Error Detection and Correction

Exercise 1: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

valid → 101
received → 110

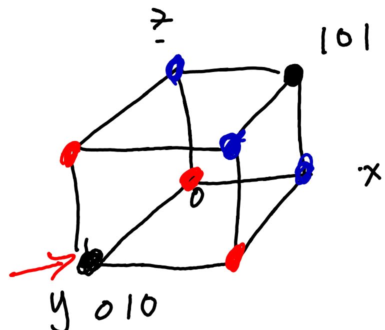
d = 2

010
110

d = 1

receiver chooses this as most likely to have been transmitted.

first bit would have been changed.



detectable

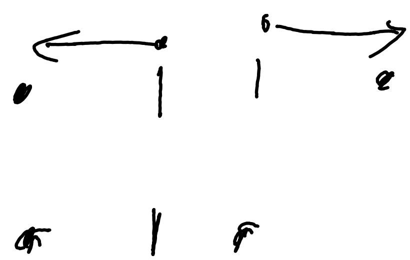
$d-1 = 2$

$2-1=1$

correctable

$\left\lfloor \frac{d-1}{2} \right\rfloor = 1$

$\left\lfloor \frac{d-1}{2} \right\rfloor = 0$



$d_{min} = 3$

$d_{min} = 2$

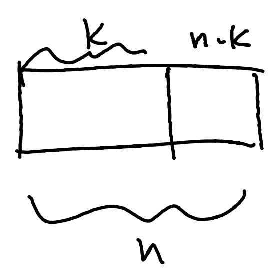
Exercise 2: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?

$d_{min} = 3$ detect = 2 correct = 1

$n = 3$

$k = \log_2(\#\text{codewords}) = \log_2(2) = 1$

$\frac{k}{n} = \frac{1}{3}$



Exercise 3: How many errors can an N-fold repetition code detect? Correct? What is the code rate?

e.g.

$d_{min} = N$

0 0	(a) 0 0 0	0 0 0 ..
0 1	(b) 0 0 0	1 1 1 .
1 0	(c) 1 1 1	0 0 0 ..
1 1	(d) 1 1 1	1 1 1 ..

can detect: $d_{min} - 1 = 3 - 1$
$= 2$

can correct: $\left\lfloor \frac{d_{min}-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = 1$

code rate $= \frac{k}{n} = \frac{k}{Nk} = \frac{1}{N}$

	a	b	c	d
a	0	3	3	6
b		0	6	3
c			0	3
d				0

$d_{min} = 3$

Exercise 4: If the rows of bits we received were 00111, 01010, 11110, 01001, 01010 which bit is likely in error?

1					
0	0	1	1	1	X
0	1	0	1	0	✓
1	1	1	1	0	✓
0	1	0	0	1	✓
0	1	0	1	0	
X	✓	✓	✓		

$$\underbrace{[\; k \mid n-k \;]}_{n}$$

$(5,3) \equiv (n, k)$.

$n = 5$

$k = 3$

Exercise 5: What is the generator matrix for the (5,3) code that computes two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where d_i is the i'th data bit?

$$G = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \qquad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$d = [d_0 \; d_1 \; d_2]$

Exercise 6: What is the parity check matrix for the code above? If data vector [101] is to be transmitted, what is the codeword? If there are no errors, what is the result of multiplying the received codeword by H? If the channel introduces an error into the second bit?

$n = 5$
$k = 3$
$n - k = 2$

$I_{n-k} = I_2$

$$H = \left[\, P^T \;\middle|\; I_{n-k} \,\right] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$d = [1 \; 0 \; 1] \cdot G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = [1 \; 0 \; 1 \; 1 \; 1]$$

$$H \cdot r = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exercise 7: How many possible correctable error patterns are there for a $(31, 26)$ Hamming code? How many possible received bit patterns?

t = number of errors the code can correct

for Hamming code $t = 1$

$$\binom{n}{t} = \binom{n}{1} = n = 31$$

number of possible received c/w $= 2^{31} \approx 2 \times 10^9$

5, 3

Exercise 8: What are the possible syndromes for the code above? What was the syndrome when the second bit was in error? Would the code correct the right bit?

