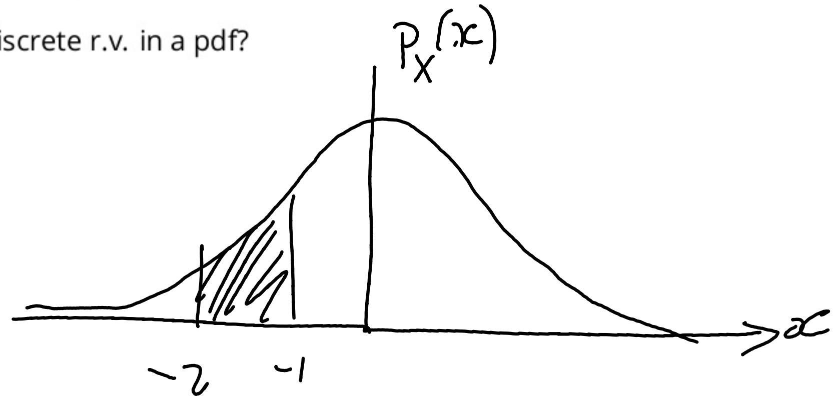
$$E[x^n] = \sum_i p(x_i) X_i^n$$

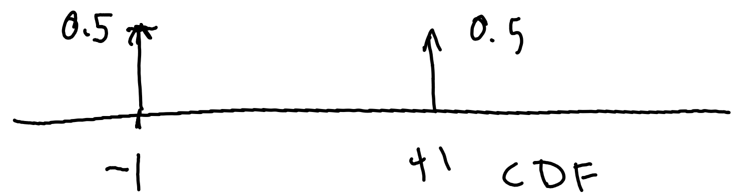
Information and Capacity

Exercise 1: How would you represent a discrete r.v. in a pdf?

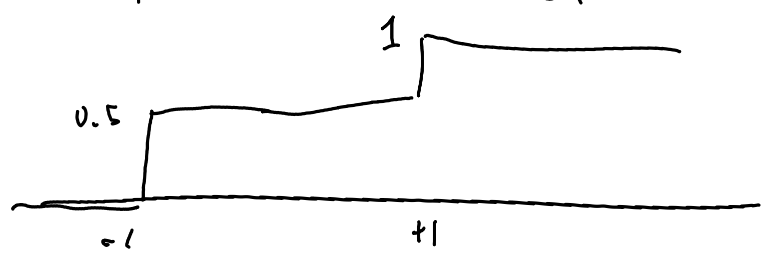
impulse fcn



pdf



CDF



$$\delta(x) = \begin{cases} \lim_{T \to 0} \frac{1}{T} \; (?) \\ 0 \text{ elsewhere} \end{cases}$$

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

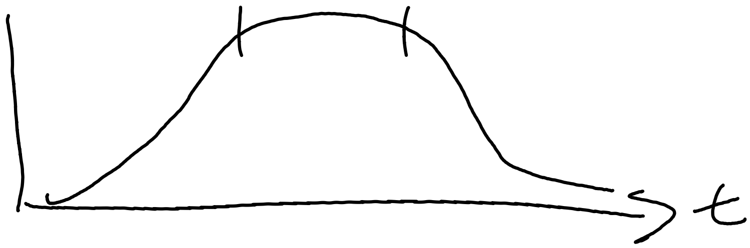
Exercise 2: Is the radio noise generated by the sun a stationary stochastic process? Under what conditions?

probably (sufficiently small time)

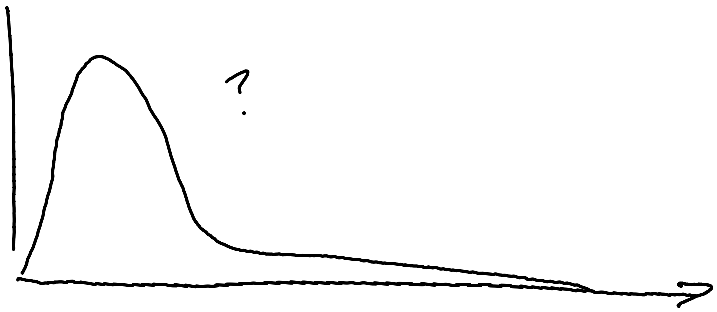
Exercise 3: Would the amount of data transmitted by cellular subscribers be an ergodic stochastic process?

probably not?

