## Error Detection and Correction

**Exercise 1**: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

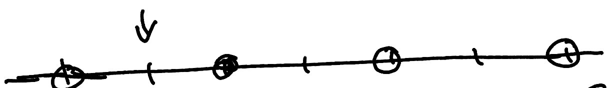
$$\begin{array}{r} 101 \\ \oplus\ 110 \\ \hline \end{array} \qquad d = 0+1+1 = 2$$

$$\begin{array}{r} 010 \\ \oplus\ 110 \\ \hline \end{array} \qquad d = 1+0+0 = 1$$

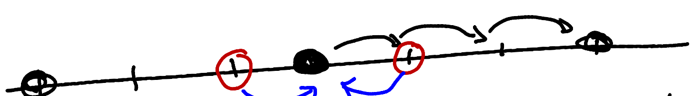
first bit was most likely in error.

receiver should decide that 010 was sent since it has smallest Hamming dist. from all valid codewords.

Yes it's possible (but less likely) that there were 2 errors & the first c/w was actually sent.



dmin = 2 guaranteed to detect 1 error



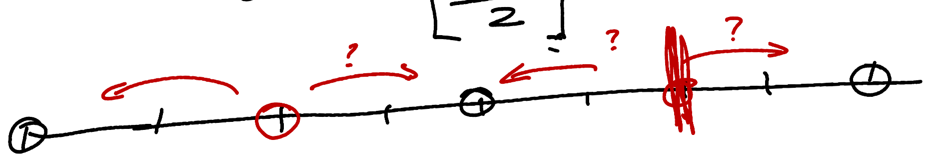
dmin = 3 " " 2 errors.

correct 1 error

If code has min distance of d.

detect d-1 errors

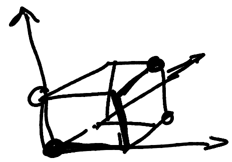
correct $\left\lfloor \frac{d-1}{2} \right\rfloor$ ← divide dmin by 2 & round down.



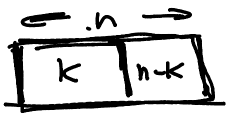
**Exercise 2**: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are $n$, $k$, and the code rate ($k/n$)?

| | 010 |
|---|---|
| 101 | 3 |

$d_{min} = 3$



- detect 2 errors
- correct $\left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error.



$n = 3$

$k = \log_2 (\#\text{ code words})$.

$= 1$

$n - k = 2$

code rate $= \frac{k}{n} = \frac{1}{3}$

**Exercise 3**: How many errors can an $N$-fold repetition code detect? Correct? What is the code rate?

$d_{min} = N$

$(n, k)$

$(Nk, k)$

can detect $N-1$ errors

correct $\left\lfloor \frac{N-1}{2} \right\rfloor$ errors.

code rate $= \frac{k}{n} = \frac{k}{Nk} = \frac{1}{N}$

**Exercise 4**: If the rows of bits we received were 00111, 01010, 11110, 01001, 01010 which bit is likely in error?

bit with error

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | ✗ |
| 0 | 1 | 0 | 1 | 0 | ✓ |
| 1 | 1 | 1 | 1 | 0 | ✓ |
| 0 | 1 | 0 | 0 | 1 | ✓ |
| 0 | 1 | 0 | 1 | 0 | ✓ |
| ✗ | ✓ | ✓ | ✓ | ✓ | |

**Exercise 5**: What is the generator matrix for the (5,3) code that computes two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where $d_i$ is the $i$'th data bit?

$\longleftarrow 5 \longrightarrow$

| $d_0$ | $d_1$ | $d_2$ | $P_0$ | $P_1$ |
|---|---|---|---|---|

$n = 5$
$k = 3$

$$\begin{bmatrix} d_0 & d_1 & d_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 \\ 0 & 1 & 0 & \vdots & 1 & 1 \\ 0 & 0 & 1 & \vdots & 0 & 1 \end{bmatrix} =$$

$I_3$ $P$

$$\begin{bmatrix} d_0 & d_1 & d_2 & d_0 \oplus d_1 & d_1 \oplus d_2 \end{bmatrix}$$

$G$

$d$

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

**Exercise 6**: What is the parity check matrix for the code above? If data vector $[101]$ is to be transmitted, what is the codeword? If there are no errors, what is the result of multiplying the received codeword by $H$? If the channel introduces an error into the second bit?

$$P = \begin{matrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{matrix}$$

$$P^T = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{matrix}$$

$$I_{n-k} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \left[ P^T \mid I_{n-k} \right] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cancel{0} & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

if no errors result is zero.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

if second bit in error.

**Exercise 7**: How many possible correctable error patterns are there for a $(31, 26)$ Hamming code? How many possible received bit patterns?

$(31, 26)$
$\;\; n \quad k$

$t = 1 \qquad \binom{31}{1} = 31$

because it's a Hamming code.

