Table A-6 Fourier Transforms of Mathematical Operations

Operation	$f(t)$	$F(\omega)$	$F(f)$
Superposition	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$	$a_1 F_1(f) + a_2 F_2(f)$
Reversal	$f(-t)$	$F(-\omega)$	$F(-f)$
Symmetry	$F(t)$	$2\pi f(-\omega)$	$f(-f)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$	$\frac{1}{\lvert a\rvert} F\left(\frac{f}{a}\right)$
Delay	$f(t - t_0)$	$\epsilon^{-jt_0\omega} F(\omega)$	$\epsilon^{-j2\pi t_0 f} F(f)$
Complex conjugate	$f^*(t)$	$F^*(-\omega)$	$F^*(-f)$
Modulation	$\epsilon^{j\omega_0 t} f(t)$	$F[(\omega - \omega_0)]$	$F\left(f - \frac{\omega_0}{2\pi}\right)$
Time differentiation	$\frac{d^n}{dt^n} f(t)$	$(j\omega)^n F(\omega)$	$(j2\pi f)^n F(f)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$	$\left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} F(f)$
Integration	$\int_0^t f_e(t)dt + \int_{-\infty}^t f_0(t)dt$	$\frac{1}{j\omega} F(\omega)$	$\frac{1}{j2\pi f} F(f)$
Integration	$\int_{-\infty}^t f(t)dt$	$\frac{1}{j\omega} F(\omega) + \pi F(0)\delta(\omega)$	$\frac{1}{j2\pi f} F(f) + \frac{1}{2} F(0)\delta(f)$
Convolution	$f_1 * f_2 = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t - \lambda) d\lambda$	$F_1(\omega) F_2(\omega)$	$F_1(f) F_2(f)$
Multiplication	$f_1(f) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\xi) F_2(\omega - \xi) d\xi$	$\int_{-\infty}^{\infty} F_1(\xi) F_2(f - \xi) d\xi$
Correlation	$\int f_1(\lambda) f_2^*(\lambda + t) d\lambda$	$F_1(\omega) F_2^*(\omega)$	$F_1(f) F_2^*(f)$

Definitions: $F(\omega) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j\omega t} dt$

$$F(f) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j2\pi f t} dt = F(\omega)\big|_{\omega = 2\pi f}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\epsilon^{j\omega t} d\omega = \int_{-\infty}^{\infty} F(f)\epsilon^{j2\pi f t} df$$

Table A-7 Fourier Transforms of Energy Signals

	$f(t)$	$F(\omega)$	$\lvert F(f)\rvert$
Rectangular pulse ($-T/2$, 0, $T/2$)	$u(t+\frac{T}{2}) - u(t-\frac{T}{2})$	$T\,\frac{\sin(\omega T/2)}{\omega T/2}$	$T\,\frac{\sin(\pi T f)}{\pi T f}$
Exponential	$\epsilon^{-at} u(t)$	$\frac{1}{j\omega + a}$	$\frac{1}{j2\pi f + a}$
Triangular ($-T/2$, 0, $T/2$)	$1 - 2\,\frac{\lvert t\rvert}{T}$, $\lvert t\rvert < \frac{T}{2}$; 0 elsewhere	$\frac{T}{2}\left[\frac{\sin(\omega T/4)}{\omega T/4}\right]^2$	$\frac{T}{2}\left[\frac{\sin(\pi T f/2)}{\pi T f/2}\right]^2$
Gaussian	$\epsilon^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a}\epsilon^{-(\omega^2/4a^2)}$	$\frac{\sqrt{\pi}}{a}\epsilon^{-(\pi^2 f^2/a^2)}$
Double exponential	$\epsilon^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + \omega^2}$	$\frac{2a}{a^2 + 4\pi^2 f^2}$
Damped sine	$\epsilon^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$\frac{\omega_0}{(a + j2\pi f)^2 + \omega_0^2}$
Damped cosine	$\epsilon^{-at}\cos(\omega_0 t)u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$\frac{a + j2\pi f}{(a + j2\pi f)^2 + \omega_0^2}$
	$\frac{1}{\beta - a}[\epsilon^{-at} - \epsilon^{-\beta t}]u(t)$	$\frac{1}{(j\omega + a)(j\omega + \beta)}$	$\frac{1}{(j2\pi f + a)(j2\pi f + \beta)}$
Cosine pulse	$\cos\omega_0 t[u(t + \frac{T}{2}) - u(t - \frac{T}{2})]$	$\frac{T}{2}\left[\frac{\sin(\omega - \omega_0)T/2}{(\omega - \omega_0)T/2} + \frac{\sin(\omega + \omega_0)T/2}{(\omega + \omega_0)T/2}\right]$	$\frac{T}{2}\left[\frac{\sin\pi T(f - f_0)}{\pi T(f - f_0)} + \frac{\sin\pi T(f + f_0)}{\pi T(f + f_0)}\right]$