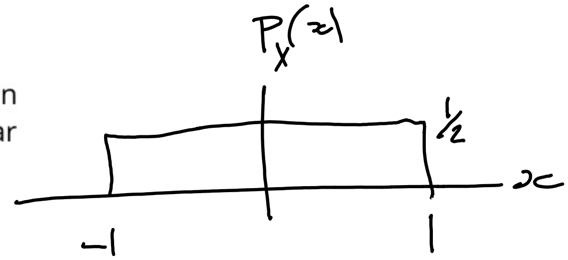
time distr.

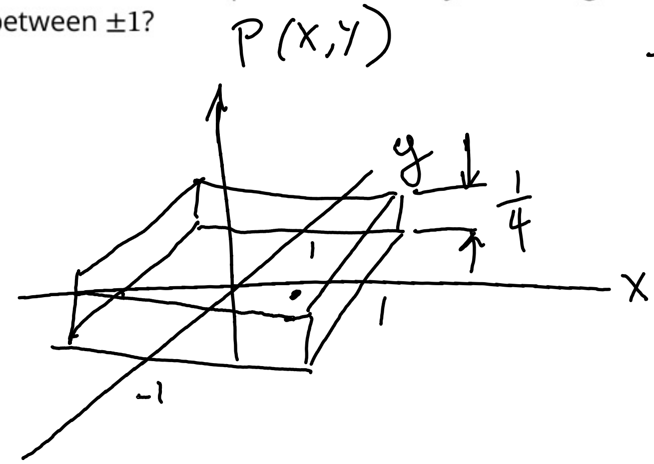


user distr.



Exercise 4: Describe the shape of the joint pdf of two zero-mean iid random variables with uniform pdfs. What if they had triangular pdfs extending between ±1?

