Solutions to Assignment 2

Question 1

(a) Since the velocity of propagation in free space is approximately 300 m/µs, a propagation path delay difference of 1800 m corresponds to a delay path difference of $\frac{1800}{300} = 6\,\mu\text{s}$. An appropriate minimum duration for the cyclic extension would thus be 6 µs.

(b) The OFDM symbol duration would have to be at least $\frac{6}{10\%} = 60\,\mu\text{s}$ to result in this cyclic extension duration being less than 10% of the OFDM symbol duration.

(c) Complex sampling a channel bandwidth of 800 kHz requires a sampling rate of $f_s \geq 800\,\text{kHz}$ which is a sample period of $T_s = \frac{1}{800\,\text{kHz}} = 1.25\,\mu\text{s}$. The smallest number of subcarriers that could be used is $N = \frac{60}{1.25} = 48$ and the next-largest power of 2 is 64.

(d) The resulting symbol has 64 samples which has a duration of $64 \times 1.25 = 80\,\mu\text{s}$.

(e) We will need at least $\frac{6\,\mu\text{s}}{1.25\,\mu\text{s}} = 4.8$ samples which would be rounded up to 5 samples.

Question 2

(a) QPSK modulation changes the signal's phase but the amplitude remains constant at $\sqrt{2}$. So the peak and average values are equal and the PAPR is 1.

(b) An OFDM signal is the sum of N sinusoidal subcarriers spaced in frequency by $1/T$. In this question each subcarrier has amplitude 1 and a random, uniformly-distributed phase. The sum is:

$$s(t) = \sum_{k=0}^{N-1} e^{j(2\pi \frac{kt}{T} + \theta_k)}$$

where θ_k is the random phase of subcarrier k. The maximum magnitude happens when the phases of all subcarriers are equal and the subcarrier magnitudes add to $N \times 1 = N$. For example, if $\theta_k = -k\pi$ then $s(T/2) = \sum_N 1 = N$. Thus the maximum possible peak power (normalized to 1 Ω) is N^2.

For large N the Central Limit Theorem applies. As discussed in Lecture 4, the probability distribution of the real and imaginary components will be normally distributed and the amplitude will be Rayleigh distributed with a parameter σ^2 which is the variance of the real (or imaginary) component. The second moment (RMS power) of the magnitude is $R_{rms}^2 = 2\sigma^2$.

The variance of the real or imaginary components, σ^2, is the variance of a sum of N sinusoids spaced at $\frac{1}{T}$. Since this makes them orthgonal over any multiple of T, the power of the sum is the sum of the powers. Each subcarrier has a power of $\frac{1}{2}$ and the sum has a power $\sigma^2 = \frac{N}{2}$.

The power (second moment) of the magnitude is thus $2\sigma^2 = N$.

Although not asked for in the question, the PAPR is $N^2/N = N$.

(c) The following Matlab (Octave) code uses an inverse FFT to compute the sum of N random-phase complex sinusoids. Each column of the matrix y is set to N random subcarrier phases. An inverse FFT is used to compute the N complex voltages of the OFDM signal. The columns are independent trials. `max(x).^2` finds the peak power for each trial and `std(x).^2` finds the mean (RMS, AC) power for each trial. We print the largest PAPR over all the trials.

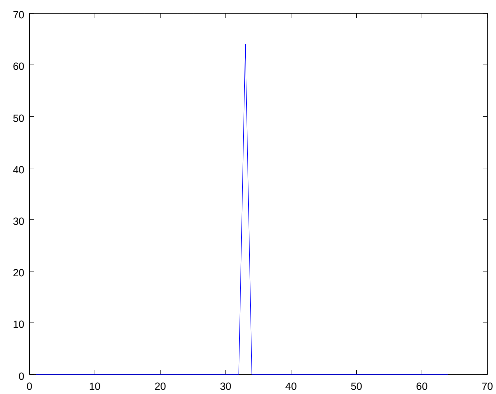
```
N=64;
y=exp(j*2*pi*rand(N,100));
x=abs(N.*ifft(y));
peak=max(max(x.^2))
avg=mean(sum(abs(x).^2)./N)
papr=peak/avg
```

Since phase offsets between the subcarriers are randomly chosen, it is unlikely that at any time they will all have the same phase. Thus the results for the peak value are different for each simulation and never reach the worst-case theoretical value of $N^2 = 4096$. However, the average is always $N = 64$.

To obtain the worst-case PAPR we can set the phases to the values that are predicted above to result in the maximum possible PAPR at $t = T/2$:

```
N=64;
y=exp(j*pi*[0:N-1]');
...
plot(abs(x))
```

which results in the following time-domain signal with a peak power of N^2, and a mean power and PAPR of N:



(d) As explained above, for large N the magnitude (voltage) of the complex OFDM signal has a Rayleigh distribution. The square (power) of a Rayleigh random variable has a chi-squared distribution with two degrees of freedom (χ_2^2).

