Solutions to Assignment 1

Question 1

The Cumulative Distribution Function (CDF) of the Rayleigh distribution is given by:

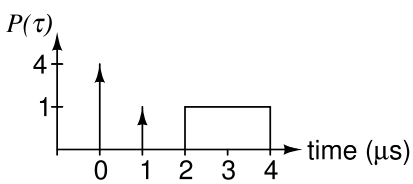
$$P(r \leq R) = \int_0^R p(r)\,dr = 1 - e^{-\rho^2}$$

where $\rho = R/R_{rms}$ where $R_{rms} = \sigma\sqrt{2}$ is the power of the signal.

In this case the power ratio is $-105 - -90 = -15$ dB which is voltage ratio of $\rho = 10^{\frac{-15}{20}} = 0.178$, $\rho^2 = 0.0316$, and thus the probability that the signal will be 15 dB lower than the mean is $1 - e^{-0.0316} \approx 3.1\%$.

Question 2

To compute the excess delay statistics we should consider that the impulse response starts at 1 μs and use the square of the impulse response:



We first need to compute the normalized power delay profile, p(?):

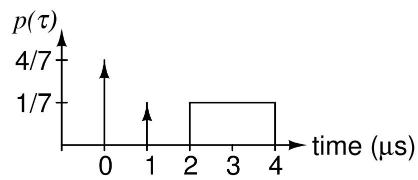
$$p(\tau) = \frac{P(\tau)}{\sum P(\tau)}$$

Since the delay profile is a continuous distribution we compute the total power as an integral instead of a sum. The delay profile and thus the integral can be expressed as a sum of three functions:

$$\int P(\tau)d\tau = \int 4\delta(0-\tau)d\tau + \int 1\delta(1-\tau)d\tau + \int_2^4 1d\tau$$

$$= 4 + 1 + [\tau]_2^4 = 4 + 1 + 2 = 7$$

and $p(\tau)$ looks as follows:



The mean excess delay is computed as the first moment of $p(\tau)$:

$$\begin{aligned}\bar{\tau} &= \int \tau p(\tau)\,d\tau \\ &= 0 \cdot \frac{4}{7} + 1 \cdot \frac{1}{7} + \int_2^4 \tau \cdot \frac{1}{7}\,d\tau \\ &= 0 + \frac{1}{7} + \frac{1}{7}\left[\frac{\tau^2}{2}\right]_2^4 \\ &= \frac{1}{7} + \frac{1}{7} \cdot \frac{16-4}{2} = \frac{1}{7} + \frac{6}{7} = 1\,\mu\text{s}\end{aligned}$$

The RMS delay spread is the square root of the second central moment of $p(\tau)$, σ^2:

$$\sigma^2 = \int (\tau - \bar{\tau})^2 p(\tau)\,d\tau$$

and as before the integral can be divided into three:

$$\sigma^2 = (0-1)^2 \cdot \frac{4}{7} + (1-1)^2 \cdot \frac{1}{7} + \int_2^4 (\tau-1)^2 \cdot \frac{1}{7}\,d\tau$$

$$= \frac{4}{7} + 0 + \frac{1}{7}\int_2^4 \left(\tau^2 - 2\tau + 1\right)\,d\tau$$

$$= \frac{4}{7} + \frac{1}{7}\left[\frac{\tau^3}{3} - \frac{2\tau^2}{2} + \tau\right]_2^4$$

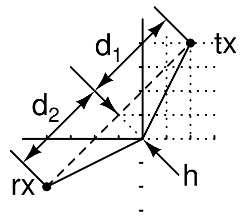
$$= \frac{4}{7} + \frac{1}{7}\left(\frac{1}{3}(4^3 - 2^3) - (4^2 - 2^2) + (4-2)\right)$$

$$= \frac{4}{7} + \frac{56}{21} - \frac{12}{7} + \frac{2}{7} \approx 1.81$$

Thus the RMS delay spread is $\sqrt{\sigma^2} = \sqrt{1.81} = 1.35\,\mu$s.

Question 3

The geometry of the diffraction path is shown below:



The height of the obstruction above the LOS path is $h = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ m, the distances to the obstruction are $d_1 = d_2 = 6\sqrt{2}$ m, the free-space distance would be $d = d_1 + d_2 = 12\sqrt{2}$ m, and the wavelength is $\lambda = c/f = 3 \times 10^8/465 \times 10^6 = 0.645$ m.

The Fresnel-Kirchoff diffraction parameter is

$$\nu = h\sqrt{\frac{2(d_1 + d_2)}{\lambda d_1 d_2}}$$

$$= 2\sqrt{2}\sqrt{\frac{2(6\sqrt{2} + 6\sqrt{2})}{0.645 \cdot 6\sqrt{2} \cdot 6\sqrt{2}}} \approx 2.42$$

and from the graph in lecture 2, the diffraction gain is approximately -22 dB.