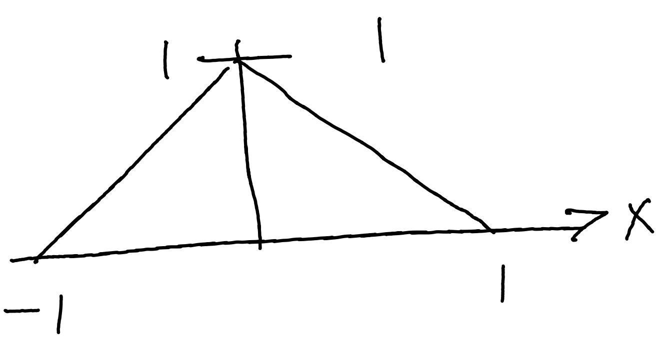


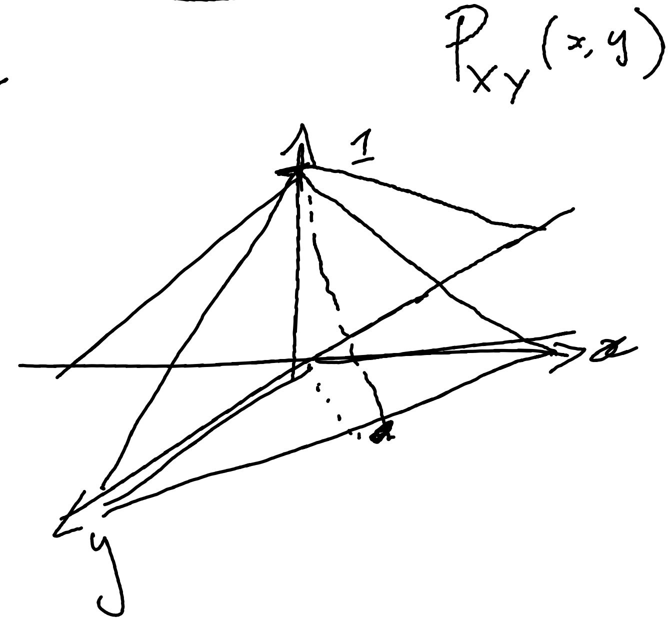


$$p(x,y) = p(x)\,p(y)$$

pdf

$$\int_x\int_y p(x,y)\,dy\,dx = 1$$





$$P_x(x) = \begin{cases} 1 - x & x \geq 0 \\ 1 + x & x < 0 \end{cases}$$

$$= P_y(y)$$

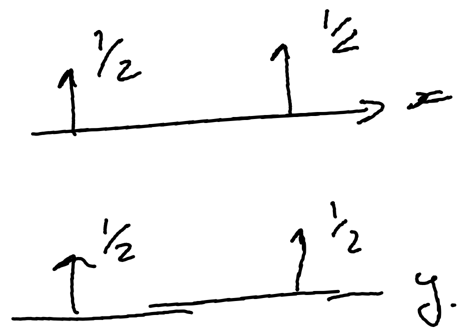
$$(1-x)\,(1-y)$$

$$P_{xy}(x,y) = 1 - x - y + xy$$

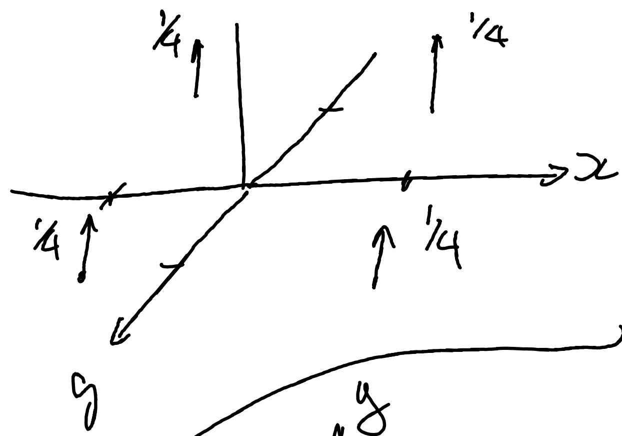
$x \geq 0$
$y \geq 0$

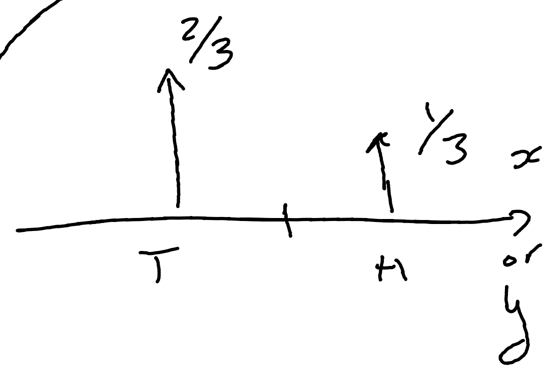
$$1 - 2x + x^2$$

Exercise 5: Two random variables, X and Y represent two flips of a coin (outcomes are H or T for each). Draw the joint pdf if the two coins are fair (unbiased) and the outcomes are independent. Draw the joint pdf if the H is twice as likely as T but the outcomes are independent. Draw the joint pdf if the coins are fair but the outcome of the second toss depends on the first and is always the opposite. Which of these are identically distributed? Which are independent r.v.? Which are i.i.d.?