[1]From Continuous and Discrete Signal and System Analysis, McGillem and Cooper, 1974

The *error function*, denoted by erf(u), is defined in a number of different ways in the literature. We shall use the following definition:

$$\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2)\, dz \tag{A7.1}$$

The error function has two useful properties:

1. $\operatorname{erf}(-u) = -\operatorname{erf}(u)$ (A7.2)
   This is known as the *symmetry relation*.
2. As u approaches infinity, erf(u) approaches unity; that is

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-z^2)\, dz = 1 \tag{A7.3}$$

The *complementary error function* is defined by

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)\, dz \tag{A7.4}$$

Table A7.1 The Error Function[a]

u	erf(u)	u	erf(u)
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.15	0.89612
0.10	0.11246	1.20	0.91031
0.15	0.16800	1.25	0.92290
0.20	0.22270	1.30	0.93401
0.25	0.27633	1.35	0.94376
0.30	0.32863	1.40	0.95229
0.35	0.37938	1.45	0.95970
0.40	0.42839	1.50	0.96611
0.45	0.47548	1.55	0.97162
0.50	0.52050	1.60	0.97635
0.55	0.56332	1.65	0.98038
0.60	0.60386	1.70	0.98379
0.65	0.64203	1.75	0.98667
0.70	0.67780	1.80	0.98909
0.75	0.71116	1.85	0.99111
0.80	0.74210	1.90	0.99279
0.85	0.77067	1.95	0.99418
0.90	0.79691	2.00	0.99532
0.95	0.82089	2.50	0.99959
1.00	0.84270	3.00	0.99998
1.05	0.86244	3.30	0.999998

[a] The error function is tabulated extensively in several references; see for example, Abramowitz and Stegun (1965, pp. 297–316).

It is related to the error function as follows:

$$\operatorname{erfc}(u) = 1 - \operatorname{erf}(u) \tag{A7.5}$$

Table A7.1 gives values of the error function erf(u) for u in the range 0 to 3.3.

A7.2 THE *Q*-FUNCTION

Consider a *standardized* Gaussian random variable X of zero mean and unit variance. The probability that an observed value of the random variable X will be greater than v is given by the *Q-function*:

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty \exp\left(-\frac{x^2}{2}\right) \tag{A7.9}$$

The Q-function defines the area under the standardized Gaussian tail. Inspection of Eqs. (A7.4) and (A7.9) reveals that the Q-function is related to the complementary error function as follows:

$$Q(v) = \frac{1}{2} \operatorname{erfc}\left(\frac{v}{\sqrt{2}}\right) \tag{A7.10}$$

Conversely, putting $u = v/\sqrt{2}$, we have

$$\operatorname{erfc}(u) = 2Q(\sqrt{2}u) \tag{A7.11}$$

[2] From Communication Systems, 3rd Ed., Simon Haykin, 1994