Information and Capacity

Exercise 1: How would you represent a discrete r.v. in a pdf?

$\delta(\cdot)$

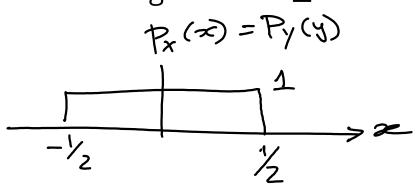
Exercise 2: Is the radio noise generated by the sun a stationary stochastic process? Under what conditions?

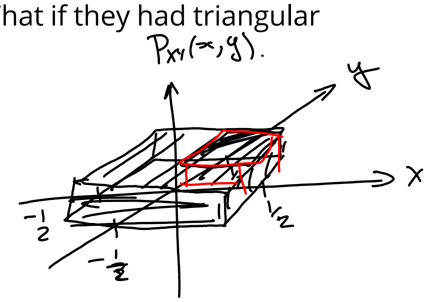
yes - over short times << day

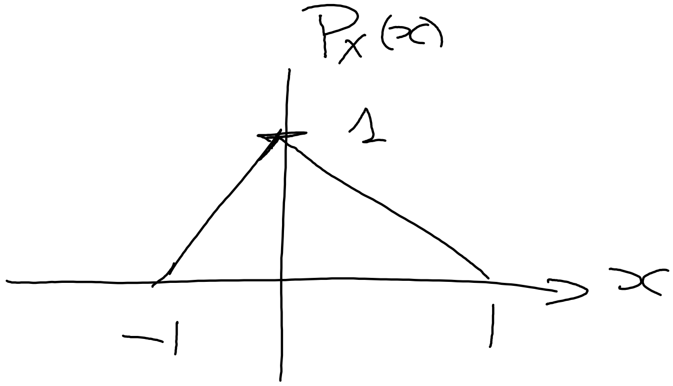
Exercise 3: Would the amount of data transmitted by cellular subscribers be an ergodic stochastic process?

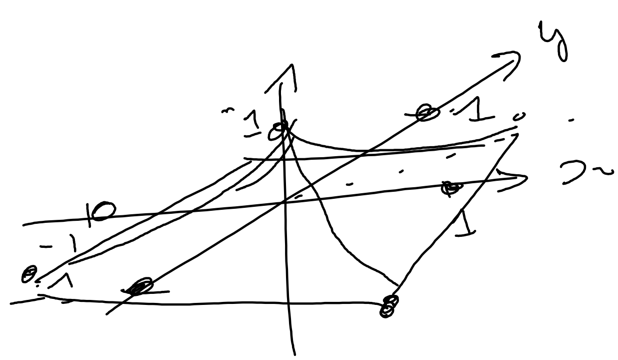
no - different users stat.s (ensemble) not same as stats for one user over time.

Exercise 4: Describe the shape of the joint pdf of two zero-mean iid random variables with uniform pdfs. What if they had triangular pdfs extending between ±1?







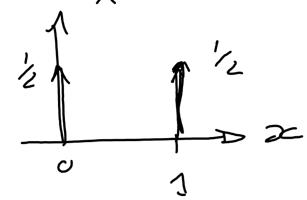


Exercise 5: Two random variables, X and Y represent two flips of a coin (outcomes are H or T for each). Draw the joint pdf if the two coins are fair (unbiased) and the outcomes are independent. Draw the joint pdf if the H is twice as likely as T but the outcomes are independent. Draw the joint pdf if the coins are fair but the outcome of the second toss depends on the first and is always the opposite. Which of these are identically distributed? Which are independent r.v.? Which are i.i.d.?