$2^{31}$ possible received codewords $\approx 2 \times 10^9$

**Exercise 8**: What are the possible syndromes for the code above? What was the syndrome when the second bit was in error? Would the code correct the right bit?

$(5,3) \quad d = ?$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$P'$

syndromes

$$He_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad He_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad He_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad He_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad He_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{\text{receive this}} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

if second bit in error.

syndrome.

$\therefore$ matches syndrome corresponding to error in second bit $(e_1)$.

Yes (In this case).

**Exercise 9**: Does a syndrome of zero indicate an error? What is the largest value of $n$ for which a ~~k~~ $n-k$ bit syndrome can correct a single error?

syndrome = 0 indicates no error.

for $n-k$ bit syndrome can only have $2^{n-k}-1$ non-zero ones.

for single error correcting code $t=1$

$$n <= \underbrace{2^{n-k}-1}$$

**Exercise 10**: What is the block size for a RS code using symbols from GF(64) in bit? In symbols?

$n = 2^m - 1 = 2^6 - 1 = 63$ symbols

$63 \times 6 = 378$ bits

**Exercise 11**: How many parity symbols would we need if we wanted to correct 8 8-bit symbol errors? What are $(n, k)$ for this code?

can correct $\frac{n-k}{2}$ errors

for 8-bit symbols $m = 8$

$n = 2^m - 1 = 255$ symbols.

$\frac{n-k}{2} = 8$ $\qquad n-k = 16$

$k = n - 16 = 255 - 16$
$= 239$

we need a $(255, 239)$ $GF(256)$ R-S code.

**Exercise 12**: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.

| | ABC | DAB | CDA | BCD |
|---|---|---|---|---|
| ABC | 0 | 3 | 3 | 3 |
| DAB | | 0 | 3 | 3 |
| CDA | | | 0 | 3 |
| BCD | | | | 0 |

$d_{min} = 3$

can correct $\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error

detect $3-1 = 2$ errors.

doesn't match any valid c/w $\Rightarrow$ error was made.

| | ADA |
|---|---|
| ABC | 2 |
| DAB | 3 |
| CDA | 1 ← pick this c/w. |
| BCD | 3 |

channel probably change C to A.

A = 00
B = 01
C = 10
D = 11

corrected 1 bit error by correcting 1 symbol error

A = 00
B = 01
C = 11
D = 10

corrected 2 bit errors by correcting 1 symbol error.

| | AAA |
|---|---|
| ABC | 2 |
| DAB | 2 |
| CDA | 2 |
| BCD | 3 |

too many errors!

**Exercise 13**: Assuming one bit at a time is input into the encoder in the diagram above, what are $k$, $n$, $K$ and the code rate?

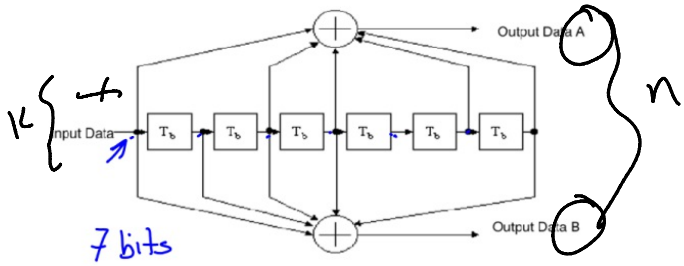



Figure 18-8—Convolutional encoder (k = 7)

$k$ = input bits = 1

$n$ = output " = 2

rate $= \frac{k}{n} = \frac{1}{2}$

$K = 7$

$$\boxed{\begin{matrix} 0 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{matrix}}$$

**Exercise 14**: Consider the convolutional encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

| bit | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| | A, | A, B | B, | A, | A, B, | B |

