ELEC 466 : DIGITAL SIGNAL PROCESSING
1995/96 WINTER SESSION TERM 2

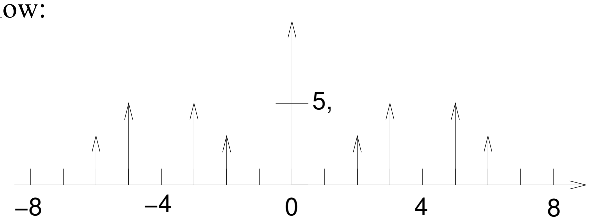
Solutions to Mid-Term Review Assignment

Question 1

The analog signal is:

$$x(t) = 3sin(2\pi 2000t) + 5sin(2\pi 5000t)$$

The effect of ideal sampling is to replicate the spectrum at multiples of the sampling rate. Thus there will be components of amplitude 3 at $8n \pm 2$ kHz and components of amplitude 5 at $8n \pm 5$ kHz. Assume the spectrum is to be plotted over the range -8 to +8 kHz. There will be components of amplitude 3 at $-6, -2, 2,$ and 6. There will be components of amplitude 5 at $-5, -3, 3,$ and 5. These are plotted below:



Question 2

Since the 5 kHz and higher signal components will be greatly attenuated by the anti-aliasing filter, the output of the filter will be approximately a sine wave at a frequency of 2 kHz.

The quantization noise for a sine wave is $q^2/12$ where $q = 2A/(2^B - 1)$ and $2A$ is the peak-to-peak ADC quantization range. To obtain the lowest quantization noise we should set $2A$ as small as possible, ideally equal to twice the amplitude of the desired (2 kHz) signal at the *output* of the filter. However, to simplify the analysis we will initially assume that the filter does not attenuate the 2 kHz signal and thus $2A = 6$volts. Thus the quantization noise level is $(6/255)^2/12 \approx 46 \times 10^{-6}$.

The power of the undesired 5 kHz signal at the output of the filter will be given by $A^2/2$ where A is the amplitude of the 5 kHz signal at the output of the filter. The amplitude of the undesired signal at the output of the filter is determined by the response of the Butterworth filter of order n which is:

$$|H(f)| = \frac{1}{\sqrt{1 + (f/f_c)^{2n}}}$$

If we substitute $A = 5$ and $f = 5$ for the amplitude and frequency of the undesired signal, and set the quantization noise power equal to the undesired signal power:

$$46 \times 10^{-6} = \frac{(5H(5))^2}{2}$$

or, solving for n,

$$n \approx \frac{2.72}{\log(5/f_c)}$$

If we assume $f_c = 4$ kHz this gives $n \approx 28$ and the amplitude of the 2 kHz component is $3H(2) \approx 3$ which justifies our original assumption.

Question 3

There are four values of n for which the convolution of the sequences $x(n) = \{2, 2\}$ and $h(n) = \{3, 1, 2\}$ is non-zero: 0, 1, 2, 3.

For $n = 0$				
k	$x(k)$	$n - k$	$h(n-k)$	$x(k) \times h(n-k)$
0	2	0	3	6
1	2	-1	0	0
				$\sum = 6$

For $n = 1$				
k	$x(k)$	$n - k$	$h(n-k)$	$x(k) \times h(n-k)$
0	2	1	1	2
1	2	0	3	6
				$\sum = 8$

For $n = 2$				
k	$x(k)$	$n - k$	$h(n-k)$	$x(k) \times h(n-k)$
0	2	2	2	4
1	2	1	1	2
				$\sum = 6$

For $n = 3$				
k	$x(k)$	$n-k$	$h(n-k)$	$x(k) \times h(n-k)$
0	2	3	0	0
1	2	2	2	4
				$\sum = 4$

The result, $y(n) = x(n) * h(n)$ is $\{6, 8, 6, 4\}$. Since convolution is a linear operator $2x(n) * 2h(n) = 4(x(n) * h(n)) = \{24, 32, 24, 16\}$.

Question 4

The inverse DFT is defined by:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W^{-kn}$$

and substituting the values of X(k),

$$x(n) = \frac{1}{4} \left(2W^{-0k} + 1W^{-3k} \right)$$

The following table shows the values of W^{-3k}

k	$-3k$	W^{-3k}
0	0	1
1	-3	$-j$
2	-6	-1
3	-9	$+j$

The inverse DFT is thus $\{1/2 + 1/4, 1/2 - j/4, 1/2 - 1/4, 1/2 + j/4\} = \{0.75, 0.5 - 0.25j, 0.25, 0.5 + 0.25j\}$.