$H \to 0$

$T \to 1$

$P_X(x)$

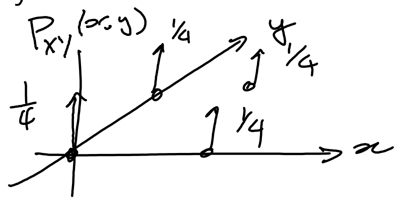


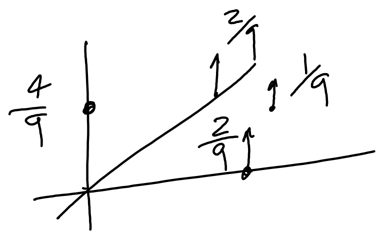
fair

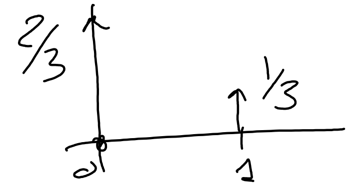
P_X & P_Y :

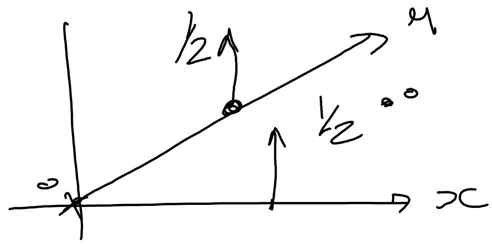
i.d.	indep	iid
Y	Y	Y
Y	Y	Y
Y	N	N

$P_{XY}(x,y)$







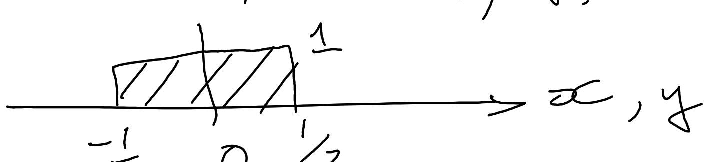


$P_X(z) = P_Y(y)$

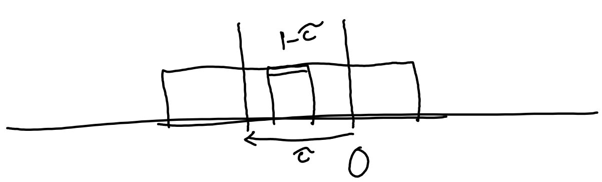
Exercise 6: What is the pdf of the sum of two zero-mean iid uniformly-distributed rv's whose pdf has a maximum value of 1?

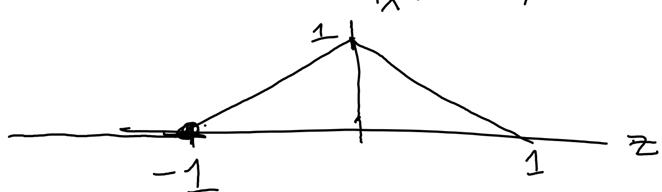
$P_X(x) = P_Y(y)$

$Z = X + Y$



$P_X(x) * P_Y(y) = P_Z(z)$





Exercise 7: Prove this.

$$E\left[(X+Y)^2\right] = E\left[X^2 + 2XY + Y^2\right]$$

$$E[a+b] = E[a] + E[b] \qquad E(a+b) = \int_{-\infty}^{\infty} \cdot \, dx$$

$$E[X^2] + \underbrace{2E[XY]}_{\substack{cov(X,Y) \\ =0 \\ (\text{independent})}} + E[Y^2]$$

✓ ✓

$$E\left[(X+Y)^2\right] = E[X^2] + E[Y^2]$$

Exercise 8: We observe a source that outputs letters. Out of 10,000 letters 1200 were 'E'. What would be a reasonable estimate of the probability of the letter 'E'?

$$\frac{1200}{10000} = 0.12$$

Exercise 9: A source generates four different messages. The first three have probabilities 0.125, 0.125, 0.25. What is the probability of the fourth message? How much information is transmitted by each message? What is the entropy of the source? What is the average information rate if 100 messages are generated every second? What if there were four equally-likely messages?

