# Solutions to Final Exam

## Question 1

A signal with a Rayleigh-distributed amplitude has a *median* amplitude of 1 mV. Assuming a 50 Ω impedance, what is the signal's RMS power? Give your answer in dBm. *Hint: The cumulative distribution function (CDF) of the amplitude is 50% at the median amplitude.*

### Answer

Setting the Rayleigh CDF to 0.5:

$$P(r < R) = 1 - e^{-\rho^2} = 0.5$$

and solving for $\rho$:

$$\rho = \sqrt{-\ln(1 - 0.5)}$$

Since $\rho$ is defined as: $\rho = {}^{R}/_{R_{rms}}$ and the question states $R = 1\,\text{mV}$ when the CDF is 0.5:

$$R_{rms} = \frac{R}{\sqrt{-\ln(1 - 0.5)}} \approx 1.2\,\text{mV}$$

Converting to power in dBm:

$$P = \frac{V^2}{R} \approx 28.9\,\mu\text{W} \approx \boxed{-45\,\text{dBm}}$$

## Question 2

A TV transmitter on Mount Seymour broadcasts 90 kW at a frequency of 600 MHz. The transmit antenna has a gain of 6 dBi and you can assume a receive antenna gain of 0 dBi. If the received signal power must be at least −40 dBm, what is the maximum distance at which the TV signal can be received? Give your answer in km.

### Answer

Solving the Friis equation for distance:

$$d = \sqrt{\frac{P_T G_T G_R}{P_R}} \left(\frac{\lambda}{4\pi}\right)$$

and substituting $P_T = 90 \times 10^6\,\text{mW}$, $P_R = 1 \times 10^{-4}\,\text{mW}$, $G_T = 4$, $G_R = 1$, and $\lambda = \frac{3\times10^8}{600\times10^6} = 0.5\,\text{m}$,

$$d \approx \boxed{75\,\text{km}}$$

## Question 3

You're driving down the highway listening to an FM broadcast station transmitting at 100 MHz and notice the signal fading six times per second. Assuming Clarke's model applies and the signal is faded when the signal level is 10 dB below the (RMS) mean level, how fast are you driving? Give your answer in km/h.

### Answer

If Clarke's model applies then the level crossing rate is:

$$N_R = \sqrt{2\pi} f_m \rho e^{-\rho^2} = \sqrt{2\pi} f_c \frac{v}{c} \rho e^{-\rho^2}$$

since $f_m = f_c \frac{v}{c}$.

We are given $N_R = 6$, the fading threshold $\rho = -10\,\text{dB} = 10^{-10/20} \approx 0.316$, and the carrier frequency $f_c = 100\,\text{MHz}$. Solving for $v$:

$$v = \frac{N_R}{\sqrt{2\pi}} \frac{c}{f_c} \frac{1}{\rho e^{-\rho^2}} \approx 25.1\,\text{m/s} \approx \boxed{90\,\text{km/h}}$$

## Question 4

A signal is received over two paths whose lengths differ by 300 m. The power received along the longer path is half of that on the shorter path. What is the (RMS) delay spread? Give your answer in microseconds.

### Answer

The normalized sum of the powers must be 1 so the normalized powers, $p(\tau)$, must be 2/3 for the first and 1/3 for the second. A path length difference of 300 m results in a delay difference of 1 µs. The excess delays are thus $\tau = 0$ and $\tau = 1\,\mu$s, and the mean delay is $\bar{\tau} = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{1}{3}$ and the RMS delay spread is:

$$\sigma = \sqrt{\sum p(\tau)(\tau - \bar{\tau})^2} = \sqrt{\frac{2}{27} + \frac{4}{27}} \approx \boxed{0.47\,\mu\text{s}}$$

## Question 5

In a NLOS environment you measure an average signal level of −40 dBm at a distance of 10 m and −80 dBm at a distance of 100 m. Assuming a power law path loss model applies, what would you estimate to be the average signal level at a distance of 20 m?

### Answer

For a power-law model the path loss in dB is:

$$PL(d) = PL(d_0) + 10n \log\left(\frac{d}{d_0}\right)$$

The path loss increased by 40 dB when the distance increased from 10 m to 100 m. Using $d_0 = 10$ m:

$$40 = 10n\left(\log\left(\frac{100}{10}\right) - \log\left(\frac{10}{10}\right)\right) = 10n$$

and $n = 40/10 = 4$. The path loss increase from $d_0 =$ 10 m to 20 m will be:

$$PL(20) - PL(10) = 10 \cdot 4 \cdot \log\left(\frac{20}{10}\right) \approx 12.0\,\text{dB}$$

and the received signal level at 20 m will be $\approx -40 - 12.0 \approx \boxed{-52\,\text{dBm}}$.

## Question 6

A wireless receiver uses maximal-ratio combining diversity on four branches. Two of the branches have an SNR of 10 dB and two have an SNR of 16 dB. What is the SNR after combining? Give your answer in dB.

### Answer

Maximal-ratio combining results in an SNR that is the sum (in linear units) of the branch SNRs. The branch SNR's in this case are 10 and 40 so the SNR after combining is $2 \times 10 + 2 \times 40 = 100 = \boxed{20\,\text{dB}}$.

## Question 7

A MIMO system uses 4 transmit and 4 receive antennas. The channel matrix has four equal, non-zero eigenvalues. The signals are received with an SNR of 13 dB, the noise is Gaussian and the bandwidth is 12 MHz. What is the (Shannon) capacity of the system? Give your answer in Mbps.

