Multipath Fading

Exercise 1: Show that R_{rms}^2 is the mean power of the signal.

(1) Using definition of second moment:

$$R_{rms}^2 = \int_0^\infty r^2 \, p(r) \, dr. \qquad p(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) & r \geq 0 \\ 0 & r < 0 \end{cases}$$

$$= \int_0^\infty \frac{r^3}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = 2\sigma^2$$ (see next page)

(2) As power of constant ($\bar{r}$) ~~+~~plus zero-mean r.v. $(r - \bar{r})$:

$$\underbrace{\bar{r}^2}_{\text{fixed}} + \underbrace{(r-\bar{r})^2}_{\text{r.v.}} =$$

$$R_{rms} = \sqrt{\left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 + \sigma^2\left(2 - \frac{\pi}{2}\right)}$$

$$= \sqrt{\sigma^2\frac{\pi}{2} + 2\sigma^2 - \frac{\pi}{2}\sigma^2}$$

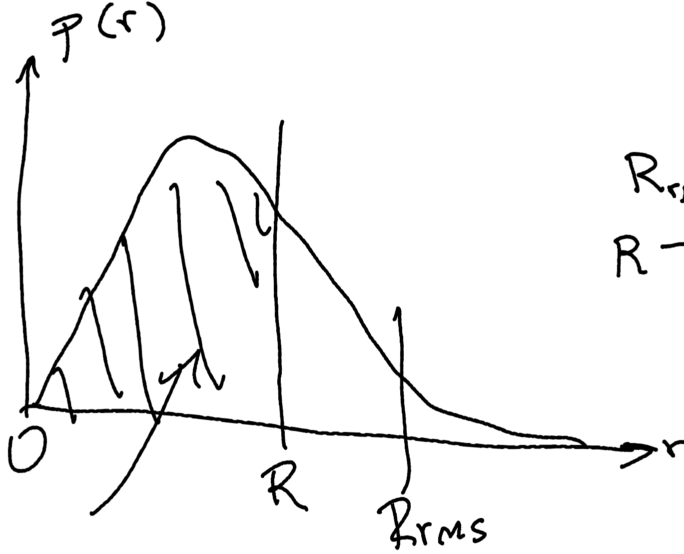
$$= \sqrt{2}\,\sigma$$

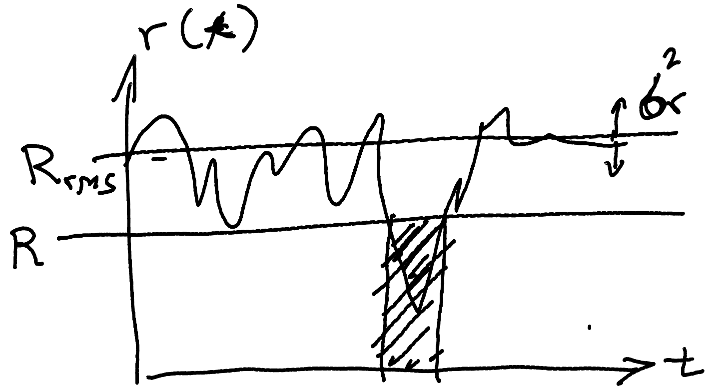
(3) As sum of two ~~independent~~ orthogonal r.v.:

$$\overline{|r|^2} = \overline{Re^2\{r\} + Im^2\{r\}} \qquad \left(\overline{Re\{r\}} = \overline{Im\{r\}} = 0\right)$$

$$= \overline{Re^2\{r\}} + \overline{Im^2\{r\}}$$

$$= \sigma^2 + \sigma^2 = 2\sigma^2$$





In [1]:
```python
from sympy import *
```

In [2]:
```python
init_session()
```

In [3]:
```python
s, r = symbols("sigma r",positive=True)
```

In [4]:
```python
# Rayleigh pdf
pdf = r/s**2*exp(-r**2/(2*s**2))
pdf
```

Out[4]:

$$\frac{re^{-\frac{r^2}{2\sigma^2}}}{\sigma^2}$$

In [5]:
```python
# first moment
rbar = integrate(r*pdf,(r,0,oo))
rbar
```

Out[5]:

$$\frac{\sqrt{2}\sqrt{\pi}\sigma}{2}$$

In [6]:
```python
# second central moment
integrate((r-rbar)**2*pdf,(r,0,oo))
```

Out[6]:

$$-\frac{\pi\sigma^2}{2} + 2\sigma^2$$

In [7]:
```python
# second moment
integrate(r**2*pdf,(r,0,oo))
```

Out[7]: $2\sigma^2$

Exercise 2: What fraction of the time is a Rayleigh-distributed signal 10dB below the mean? 20dB? 30dB? This is a useful result to remember.