The chi-squared distribution has a single parameter k. The mean of the χ_k^2 distribution is k ($= 2 = 3\,\text{dB}$). A level of 9 dB ($= 10^{9/10} = 8$) more than the average has a value of $2 \times 8 = 16$. Thus $1 - \chi_2^2(16) \approx 3.4 \times 10^{-4}$ is the probability that the signal's power is 9 dB greater than its mean power.

We can also compute this probability by finding the probability that the signal voltage exceeds the RMS voltage (square root of the mean power) by 9 dB. In Lecture 4 the Rayleigh CDF is given as:

$$P(r \leq R) = 1 - e^{-\rho^2}$$

where ρ is $R/R_{rms} = 9\,\text{dB} = 10^{9/20} = 2.8$ in this case. The probability of exceeding the RMS voltage is the complementary probability or $e^{-(2.8^2)} \approx 3.6 \times 10^{-4}$ as above.

The above values can be found with the Matlab/Octave expressions `1-chi2cdf(16,2)` or `1-raylcdf(2.8,1/sqrt(2))`.

Question 3

This question was not marked.

(a) The integral from 0 to T of the product of subcarriers n and $n+1$ with a frequency offset of δ (and assuming no phase difference) is:

$$\int_0^T \sin\left(2\pi(\frac{n}{T} + \delta)t\right) \cdot \sin\left(2\pi(\frac{n+1}{T} + \delta)t\right)\, dt$$

using $\sin\theta \sin\varphi = \frac{1}{2}(\cos(\theta - \varphi) - \cos(\theta + \varphi))$:

$$= \frac{1}{2}\int_0^T \cos(2\pi\frac{-t}{T})\, dt - \frac{1}{2}\int_0^T \cos\left(2\pi(\frac{2n+1}{T} + 2\delta)\, t\right)\, dt$$

and since the integral of cos() over one period is zero, the first term is zero:

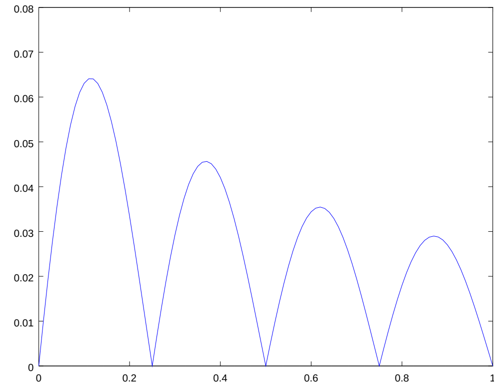
$$= -\frac{1}{2}\frac{1}{2\pi(\frac{2n+1}{T} + 2\delta)}\left[\sin\left(2\pi(\frac{2n+1}{T} + 2\delta)\, t\right)\right]_0^T$$

$$= \frac{-T\sin(4\pi\delta T)}{8\pi n + 4\pi + 8\pi\delta T}$$

(b) Since the values of T and n were not given in the question, $T = 1$ and $n = 0$ were used to obtain a plot and the absolute value of this function:

$$\left|\frac{-\sin(4\pi\delta)}{4\pi + 8\pi\delta}\right|$$

is plotted below as a function of δ from 0 to 1:



Question 4

The link budget has one unknown: the transmit antenna gain, G_T.

The received noise power can be computed as $N = kT_0BF$ from $kT_0 = -174$ dBm/Hz, the noise bandwidth ($B = 100$ MHz or 80 dB-Hz), and the noise figure $F = 4$ dB as $N = -174 + 80 + 4 = -90$ dBm.

Since the required SNR is $\gamma = S/N = 6$ dB, the required received signal power is $P_R = \gamma N = 6 - 90 = -84$ dBm.

Fading causes the path loss to be normally distributed when measured in dB. The distribution has standard deviation of $\sigma = 6\,$dB. The required SNR for service is 6 dB so the mean signal level at the given distance[1] d must be sufficiently above the mean so that the probability of being below this level is $1 - 95\% = 5\%$ to obtain a 95% probability of service.

From a table of the CDF of a normal distribution we can find that a mean that is $\approx 1.645\sigma = 9.8$ dB above the required minimum signal level will give the required service probability. Thus the required fading margin is $M = 9.8$ dB.

The received signal power is given by the Friis equation with additional losses and fade margin:

$$P_R = P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2 \frac{1}{LM}$$

where L represents the additional (atmospheric and transission line) losses and M is the required fade margin. We can solve for for the required transmit antenna gain:

$$G_T = \frac{P_R}{P_T G_R} \left(\frac{\lambda}{4\pi d}\right)^{-2} LM$$

[1] This assumes the service probability is defined at the cell edge, in some cases it's defined over the whole cell.

Given $P_R = -84$ dBm, $P_T = 3\,\text{W} = 34.8\,$dBm, $G_R = 10$ dB, $d = 2000$ m and $\lambda = c/f = 3 \times 10^8/30 \times 10^9 = 0.01$ m, $L = 3.5 + 1.5 = 5$ dB, and $M = 9.8$ dB we can find $G_T = -84 - 34.8 - 10 + 128 + 5 + 9.8 \approx 14$ dB.