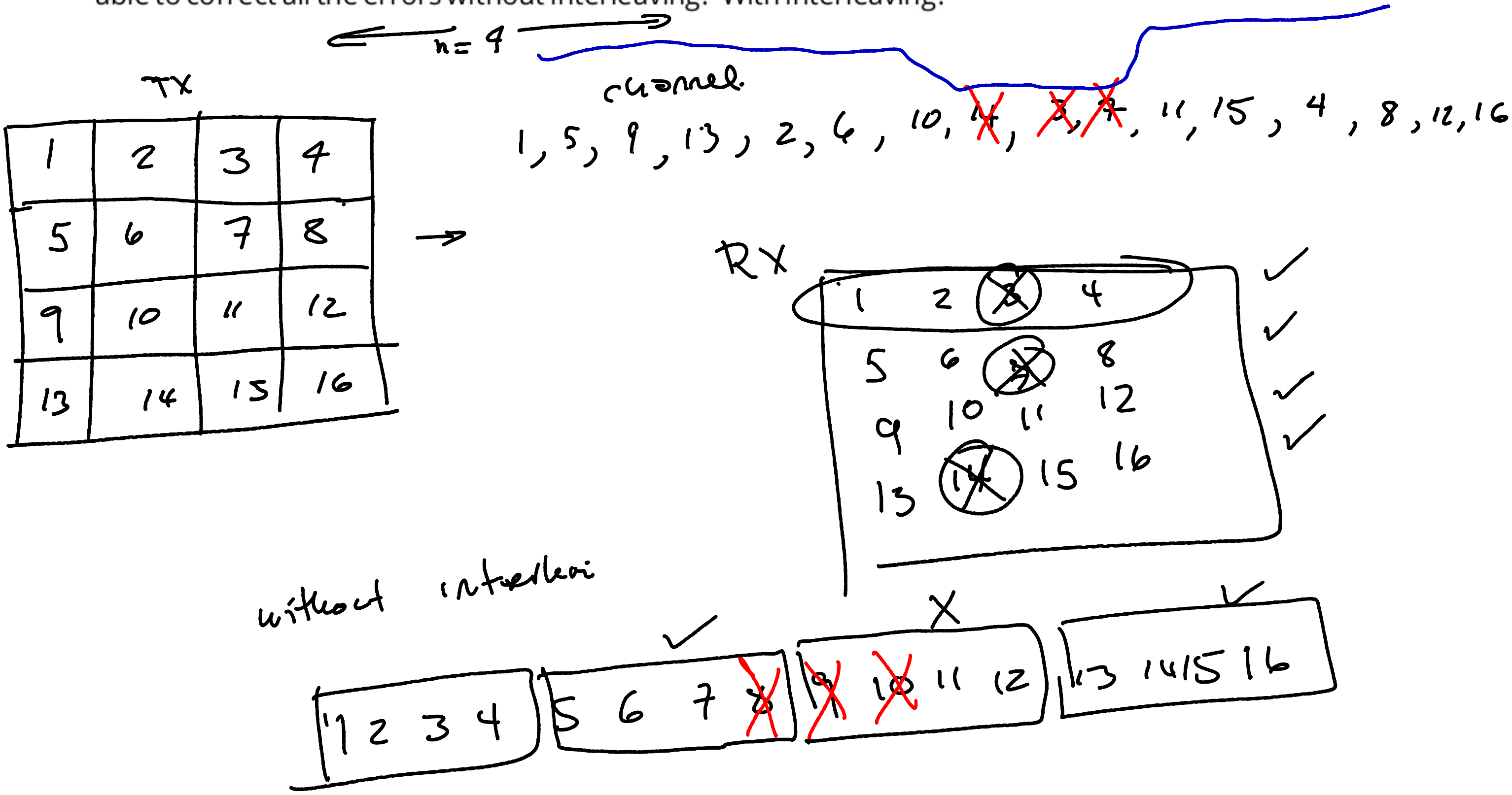
$\frac{k}{n} = \frac{3}{4}$ $\left( \frac{6}{8} = \frac{3}{4} \right)$

code rate $= \frac{3}{4}$

**Exercise 15**: A code uses symbols drawn from GF(8) and is implemented using modulo-8 arithmetic with digits 0 to 7. If we implement parity sum and the three symbols are 2, 4 and 5, what is the value of the parity symbol? If we receive the sequence 1, 2, 3, 4 was there an error? If we know the error was in the first symbol, what was the correct value?

2 4 5

**Exercise 16**: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?




**Exercise 17**: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

Interleave <u>bytes</u> to take advantage of ability of RS codes to correct multiple error per symbol.

**Exercise 18**: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Energy : Joules

Power : W.

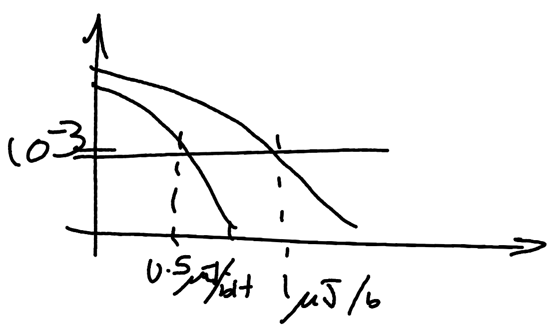
Bit Period : s

$J = W \cdot s$

$E_b$ : Power. Bit Period.

$= S \cdot T_b$

**Exercise 19**: A system needs to operate at an error rate of $10^{-3}$. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is $E_b$ in each case? What is the coding gain?




w/o FEC: $\frac{1\,W}{1\,Mb/s}$ $1W \cdot 1\mu s$

$E_b = 1\,\mu J/bit$

↑ information (data) bits

w/ FEC: $\frac{0.5\,W}{1\,Mb/s}$ $E_b = 0.5\,\mu J/b.$

| | w/o FEC | w/ FEC |
|---|---|---|
| bit rate over channel | 1 Mb/s | 2 Mb/s |
| data (info) " " | 1 Mb/s | 1 Mb/s. |
| | 1 μJ/b | 0.5 μJ/b. |

coding gain $= \frac{1}{0.5} = 2$ (3 dB).