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccc} e_0 & e_1 & e_2 & & e_4 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

$$H \cdot e_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad H \cdot e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad H \cdot e_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H \cdot e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow \qquad H \cdot e_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d = [0 \ 0 \ 1] \cdot G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = [0 \ 0 \ 1 \ 0 \ 1]$$

$$H \cdot r = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \times \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Exercise 9: Does a syndrome of zero indicate an error? What is the largest value of n for which a ~~k~~-bit syndrome can correct a single error?

$n-k$

$$\underbrace{\boxed{k \mid n-k}}_{n}$$

2^{n-k} possible syndromes.

$2^{n-k}-1$ possible error indicates. (error locations).

$n = 2^{n-k}-1$ e.g. $(7,4)$ Hamming code $n=7$

$k=4$

$n-k=3$

$n = 2^3-1 = 8-1=7$

Exercise 10: What is the block size for a RS code using symbols from GF(64) in bit? In symbols?

GF(64) $\rightarrow$ 6 bits / symbol

$n = 2^6 - 1 = 63$ symbols.

$\rightarrow 63 \cdot 6 = 378$ bits

Exercise 11: How many parity symbols would we need if we wanted to correct 8 8-bit symbol errors? What are (n, k) for this code?

$$t = \frac{n-k}{2}$$ correctable errors.

8-bit symbols $\rightarrow$ GF(256)

$k = ?$

$n = 2^8 - 1 = 255$ symbols

$$t = 8 = \frac{255-k}{2}$$

$= 255 \cdot 8 = 2040$ bits

$k = 255 - 16 = \underline{239}$ data symbol

255 total symbols

& can correct up to 8 symbols.

Exercise 12: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.

	ABC	DAB	CDA	BCD
ABC	0	3	3	3
DAB	3	0	3	3
CDA	3		0	3
BCD				0

$d_{min} = 3$

can correct $\left\lfloor \frac{d_{min}-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error

can detect $d_{min} - 1 = 3 - 1 = 2$ error.

	ABC	DAB	CDA	BCD
ADA	2	3	1	3

yes, error made

yes can correct because CDA is at distance 1 & the error would be in first symbol (C changed to A) corrected 1 symbol → If A 00, C 11 → 2 bit errors correcte

A 00
B 01
C 10
D 11

if A 00, C 01 → 1 bit error corrected.

	ABC	DAB	CDA	BCD
AAA	2	2	2	3

min distance ≥ 2 ∴ exceeded error correcting ability of the code.

Exercise 13: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?

K = input data bits = 1

n = Output bits = 2

K = # bits considered per o/p = 7

Exercise 14: Consider the convolutional encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

$$\frac{k}{n} = \frac{3}{4} \frac{\text{input bits}}{\text{output bits}}$$

A_0, A_1, A_2

$\vec{x_0, x_1, x_2}$ B_0, B_1, B_2

Exercise 15: A code uses symbols drawn from GF(8) and is implemented using modulo-8 arithmetic with digits 0 to 7. If we implement parity sum and the three symbols are 2, 4 and 5, what is the value of the parity symbol? If we receive the sequence 1, 2, 3, 4 was there an error? If we know the error was in the first symbol, what was the correct value?

$6 + 4 \rightarrow 2$

$3 \times 5 \rightarrow 7$

$2 + 4 + 5 = 11 \bmod 8 = 3$

checksum $= 8 - 3 = \overline{5}$

c.s. $2 + 4 + 5 + 5 = 16 \bmod 8 = 0$ ✓

$1 + 2 + 3 + 4 = 10 \bmod 8 = 2 \neq 0 \therefore$ error.

if error in first symbol (1)

then $\underbrace{1 + x}_{\text{correction}} + 2 + 3 + 4 \bmod 8 = 0$

$10 + x \bmod 8 = 0$

$x = 6$

$\therefore$ first symbol should be $1 + x = 7$

Exercise 16: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

1 c/w

4 c/w

1	2	~~3~~	4
5	6	~~7~~	8 X
9 X	10 X	11	12
13	~~14~~	15	16

bits come out column-wise:

1, 5, 9, 13, 2, 6, 10, ~~14~~

~~3~~, ~~7~~, 11, 15, 4, 8, 12, 16.

X – if not interleaved

X – if interleaved

without interleaving could not correct c/w 3 (2 errors)

w/ interleaving can correct all error (0 or 1 error per c/w).

Exercise 17: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

interleave bytes because can correct any number of errors in each byte.

Exercise 18: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Joules, watts, seconds

$E_b = P \cdot T_b$

Exercise 19: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is E_b in each case? What is the coding gain?