric circuit with an isolation transformer. Connecting a unity gain isolation transformer to a grounded capacitor, as shown in Figure 4, results in a system with the equivalent circuit of an ungrounded capacitor. Since a electrical transformer does have a mechanical equivalent, a similar approach may be used to arrive an ungrounded, or isolated mass.

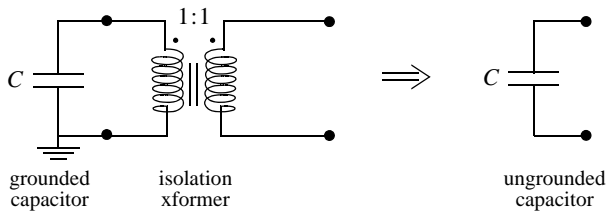


Fig. 4. Achieving relative mass with a mechanical isolation transformer.

A transformer scales voltage by its winding ratio and current by the inverse. Similarly, a gear and pinion scales velocity by its tooth count ratio and force by the inverse. Of course, a gear and pinion is one of many methods to obtain mechanical speed reduction. Others include a linear rack and pinion, a planetary gear, a lead screw, a worm gear, and a cable and pulley system, among others.

Isolating a mass using the ideal cable/pulley transmission shown in Figure 5 results in the relationship shown in (8). Note that only the component of differential velocity between nodes n_1 and n_2 will translate the mass. All common mode velocity is cancelled by motion of the ideal (i.e. massless, frictionless) pulley block. It can be shown that the transmission system shown in Figure 5 is the mechanical equivalent of a unity gain isolation transformer.

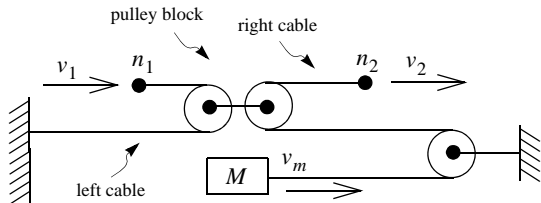


Fig. 5. Isolated mass model.

$$v_m = v_2 - v_1 \quad (8)$$

Differentiating (8) results in (9,10) with the Laplace transform shown in (11) where F corresponds to the tensile force present in both cables (left and right) in Figure 5.

$$\dot{v}_m(t) = a_m(t) = \frac{f}{M} \quad (9)$$

$$\dot{v}_2 - \dot{v}_1 = \frac{f}{M} \quad (10)$$

$$V_2(s) - V_1(s) = F(s) \frac{1}{sM} \quad (11)$$

Equation (11) is merely Ohm's Law for a mass (7) with the implicit $V_I(s) = 0$ constraint removed. This mirrors Ohm's Law for a "not necessarily grounded" capacitor (6).

The symbol shown in Figure 6 is used here to represent the isolated mass element. Unlike the mass in Figure 1, both terminals of the isolated mass (n_1 and n_2) may be connected to any node in a mechanical circuit. It is not necessary to connect either to ground.

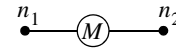


Fig. 6. Block diagram of isolated mass.

The instantaneous power flowing into the isolated mass $P_d(t)$ (12) is computed by substituting (9) into the equation for mechanical power. This corresponds to the instantaneous power flowing into a capacitor $P_c(t)$ (13). From (11), the complex impedance $Z_d(s)$ of the isolated mass is obtained directly (14).

$$P_d(t) = \frac{\text{Work}}{\text{Unit Time}} = f(t)v(t) = Ma_m(t)v_m(t) \quad (12)$$

$$= M \frac{f(t)}{M} (v_2(t) - v_1(t)) = f(t)(v_2(t) - v_1(t))$$

$$P_c(t) = i(t)(e_2(t) - e_1(t)) \quad (13)$$

$$Z_d(s) = \frac{1}{Ms} \quad (14)$$

The isolated mass model has two symmetric, interchangeable terminals, and is shown to obey Ohm's law, and to satisfy the same differential equation as a conventional mass but with an arbitrary reference velocity. It is, therefore, a general mechanical equivalent of a capacitor.

Unlike a pure mass, an isolated mass can be used to model C_I in the band-pass filter example in Figure 3. Substituting each capacitor with an isolated mass results in Figure 7 which may be analyzed just like an electric circuit. At very low frequencies (15), the impedance of both the conventional and isolated mass approach infinity (16), and due to the finite impedance of damper B_2 , all of the input velocity is "dropped" across the isolated mass M_2 (17). In other words, the masses simulate mechanical open circuits, just like the capacitors in Figure 3.