$$\rho = \frac{R}{R_{rms}} = -10\text{dB}$$

$$\rho = 10^{\left(\frac{-10}{20}\right)} = 0.316$$

$$1 - e^{-\rho^2} = 1 - e^{-0.316^2} \approx 0.1$$

$1-e^{-ANS^2} =$

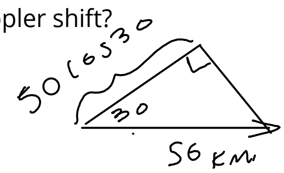
0.095162581

-20 ≈ 0.01

-30 ≈ 0.001

Exercise 3: A receiver in a car receives a 1.8 GHz signal while travelling on a road at 50 km/h. The road is at an angle of 30 degrees relative to the direction of arrival of the signal. What is the velocity relative to the direction of arrival of the signal? By how much does the path length change each second (in meters)? In wavelengths? What is the Doppler shift?

$f_c = 1.8 \times 10^9$



$$v = \frac{50 \times 10^3}{3600} \cdot \cos 30 \approx 12. \ \frac{m}{s}.$$

12 m/s.

$$\lambda = c/f = \frac{3 \times 10^8}{1.8 \times 10^8} \approx 0.17 \text{ m/wave.}$$

$$\frac{12 \text{ m/s}}{0.17 \text{ m/w.}} = 72 \text{ wavelengths/s.}$$

$$f_D = \frac{v}{c} f_c = \frac{12}{3 \times 10^8} \ 18 \times 10^8 = 72 \text{ Hz}$$

Exercise 4: How often will the signal drop 10dB below the mean if the carrier frequency is 1.8 GHz and the velocity is 100 km/h? On average, how long will each of these fades last?

$$\rho = -10\,dB \quad = 20 \log \frac{R}{R_{rms}} = 20 \log \rho$$

$$f_m = \frac{v}{c} f_c = \frac{100 \times 10^3}{3600} \cdot \frac{1}{3 \times 10^8} \cdot 1.8 \times 10^9 = 167\ Hz$$

$$N_R = \sqrt{2\pi}\, f_m\, \rho\, e^{-\rho^2}$$

$$\rho = 10^{-\frac{10}{20}} = \frac{1}{\sqrt{10}} = 0.316$$

$$N_R = \sqrt{2\pi} \cdot 167 \cdot 0.316 \cdot e^{-\frac{1}{10}}$$

$$= 120\ Hz$$

$$\bar{\tau} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \frac{e^{\frac{1}{10}} - 1}{\frac{1}{\sqrt{10}} \cdot 167 \sqrt{2\pi}}$$

$$= 1.6\ ms.$$

Exercise 5: What is the product of N_R and $\overline{\tau}$? How does this compare to $P(r \leq R)$?

$$N_R = \cancel{\sqrt{2\pi}}\,\cancel{f_m}\cancel{\rho}e^{-\rho^2} \qquad \overline{\tau} = \frac{e^{\rho^2} - 1}{\cancel{\rho}\cancel{f_m}\cancel{\sqrt{2\pi}}}$$

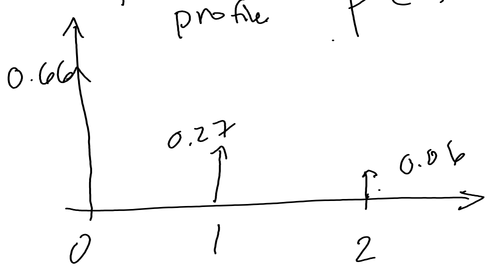
$$N_R \cdot \overline{\tau} = e^{-\rho^2} \cdot e^{\rho^2} - e^{-\rho^2} = 1 - e^{-\rho^2}$$

Exercise 6: A channel has three multipath components with delays of 1, 2 and 3 μs and amplitudes of 10, 6 and 0 dBm respectively. What are the excess delays, the power delay profile, the normalized power delay profile, the mean excess delay and the RMS delay spread?

delay	power (dBm)	excess delay (τ)	power (mW) $P(\tau)$	$p(\tau)$
1	10	0	10	$10/15 = 0.66$
2	6	1	4	$4/15 = 0.27$
3	0	2	1	$1/15 = 0.07$
			15	