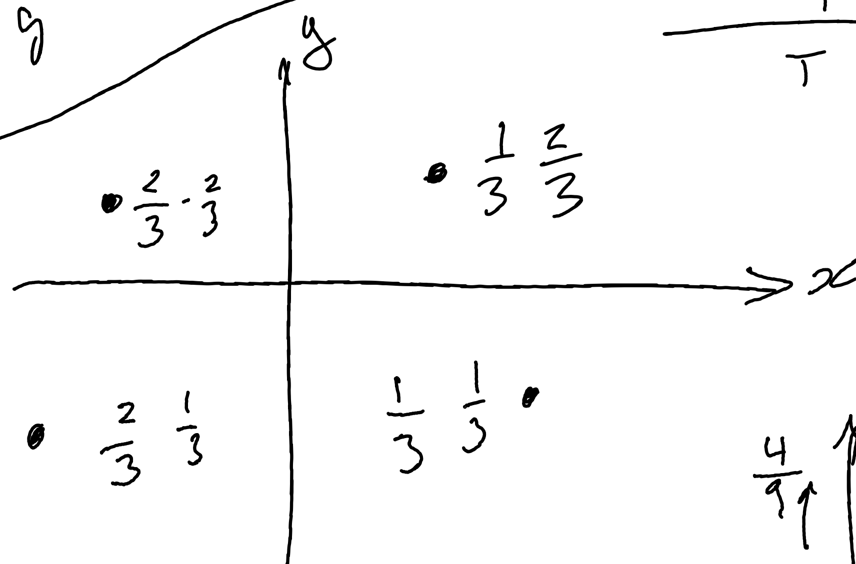


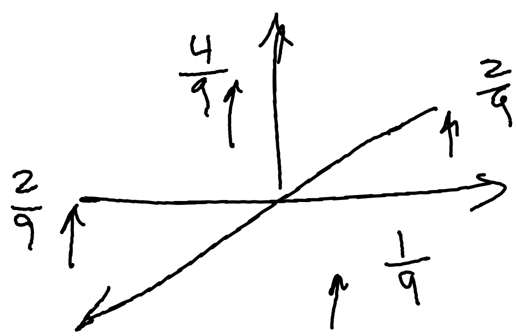
1





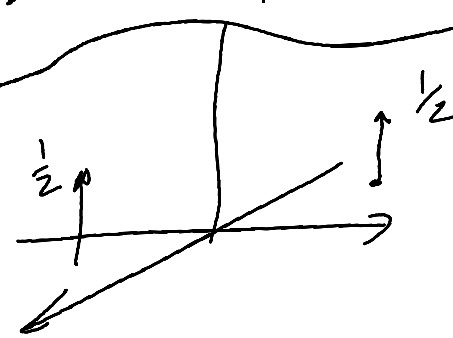
2





3

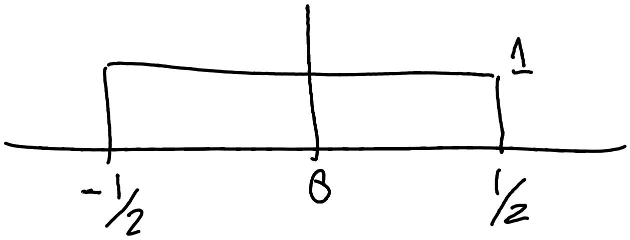
x \ y	H	T
H	0	$\frac{1}{2}$
T	$\frac{1}{2}$	0

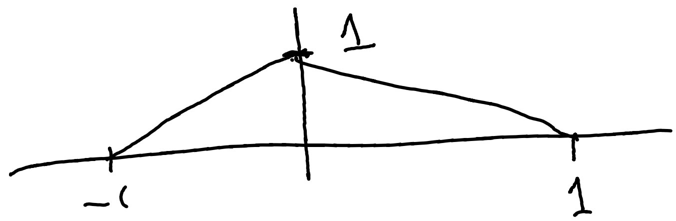


$$P_{XY} = P_X P_Y$$

Exercise 6: What is the pdf of the sum of two zero-mean iid uniformly-distributed rv's whose pdf has a maximum value of 1?