The free-space path gain is given by:

$$\left(\frac{\lambda}{4\pi d}\right)^2 = \left(\frac{0.645}{4\pi \cdot 12\sqrt{2}}\right)^2 = 9.15 \times 10^{-6} = -50.4\,\text{dB}$$

Thus the total loss, including both diffraction and distance effects would be approximately 22 + 50 = 72 dB.

Question 4

For a receiver with two diversity branches with SNRs on the two branches of 10 dB and 6 dB, the signal powers on the two branches would be $10N$ and $\approx 4N$ where N is the noise power. The voltages would be proportional to the square roots of these, $3.16\sqrt{N}$ and and $2\sqrt{N}$ respectively.

(a) (i) With selection diversity the branch with the best SNR would be chosen: 10 dB.

(ii) For equal-gain *non-coherent* combining the signal powers and the noise powers add. The signal power would be $(10 + 4)N$ and the noise power $2N$ for an SNR of $7 = 8.4$ dB. This is worse than selection diversity but simpler to implement and better than the worst branch.

For equal-gain *coherent* combining the signal voltages add and the noise powers add. The signal voltage would be $5.16\sqrt{N}$, the noise power $2N$ and the SNR $\frac{5.16^2}{2N} = 13.3 = 11.2$ dB. This is better than selection diversity.

(iii) For maximal-ratio combining the branches are weighted according to the signal voltage to noise power ratios. These ratios are $3.16\sqrt{N}/N = 3.16/\sqrt{N}$ and $2\sqrt{N}/N = 2/\sqrt{N}$.

With coherent[1] maximal-ratio combining the combined signal voltage would be $(3.16\sqrt{N} \cdot 3.16/\sqrt{N} + 2\sqrt{N} \cdot 2/\sqrt{N}) = (10 + 4) = 14$, and the combined signal power would be 14^2.

The noise powers on the two branches are scaled by the squares of the voltage scaling factors: $10/N$ and $4/N$ and result in powers of 10 and 4 and combined power of 14.

The SNR would be $14^2/14 = 14 = 11.5$ dB (the sum of the branch SNRs). This is the optimum linear combiner and slightly better, in this case, than equal-gain coherent combining.

(b) With switching diversity if the switching threshold was 8 dB then the SNR would be 10 dB as the receiver would be guaranteed to switch away from the branch with a 6 dB SNR. However, if the threshold was 5 dB then the receiver would not switch and the SNR could be either 6 dB or 10 dB depending on which branch had been selected most recently.

Question 5

The base station transmits $P_T = 20$ dBm $= 100$ mW. The antenna gains are not given and will be assumed to be 0 dBi = 1. The wavelength at 1.8 GHz is $\lambda = c/f = 3 \times 10^8/1.8^9 = 0.166$ m. In this case the path loss at a distance of 100 m matches that of free-space.

[1] For maximal-ratio combining, coherent combining is implied.

Thus the signal level will be given by the Friis equation:

$$P_{R(100\text{ m})} = P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2$$

$$= 100 \cdot 1 \cdot 1 \left(\frac{0.166}{4\pi 100}\right)^2 = 1.75 \times 10^{-6}\text{ mW}$$

The signal is affected by log-normal shadowing with a standard deviation of 10 dB. The mean of this path loss is zero and the standard deviation is 10 dB. The Gaussian CDF has a value 0.95 at 1.7 standard deviations above the mean (17 dB in this case). So we will find the distance at which the mean signal level is 17 dB above the minimum required: 17 dB + −105 dBm = −88 dBm = 1.59×10^{-9} mW.

If the distance-dependent path loss is defined by a power law with an exponent of −2.7, the signal level at a distance d will be:

$$P_R(d) = P_{R(d_0=100\text{ m})} \cdot \left(\frac{d}{d_0}\right)^{-2.7}$$

and solving for d:

$$d = d_0 \left(\frac{P_R(d)}{P_{R(100\text{ m})}}\right)^{\frac{-1}{2.7}}$$

$$= 100 \left(\frac{1.59 \times 10^{-9}}{1.75 \times 10^{-6}}\right)^{\frac{-1}{2.7}} \approx 1.3\text{ km}$$

Question 6

(a) If the two coins are fair (unbiased) the marginal probabilities are 0.5 for H and T. If the outcomes are independent then we can obtain the joint pdf by multiplying the two marginal probabilities and the probability of each outcome will be 0.25.