At very high frequencies (18), the impedances of both the conventional and isolated masses approach zero (19) and all of the input velocity is "dropped" across damper B_I (20). In other words, the masses simulate mechanical short circuits, just like the capacitors in Figure 3. At all other frequencies the output

velocity v_o is non-zero and finite, and a mechanical band-pass filter is realized.

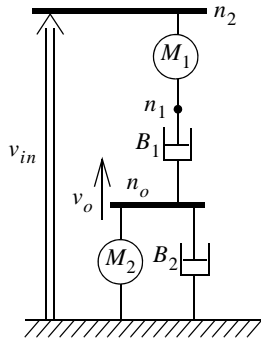


Fig. 7. Mechanical model of a band-pass filter.

$$\begin{aligned} \text{low frequency (DC)} \quad & \left\{ \begin{aligned} v_2 &= v_{in}(\omega = 0) & (15) \\ Z_{M2}(s) &= \frac{1}{M_2(j0)} = \infty & (16) \\ v_2 - v_1 &= v_{in} & (17) \end{aligned} \right. \\ \text{high frequency} \quad & \left\{ \begin{aligned} v_2 &= v_{in}(\omega = \infty) & (18) \\ Z_{M1}(s) = Z_{M2}(s) &= \frac{1}{M(j\infty)} = 0 & (19) \\ v_1 - v_o &= v_{in} & (20) \end{aligned} \right. \end{aligned}$$

IV. INTUITION DEVELOPMENT

The benefit of the isolated mass model is not limited to an expanded ability for transforming electric circuits. It also overcomes what is almost certainly the largest obstacle to the development of intuition and understanding, topological dissimilarity. Consider Figure 8 which shows what appears to be a series connection between a spring, mass and damper and its electrical equivalent.

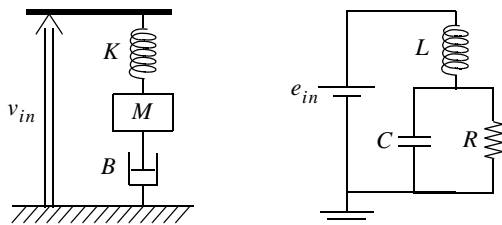


Fig. 8. A conventional MBK circuit and its RLC equivalent.

The mechanical circuit appears like it contains a single force loop when, in fact it contains two as is evident from its electrical equivalent. The reason for its deceptive appearance is the implicitly grounded mass. Since it only has one explicit node, the entire symbol is typically used to represent that node, just as it is in a free body diagram. Adjacent components are connected from all sides, providing the false impression of series connections that do not actually exist.

An isolated mass, on the other hand, has two distinct terminals, just like any other mechanical or electrical component. This eliminates any ambiguity between series and parallel connections. In addition, transforming any circuit from one domain to the other simply involves replacing each component with its associated equivalent. The topology of the circuit remains completely unchanged, without exception. Consider Figure 9 where the conventional mass from Figure 8 is replaced by an isolated mass. In Figure 9, the mechanical system behaves exactly the way it looks, like a series connection. In addition, its electrical equivalent mirrors it in every detail, including the particular choice of reference node.

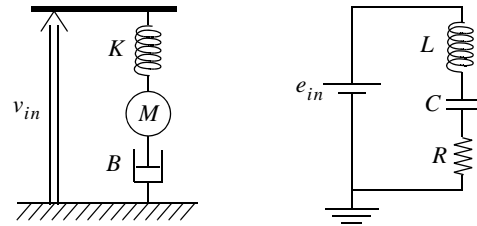


Fig. 9. An MBK circuit with an isolated mass and its RLC equivalent.

From the perspective of the student, this is much more intuitive. Although it does take some effort to develop an understanding of the difference between a conventional and isolated mass, the payoff is well worth the effort, even for simple circuits like Figure 9. Although duality in Figure 8 may be mathematically proven, duality in Figure 9 is obvious.

V. CONCLUSION AND IMPACT

As stated earlier, the domain transformation technique is particularly valuable when teaching electric circuits since mechanical interactions are observable with the naked eye and are a normal part of everyday life. Developing a deep understanding of capacitance is particularly cumbersome since no mechanical model exists that fully describes it. The isolated mass model presented here fills that gap. It mimics the behaviour of a capacitor in the general case, with a practical system that can actually be built using simple parts so that students do not have to rely on a mathematical proof and their imagination to develop intuition. They can “feel” capacitance with their own two hands, just as they are able to feel inductance by stretching a spring or feel resistance by compressing a shock absorber.

This work may be used to improve circuit analysis and control theory courses in science, electrical, mechanical and other engineering disciplines by filling in the one remaining hole in the analogy that links the behaviour of a mechanical proportional / integral / differential (PID) system to a PID system in any other domain (electrical, fluid, chemical, biological, etc.)

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