$\sum P(\tau)$

normalized power-delay profile $p(\tau)$



$$\overline{\tau} = \sum p(\tau)\tau = 0.66 \cdot 0 + 0.27 \cdot 1 + 0.06 \cdot 2$$

$$= 0.27 + 0.12 = 0.39 \ \mu s$$

$$\sigma = \sqrt{p(\tau)(\tau - \overline{\tau})^2} =$$

$$\sigma^2 = 0.66(0-0.39)^2 + 0.27(1-0.39)^2 + 0.06(2-0.39)^2$$

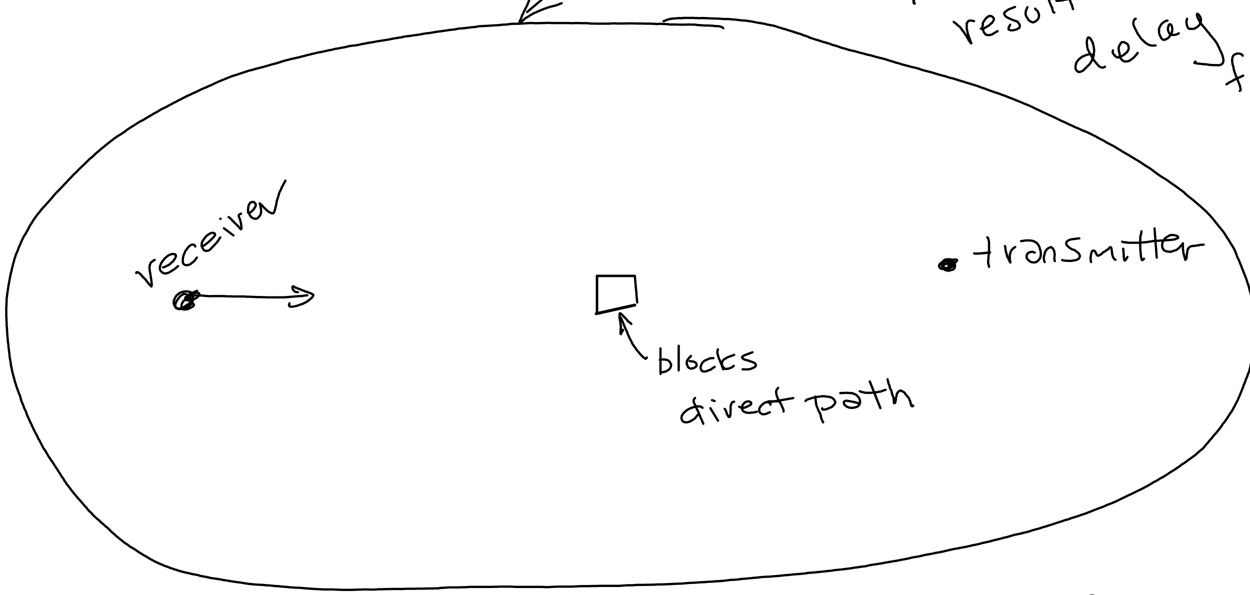
$$\sigma = 0.61 \ \mu s.$$

Using Octave:

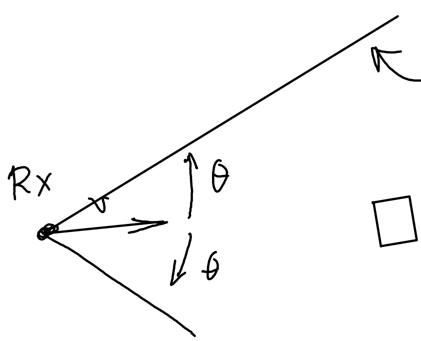
```
>> t=[1,2,3];
>> t=t-min(t);
>> p=[10,6,0];
>> p=10.^(p/10);
>> p=p/sum(p);
>> tbar = sum(t.*p)
tbar =  0.39924
>> s = sqrt(sum(p.*(t-tbar).^2))
s =  0.61102
```

Exercise 7: Imagine a receiver traveling in a straight line towards a transmitter but with no LOS path. How could you arrange reflecting objects such that there was no time dispersion (flat fading)? What arrangement would result in no time-varying fading? Neither?

Scattering objects on ellipse(oid) all result in same delay (flat fading).

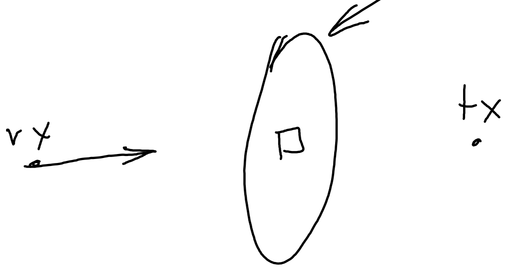


objects at same angle to direction of motion have same Doppler shift ∴ not time-varying (but frequency-selective)



tx

in 3D a ring of scatterers concentric to direction of motion would have a single delay & single Doppler



(each path is same length & all change simultaneously — indistinguishable from L.O.S propagation.