(NOT TO SCALE)

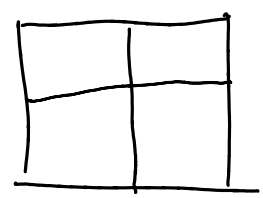




Exercise 7: Prove this.

$$E[(x+y)^2] = E[x^2] + \cancelto{0}{E[xy]} + E[y^2]$$

$$E[xy] = 0$$



(see previous exercise solutions)

Exercise 8: We observe a source that outputs letters. Out of 10,000 letters 1200 were 'E'. What would be a reasonable estimate of the probability of the letter 'E'?

$$\approx 12\% = \frac{1{,}200}{10{,}000}$$

Exercise 9: A source generates four different messages. The first three have probabilities 0.125, 0.125, 0.25. What is the probability of the fourth message? How much information is transmitted by each message? What is the entropy of the source? What is the average information rate if 100 messages are generated every second? What if there were four equally-likely messages?

$$P_0 = \frac{1}{8} \quad P_1 = \frac{1}{8} \quad P_2 = \frac{1}{4} \quad P_3 = 1 - \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{4}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$I_0 = 3 \quad I_1 = 3 \quad I_2 = 2 \quad I_3 = 1$$

$$H = \frac{3}{8} + \frac{3}{8} + \frac{2}{4} + \frac{1}{2} = 1\frac{6}{8} = 1.75 \text{ bits/msg}$$

rate = 100 msg/s

information rate = 175 bits/second

for 4 equally likely messages

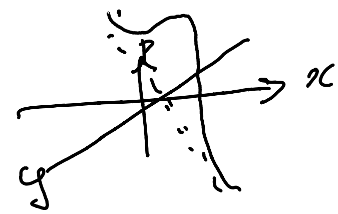
$$P_i = \frac{1}{4}$$

$$I_i = 2$$

$$H = \sum 2 \cdot \frac{1}{4} = 4 \cdot 2 \cdot \frac{1}{4} = 2 \text{ bits/msg.}$$

infor. rate = 200 bits/second.

Exercise 10: What is the mutual information if X and Y are independent? If they are the same?

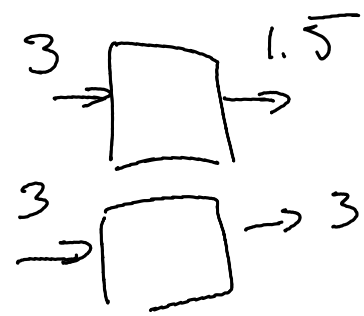


$$\mathrm{I}(X;Y) = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} p(x,y) \log_2 \left(\frac{p(x,y)}{p(x)\,p(y)} \right) \ \frac{\text{bits}}{\text{channel use}}$$

if independent $P(x,y) = P(x)\,P(y)$

then $\log_2(1) = 0$ & $I(x;y) = 0$

if $X = Y$ $\quad p(x) \boxed{\log_2 \left(\frac{1}{p(x)} \right)}$



$= -\log_2(p(x))$

$= I(x)$

$I(x;y) = H(X).$

Exercise 11: What is capacity of a binary channel with a BER of $\frac{1}{8}$ (assuming the same BER for 0's and 1's)?

$$C = 1 - (-p \log_2 p - (1-p)\log_2(1-p))$$

$= 1 - \left(-\frac{1}{8} \log_2 \frac{1}{8} - \left(1 - \frac{1}{8}\right) \log_2 \left(1 - \frac{1}{8}\right) \right)$

$= 1 - \left(\frac{3}{8} - \frac{7}{8} \log_2 \frac{7}{8} \right) \overset{?}{=} 0.45$

Exercise 12: What is the channel capacity of a AWGN 4 kHz channel with an SNR of 30dB?

$$C = B \log_2\left(1 + \frac{S}{N}\right)$$

$$= 4000 \log_2 (1 + 1000)$$

$$\approx 40 \text{ kb/s}$$

Exercise 14: You receive 1 million frames, each of which contains 100 bits. By comparing the received frames to the transmitted ones you find that 56 frames had errors. Of these, 40 frames had one bit in error, 15 had two bit errors and one had three errors. What was the FER? The BER?

$$FER = \frac{56}{10^6} = 56 \times 10^{-6} = 5.6 \times 10^{-5}$$

$$BER = \frac{40 \times + 15 \times 2 + 1 \times 3}{100 \times 10^6} = 73 \times 10^{-8} = 7.3 \times 10^{-7}$$