Question 5

Similarly, the (forward) DFT reduces to

$$X(k) = \left(2W^{0k} + 1W^{3k} \right)$$

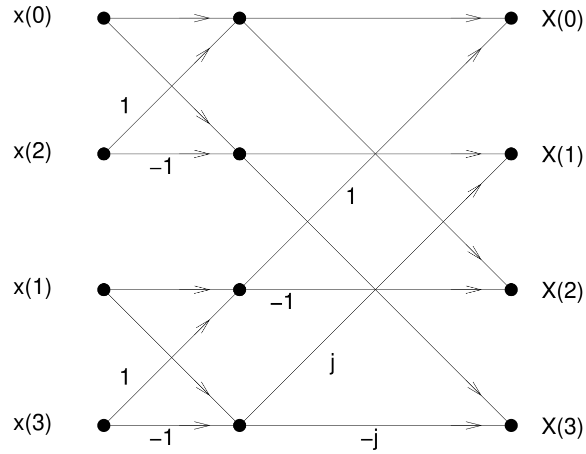
where

k	$3k$	W^{3k}
0	0	1
1	3	j
2	6	-1
3	9	$-j$

and $X(k) = \{2 + 1, 2 + j, 2 - 1, 2 - j\} = \{3, 2 + j, 1, 2 - j\}$.

Question 6

The signal flow graph for a 4-point FFT that computes the DFT, $X(k)$ of a sequence $x(n)$ is as follows:

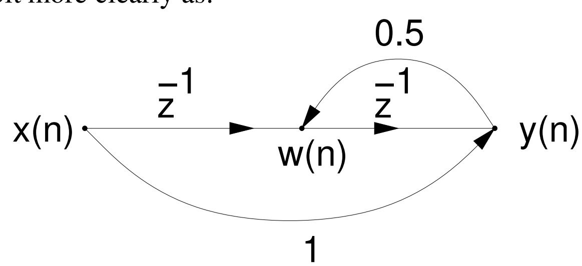


Question 7

From the definition, the z-transform of the sequence $\{1, 0, 2\}$ is $1z^{-0} + 2z^{-2}$.

Question 8

The signal flow graph for the system can be redraw a bit more clearly as:



From the signal flow graph:

$$W(z) = 0.5Y(z) + z^{-1}X(z)$$

and

$$Y(z) = z^{-1}W(z) + X(z)$$

so that

$$Y(z) = z^{-1} \left(0.5Y(z) + z^{-1}X(z) \right) + X(z)$$

$$Y(z)(1 - 0.5z^{-1}) = X(z) \left(1 + z^{-2} \right)$$

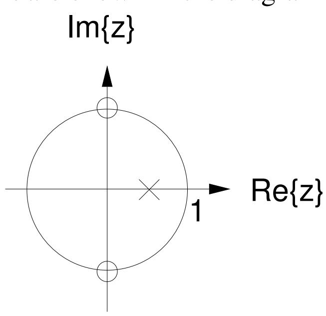
$$
\begin{array}{rr|rrrr}
 & & +1z^{-0} & +0.5z^{-1} & +1.25z^{-2} & \\
\cline{3-5}
+1z^{-0} & -0.5z^{-1} & +1z^{-0} & +0z^{-1} & +1z^{-2} & \\
 & & +1z^{-0} & -0.5z^{-1} & & \\
\cline{3-4}
 & & & +0.5z^{-1} & +1z^{-2} & \\
 & & & +0.5z^{-1} & -0.25z^{-2} & \\
\cline{4-5}
 & & & & +1.25z^{-2} & +0z^{-3} \\
 & & & & +1.25z^{-2} & -0.75z^{-3} \\
\cline{5-6}
 & & & & & +0.75z^{-3}
\end{array}
$$

Figure 1: Impulse response computed by polynomial division.

and the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 - 0.5z^{-1}}$$

(a) By inspection, the transfer function has a pole at $z = 0.5$ and two zeros at $z = \pm j$. The pole/zero positions are shown in the diagram below:



(b) The system will be stable since all poles lie inside the unit circle.

(c) The first 3 values of the impulse response of the system can be found by:

(i) long division of the transfer function as shown in Figure 1.

(ii) computing the impulse response from the difference equation:

$$y(n) = 0.5y(n-1) + x(n-2) + x(n)$$

as follows:

n	$x(n)$	$y(n)$
0	1	$0.5 \cdot 0 + 0 + 1 = 1$
1	0	$0.5 \cdot 1 + 0 + 0 = 0.5$
2	0	$0.5 \cdot 0.5 + 1 + 0 = 1.25$

In both cases the impulse response is $\{1, 0.5, 1.25\}$.

Question 9

Since speech does not require linear-phase filtering an FIR filter is not essential. Cost would be an important consideration in the design of a toy so a filter with as few coefficients as possible would be desirable. For these reasons an IIR design would probably be more suitable.