Since the joint pdf has only four non-zero values it can be represented as table with the outcome of the first coin as the row and the outcome of the second coin as the column:

	H	T
H	0.25	0.25
T	0.25	0.25

(b) If H is twice as likely as T, then P(H)=2/3 and P(T)=1/3. If the outcomes are independent we can multiply these probabilities and obtain P(HH)=4/9, P(HT)=P(TH)=2/9 and P(TT)=1/9.

	H	T
H	4/9	2/9
T	2/9	1/9

(c) If the coins are fair but the outcome of the second toss depends on the first and is always the opposite there are only two outcomes: HT and TH and each will have a probability of 1/2. The probability of HH and TT will be zero.

	H	T
H	0	0.5
T	0.5	0

(d) X and Y (the two coins) are identically distributed if they have the same (marginal) distributions. This is true for (a) and (c). They are independent if the joint distribution is the product of the marginal distributions. This is true for (a) and (b). They are i.i.d. (both independent and identically distributed) only for (a).

Question 7

To find the capacity we find the input probabilities ($p(X)$) that maximize the mutual information.

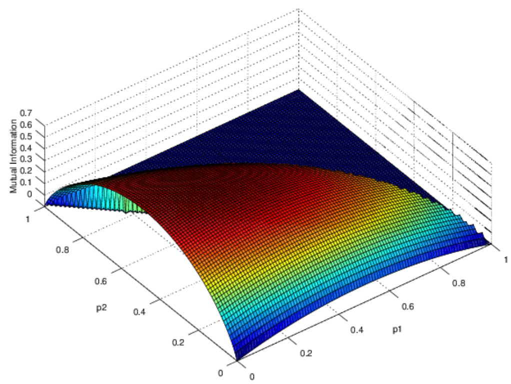
This solution uses the Octave (or Matlab) function `fminsearch()` by passing it an objective function that computes the (negative) of the entropy and an initial guess for the first two probabilities (e.g. 0.3, 0.3).

This function computes the remaining probability and returns an appropriate value (0) for invalid locations in the search space (e.g. if $p(x=1) + p(x=2) > 1$).

The probabilities of the input values that maximizes the mutual information between the input and output and the resulting channel capacity (in bits per channel use) are given by the program as:

```
MI maximized for p(X)= 0.170994 0.438680 0.390327
Capacity = 0.682955 bits/use
```

A surface plot of mutual information versus the first two probabilities shows the maximum at approximately these values:



The code used is shown below:

```
function asg1_6

   % disable some Octave-specific warnings
   warning("off", "Octave:broadcast");

   % channel transition probabilities
   global c

   % test/example channels:

   % ideal channel
   c=[1 0 0 0;0 1 0 0;0 0 1 0] ;

   % "noisy typewriter" channel
   c=[1 0 0 0;0 1 0 0;0 0 .5 .5] ;

   % random channel
   c=[1/4 1/4 1/4 1/4;1/4 1/4 1/4 1/4;1/4 1/4 1/4 1/4] ;

   % second row provides no information - not used
   c=[.5 .5 0 0;1/4 1/4 1/4 1/4;0 0 .5 .5] ;

   % the problem
   c=[.6 .0 .2 .2;.2 .7 .0 .1;.2 .0 .6 .2] ;

   % check
   assert(all(sum(c')==1))

   % maximize mutual information
   [x,fval]=fminsearch(@f, [1/3,1/3]);

   % show results
   disp(c) ;
   fprintf(1,"MI maximized for p(X)= %f %f %f\n", ...
              x(1), x(2), 1-sum(x)) ;
   fprintf(1,"Capacity = %f bits/use\n",-fval) ;

   % plot mutual information vs p(X)
   n=100;
   r=linspace(0,1,n);
   z=zeros(n);
   for x=1:n
      for y=1:n
         z(y,x)=-f([r(x),r(y)]);
      end
   end
   [xx,yy]=meshgrid(r);
   surf(xx,yy,z);
   xlabel('p1')
   ylabel('p2')
   zlabel('Mutual Information')
   % print(1,'sol1-1.png','-S640,480')


end

% function to minimize: negative of mutual information

function in = f(px)

   global c

   % channel input probabilities
   p1=px(1) ;
   p2=px(2) ;
   p3=1-p1-p2 ;

   % avoid non-feasible solutions
   if p1<0 || p2<0 || p3<0
      in = 0 ;
      return
   end

   % return the negative of the mutual information
   in = -mi([p1;p2;p3],c) ;
end

% mutual information for channel input and transition
% probabilities

function in = mi(px,c)
   pxy=px.*c ;
   py=sum(pxy) ;
   h=(pxy.*log2(pxy./(px*py))) ;
   in=sum(h(pxy>0)) ;
end
```