### Answer

For the special case of $N$ transmit and receive antennas, and equal-amplitude channel matrix eigenvalues, the capacity is $N$ times the capacity of one channel. In this case $N = 4$ and the channel is an AWGN channel with bandwidth $B = 12$ MHZ and an SNR $S/N = 13\,\text{dB} = 10^{13/10} \approx 20$ so the capacity of this MIMO channel is:

$$N \cdot B \log_2(1 + S/N) \approx \boxed{210\,\text{Mb/s}}$$

## Question 8

(a) What is a generator matrix for a systematic (6,2) repetition code? The parity bits may be transmitted in any order. Follow the conventions used in the lecture notes.

### Answer

In this question $(n, k) = (6, 2)$ so each codeword has $n = 6$ bits, $k = 2$ data bits and $n - k = 4$ parity bits.

For a systematic code the data bits are transmitted first followed by the parity bits. For a repetition code the parity bits are the same as the data bits. In the lecture notes the data bits are transmitted first by using an identity matrix as the first part of the generator matrix, followed by the parity bits:

$$G = \left[I_k \,|\, P\right]$$

To use the data bits as the parity bits we can use multiple identity matrices:

$$\boxed{G = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]}$$

We could also use any other ordering of the columns of $P$.

(b) What is the corresponding parity check matrix?

### Answer

The parity check matrix is given by:

$$H = \left[P^T \,|\, I_{n-k}\right]$$

For the ordering of the parity bits shown above,

$$\boxed{H = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]}$$

For a different ordering of the parity bits the first two columns must be the transpose of $P$.

## Question 9

A communication system is transmitting at 16 Mb/s, including parity bits, using a rate 3/4 code over a bandwidth of 12 MHz. The SNR is 10 dB. What is $E_b/N_0$? Give your answer in dB.

### Answer

The energy per bit, $E_b$, is the product of the signal power and the *information* bit duration. For a rate 3/4 code the information rate is 3/4 of the bit rate: $R = \frac{3}{4} \cdot 16\,\text{Mb/s} = 12\,\text{MHz}$.

The noise power spectral density, $N_0$, is the noise power divided by the receiver bandwidth. The question gives the bandwidth as $B = 12$ MHz.

Thus:

$$\frac{E_b}{N_0} = \frac{S}{N}\frac{B}{R} = 10\frac{12\,\text{MHz}}{12\,\text{MHz}} = 10 = \boxed{10\,\text{dB}}$$

## Question 10

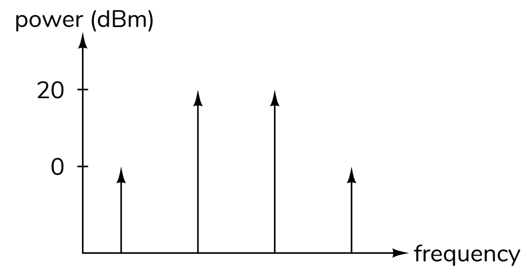
An OFDM system is being designed to cope with multipath propagation in an environment where the maximum path length differences are expected to be 60 m. The sample rate is 50 MHz. What minimum duration of cyclic prefix should be used? Give your answer in number of samples.

### Answer

The cyclic prefix duration should be at least as long as the maximum path length difference so that ISI due to one symbol will not affect the next one. In this case the maximum path length difference is 60 m which corresponds to a time difference of $\tau = c/d = 3 \times 10^8\,\text{m/s}/60\,\text{m} = 200\,\text{ns}$. At a clock rate of 50 MHz this corresponds to $50 \times 10^6$ samples/s $\times\, 200 \times 10^{-9}$ s $= \boxed{10\text{ samples}}$.

## Question 11

You measure the following spectrum at the output of an RF amplifier for a two-tone input. Assuming the amplifier is operating in its linear region, what is the amplifier's OIP3? Give your answer in dBm.




### Answer

For a reduction in output power of $\Delta$ dB from the OIP3, the third-order products decrease by $3\Delta$ dB and thus decrease by $2\Delta$ dB relative to the (desired) output power. In the diagram $2\Delta = 20 - 0 = 20$ dB and thus the OIP3 is $20 + 2\Delta/2 = 20 + 10 = \boxed{30\,\text{dBm}}$.

## Question 12

The input of an RF amplifier is connected to a calibrated noise source that outputs broadband noise with a power spectral density of $-164$ dBm/Hz when it is turned on.

(a) What is the ENR of the noise source? Give your answer in dB.

### Answer

The ENR is the increase in noise temperature above $T_0$. The PSD at a noise temperature of $T$

is $kT$. The output PSD at $T_0$ is $kT_0 = -174$ dBm/Hz. An output power of 164 dBm/Hz corresponds to an ENR of $-174 - (-164) = \boxed{10\,\text{dB}}$.

(b) When the noise source is turned on, the power spectral density of the noise at the output of the amplifier increases by 7 dB. What is the amplifier's noise figure? Give your answer in dB.

### Answer

The noise figure can be computed using the "Y-factor" method where $Y$ is the increase in output noise PSD when the input noise PSD is increased by the noise source ENR. In this case $Y$ is specified as $7\,\text{dB} = 10^{7/10} \approx 5$ and the noise factor is:

$$F = ENR - 10\log(Y-1) \approx 10 - \log(5-1) \approx \boxed{4\,\text{dB}}$$