Solutions to Final Exam

Question 1

A fading signal with a Rayleigh distribution has a field strength greater than 100 μV_{rms} 10% of the time. What is the mean field strength in μV_{rms}?

Answer

The cumulative distribution (CDF) of a Rayleigh random variable can be written as:

$$P(r \leq R) = \int_0^R p(r)\, dr = 1 - e^{-\rho^2}$$

where

$$\rho = R/R_{rms}$$

$P(r \leq R) = 1 - P(r > R) = 1 - 10\% = 0.9$. Solving $1 - e^{-\rho^2} = 0.9$ gives $\rho = \sqrt{-\ln(1 - P(r \leq R))}$. The question gives $R = 100\ \mu V$ so the mean field strength is:

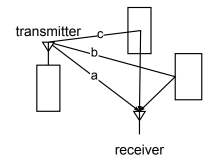
$$R_{\text{rms}} = \frac{100}{\sqrt{-\ln(1-0.9)}} \approx 65.9\ \mu V$$

Many, perhaps most, students did not notice that the question gives the probability that the signal is *greater* than a threshold while the CDF is defined as the probability that a random variable is *less than* a threshold.

Although the question asks for the RMS voltage, many students converted to amplitude instead using a factor of $\sqrt{2}$. Others obtained the correct value and then computed the RMS voltage of the envelope (which is not the same as the RMS voltage of the signal or "field strength") in which case marks were not deducted.

Question 2

(a) A signal transmitted from a base station arrives at a mobile via the three paths shown below: (a) a direct path of length 400 m, (b) a signal reflected back from a building behind the receiver with total path length 700 m at a level of -6 dB relative to the direct path, and (c) a signal reflected from an adjacent building with a total path length of 1 km and a level of -10 dB relative to the direct path. What is the (RMS) delay spread?



(b) A wireless communication system operating over the channel above uses OFDM with a guard time of 1 μs. Will this system's performance over this channel be degraded in any way by ISI? Briefly explain why or why not.

Answer

(a) The mean excess delay is defined as:

$$\bar{\tau} = \sum p(\tau)\tau$$

where

$$p(\tau) = \frac{P(\tau)}{\sum P(\tau)}$$

and τ_i are the excess delays.

The "RMS delay spread" of the channel is

$$\sigma = \sqrt{\sum p(\tau)(\tau - \bar{\tau})^2}$$

The (absolute) delays can be computed from the path lengths as d/c. These are converted to excess delays (τ) by subtracting the minimum delay. The un-normalized signal levels (P) can be computed as the negatives (in dB) of the attenuations. Note that the actual signal powers don't affect the values of mean delay and the delay spread because these use are computed using the normalized power profile (total power = 1).

The mean excess delay and RMS delay spreads can be computed using a spreadsheet as shown in Figure 1 or the following Octave script:

```
format short eng
att=[0, 6, 10];
```

d	delay	tau	att (dB)	P	p	p(tau)*tau		p(tau)*(tau-taubar)^2
400	1E-6	0E+0	0E+0	1E+0	740E-3	0E+0		83E-15
700	2E-6	1E-6	6E+0	251E-3	186E-3	186E-9		82E-15
1000	3E-6	2E-6	10E+0	100E-3	74E-3	148E-9		205E-15
min:	1E-6		*sum:*	1E+0	*mean excess delay:*	334E-9	*rms delay spread:*	609E-9

Figure 1: Computation of RMS delay spread for Question 2.

```
d=[400,700,1000];
tau=(d-min(d))/3e8;
P=10.^(-att/10);
p=P/sum(P);
taubar=sum(p.*tau)
sigma=sqrt(sum(p.*(tau-taubar).^2))
```

which outputs:

```
taubar =   333.9198e-009
sigma =   608.6339e-009
```

In both cases the units are seconds.

(b) Despite the fact that the delay spread is less than 1 μs, the path with a delay of 2 μs will introduce ISI from one OFDM symbol into the subsequent symbol. Although the power of the ISI is approximately 10 dB less than the power of the signal, it is not zero.

Question 3

(a) A wireless communication system uses satellites that orbit at an altitude of 500 km and operate at a frequency of 12 GHz. What is the path loss, in dB, between a ground station and the satellite when the satellite is directly overhead?

(b) Assuming this system uses circular antennas with a 50 cm diameter and that the effective area of the antenna is equal to its physical area, what is the gain of the antenna in dB?

(c) If the uplink transmit power is 1 W and both transmit *and* receive antennas have the gain calculated above, what power is received at the satellite? Give your answer in dBm.

Answer

(a) The wavelength is $\lambda = {}^{c}/_{f} = {}^{3\times10^8}/_{12\times10^9} = 0.025$ m and the path loss is:

$$\left(\frac{4\pi d}{\lambda}\right)^2 = \left(\frac{4\pi\cdot 500\times10^3}{0.025}\right)^2 \approx 168 \text{ dB}$$

(b) The physical, and assumed effective area, is $A_e = \pi r^2 = \pi\left(\frac{0.5}{2}\right)^2 = \pi/16$. Substituting A_e and λ in the equation for gain as a function of effective area gain is:

$$G = \frac{4\pi A_e}{\lambda^2} = \frac{\pi^2/4}{0.025^2} \approx 3.95\times10^3 \approx 36 \text{ dB}$$

(c) From the Friis equation $P_R = P_T G_T G_R / PL$. $P_T = 1\,\text{W} = 30\,\text{dBm}$ and $P_R \approx 30{+}36{+}36{-}168 = -66$ dBm.