$$P_i = \underbrace{\frac{1}{8}, \frac{1}{8}, \frac{1}{4}}_{1/2}, \ 1/2 \qquad \sum P_i = 1 \Rightarrow P_3 = 1/2$$

$$I_0 = -\log_2\left(\frac{1}{8}\right) = -\log_2\left(2^{-3}\right) = --3 = 3 \text{ bits}$$

$$I_1 = 3 \text{ bits}$$

$$I_2 = -\log_2\left(\frac{1}{4}\right) = 2 \text{ bits}$$

$$I_3 = -\log_2\left(1/2\right) = 1 \text{ bit}$$

$$H = \sum I_i \cdot P_i = 3 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}$$

$$= \frac{3}{8} + \frac{3}{8} + \frac{4}{8} + \frac{4}{8} = \frac{14}{8} = 1.75 \text{ bits/message.}$$

$$\dot{H} = 1.75 \ \frac{\text{bits}}{\text{msg}} \cdot 100 \ \frac{\text{msg}}{\text{s}} = 175 \text{ bit/s}$$

$$\text{for } P_i = 1/4 \qquad I_i = -\log_2\left(\frac{1}{4}\right) = 2 \text{ bits}$$

$$\dot{H} = 2 \text{ bits/msg}$$

$$= 200 \text{ b/s.}$$

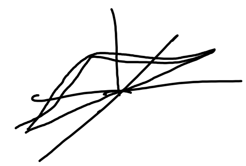
Exercise 10: What is the mutual information if X and Y are independent? If they are the same?

$p(x)p(y)$

$$I = \sum\sum p(x,y)\log_2\left(\frac{p(x,y)}{\underbrace{p(x),p(y)}_{1}}\right) = 0.$$

when X & Y independent.

$p(x) = p(y)$



$$I = \sum\sum p(x,y)\log_2\left(\frac{p(x,y)}{p(x)\,p(y)}\right)$$

$$= \underbrace{\sum p(x)\log_2\left(\frac{1}{p(x)}\right)}_{H(x)}$$

Exercise 11: What is capacity of a binary symm. channel with a BER of $\frac{1}{8}$ (assuming the same BER for 0's and 1's)?

$$C = 1 - (-p\log_2 p - (1-p)\log_2(1-p))$$

$$C = 1 - \left(-\frac{1}{8}\log_2\frac{1}{8} - \left(1-\frac{1}{8}\right)\log_2\left(1-\frac{1}{8}\right)\right)$$

$$= 1 - \left(\frac{3}{8} - \frac{7}{8}\log_2\left(\frac{7}{8}\right)\right)$$

$$= 0.456 \text{ bits / bit transmitted}$$

Exercise 12: What is the channel capacity of a 4 kHz channel with an SNR of 30dB?

$$30\,dB = 10^{\frac{30}{10}} = 1000$$

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

$$= 4000 \log_2(1 + 1000)$$

$$\approx 40\ kb/s$$

Exercise 13: Can we achieve the capacity of a AWGN channel by transmitting an NRZ waveform? Can we achieve the capacity of a BSC channel?

- No, X needs to be Gaussian.
- Yes, X has only two values.

Exercise 14: You receive 1 million frames, each of which contains 100 bits. By comparing the received frames to the transmitted ones you find that 56 frames had errors. Of these, 40 frames had one bit in error, 15 had two bit errors and one had three errors. What was the FER? The BER?

10^6 frames · 100 bits

56 errors :

$40 \times 1 = 40$

$15 \times 2 = 30$

$1 \times 3 = 3$

73 total bit errors

$$FER = \frac{56}{10^6} = 56 \times 10^{-6}$$

$$BER = \frac{73}{100 \cdot 10^6} = 0.73 \times 10^{-6}$$

Exercise 15: The BER over a channel that sees independent bit errors is 10^{-5}. What is the FER for 128-byte frames? For 9000-byte frames?

$P_e = 10^{-5}$

$N = 128 \cdot 8 \simeq 10^3$

$= 9000 \cdot 8 \simeq 10^5$

$$FER(128 \text{ byte frames}) = 1 - (1 - 10^{-5})^{10^3} = 9.9 \times 10^{-3}$$

$$= 1 - (1 - 10^{-5})^{10^5} = 0.632$$