Question 4

A diversity receiver uses maximal-ratio combining and three antennas. The SNRs received on the three antennas are 10 dB, 6 dB and 0 dB.

(a) What weights are applied to the signals from the three antennas? Give your answer in linear (voltage gain) units with the largest gain normalized to 1.

(b) What is the SNR after diversity combining? Give your answer in dB.

Answer

1. To obtain the highest SNR the signals should be weighted according to their voltage ratios (square roots of the SNRs). These are $10^{10/20} = \sqrt{10} \approx 3.16$, $10^{6/20} \approx 2$, and $10^{0/20} = 1$, which normalized to a maximum of 1 are ≈ 1, 0.633, and 0.316.

2. After maximal-ratio diversity combining the SNR is the sum (in linear power units) of the SNRs. This is $10^{10/10} + 10^{6/10} + 10^{0/10} = 10 + 4 + 1 = 15$ which is ≈ 11.8 dB.

Question 5

A communication system transmits bits (0 or 1) with a probability of error equal to 10^{-2}. The error rate is the same for 0's and 1's. You will be adding FEC coding to ensure a negligible (arbitrarily small) BER. What is the highest code rate you would ever expect to be able to achieve? Show your work.

Answer

From the description, this is a binary symmetric channel with $p = 0.01$. The capacity of the channel is thus given by:

$$
\begin{aligned}
C &= 1 - (-p \log_2 p - (1-p)\log_2(1-p)) \\
&= 1 - (-0.01 \log_2 0.01 - (1-0.01)\log_2(1-0.01)) \\
&\approx 0.92
\end{aligned}
$$

The code rate is the ratio of information bits to transmitted bits. This is also the definition of capacity for the BSC. The Shannon capacity theorem requires the information rate be less than capacity to achieve an arbitrarily low error rate. Thus the highest code rate we can expect to achieve is approximately 0.92.

Question 6

(a) What is the generator matrix for a *systematic*[1] (4,2) repetition code?

(b) What is the corresponding parity check matrix?

Answer

(a) A repetition code simply outputs each bit multiple times. For a $(4,2)$ code each bit must be repeated once. A systematic code outputs the data bits at the start of the codeword (i.e. the first part of the generator matrix is a $k \times k$ identity matrix). To output each bit once and then repeat each bit we can make the generator matrix equal to two identity matrices:

$$
G = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]
$$

An alternative would be where the bits are repeated in reverse order:

$$
G = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]
$$

(b) The parity check matrix is thus the same as the generator matrix:

$$
\begin{aligned}
H &= [P^T \,|\, I_{n-k}] \\
&= \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]
\end{aligned}
$$

For the alternative generator matrix the parity check matrix would then be:

$$
\begin{aligned}
H &= [P^T \,|\, I_{n-k}] \\
&= \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]
\end{aligned}
$$

Some solutions had two ones in the same column of the generator matrix. This outputs the modulo-two sum of both data bits. This is a reasonable FEC code but it's not a repetition code. Others had two ones in the same row. This includes each data bit in multiple parity bits which, while also reasonable, is also not a repetition code.

Question 7

The third-order intermodulation products at the output of a receiver must be 30 dB below the desired output. The desired (in-channel) output level is 1 V_{rms} at an impedance level of 50 ohms (Hint: $P = V^2/R$).

(a) What is the minimum required output IP3 of the receiver in dBm?

(b) If the receiver has 80 dB of gain from the input to the output, what is the input IP3 of the receiver?

[1]The data bits are transmitted at the beginning of a codeword.

Answer

(a) A voltage of 1 V at an impedance of 50 Ω corresponds to a power of $1^2/50 = 20$ mW $= 13$ dBm. The output IP3 must be $30/2 = 15$ dB above the output level at $13 + 15 = 28$ dBm.

(b) The input and output IP3 are related by the gain of the device so the input IP3 must be 80 dB lower or $28 - 80 = -52$ dBm.

Question 8

Which type of spread-spectrum system, DSSS or FHSS, would be able to avoid interference to nearby narrow-band users of overlapping spectrum? Explain briefly.

Answer

FHSS would be able to avoid interference to nearby narrow-band users of overlapping spectrum by avoiding hopping to the frequencies they were using.

DSSS reduces the power spectral density of the signal and thus reduces but cannot *avoid* interference to users of overlapping spectrum. Due to the (typically) power-law relationship between distance and path loss, the reduction is likely to be insufficient to avoid interference to *nearby* users.

Question 9

By how much could a MIMO system increase throughput to one user if the transmitter had four antennas and the receiver had three?

Answer

A MIMO system can increase throughput to one user by the minimum of the number of transmit and receive antennas. In this case the throughput improvement could be up to $\min(4, 3) = 3$.