Solutions to Assignment 4

Question 1

(a) We first express the numerator and denominator in powers of z^{-1}:

$$H(z) = \frac{z^{-3}}{1 + z^{-1} - z^{-2} - z^{-3}}$$

The two polynomials are then divided as shown in Figure 1 on the following page. The impulse response is given by the coefficients of the powers of z^{-1} and are $\{0, 0, 0, 1, -1, 2, -2, 3, -3, \ldots\}$.

(b) The documentation of the `filter()` function defines the output of the function, $y(n)$, as:

```
y(n) = b(1)*x(n) + b(2)*x(n-1) +
         ... + b(nb+1)*x(n-nb)
       - a(2)*y(n-1) -
         ... - a(na+1)*y(n-na)
```

The values $b(i), i = 1, \ldots, n_b + 1$ are thus the constants that scale the inputs to the filter after delay of $i - 1$ samples. The $b(i)$'s correspond to the coefficients of the powers of z^{-n} for $n = 0, \ldots, N - 1$ in numerator of the transfer function (see the derivation in the lecture notes). The values $-a(i), i = 2, \ldots, n_a + 1$ are the constants that scale the previous outputs of the filter before they are fed back into the sum which forms the filter output. The $a(i)$'s correspond to the coefficients of the powers of z^{-n} for $n = 0, \ldots, M - 1$ in the denominator of the transfer function.

Note the following:

- Matlab uses 1-based indexing so that $b(1)$ corresponds to b_0 in the equations the textbook and lecture notes
- $b(1)$ must be 1 because it is (implicitly) the coefficient of $y(n)$
- the signs of the feedback coefficients as given in the diagrams in the lecture notes are missing the minus signs
- Matlab (and every other DSP book) reverses the meanings of the variables a and b.

The numerator of the transfer function written in powers of z^{-1} can be written as:

$$0z^{-0} + 0z^{-1} + 0z^{-2} + 1z^{-3}$$

from which the values of $b(i)$ are: $\{0, 0, 0, 1\}$ and the denominator of the transfer function can be written as:

$$1z^{-0} + 1z^{-1} - 1z^{-2} - 1z^{-3}$$

from which the values of $a(i)$ are: $\{1, 1, -1, -1\}$.

To find the impulse response using the `filter` function we set b and a as above and supply an input vector which is an impulse. Matlab displays the output of the filter which is, by definition, the impulse response. In order to display the first 20 values of the impulse response we append 19 zeros to the impulse using the `zeros()` function.

```
>> b=[0 0 0 1];
>> a=[1 1 -1 -1];
>> x=[1 zeros(1,19)];
>> filter(b,a,x)
```

The result is $\{0, 0, 0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9\}$. As expected, the first few terms match the ones computed by hand.

(c) We first re-write the transfer function as a ratio of two polynomials in powers of z^{-1}. The coefficients of the denominator polynomial can be found by expanding the factors by hand or by using the `conv` function:

$$
\begin{array}{r|rrrrrrrrr}
 & +1z^{-3} & -1z^{-4} & +2z^{-5} & -2z^{-6} & +3z^{-7} & -3z^{-8} & & & \\
\cline{2-7}
+1z^{-0}\ +1z^{-1}\ -1z^{-2}\ -1z^{-3} & +1z^{-3} & & & & & & & & \\
 & +1z^{-3} & +1z^{-4} & -1z^{-5} & -1z^{-6} & & & & & \\
\cline{2-5}
 & & -1z^{-4} & +1z^{-5} & +1z^{-6} & & & & & \\
 & & -1z^{-4} & -1z^{-5} & +1z^{-6} & +1z^{-7} & & & & \\
\cline{3-6}
 & & & +2z^{-5} & +0z^{-6} & -1z^{-7} & & & & \\
 & & & +2z^{-5} & +2z^{-6} & -2z^{-7} & -2z^{-8} & & & \\
\cline{4-7}
 & & & & -2z^{-6} & +1z^{-7} & +2z^{-8} & & & \\
 & & & & -2z^{-6} & -2z^{-7} & +2z^{-8} & +2z^{-9} & & \\
\cline{5-8}
 & & & & & +3z^{-7} & +0z^{-8} & -2z^{-9} & & \\
 & & & & & +3z^{-7} & +3z^{-8} & -3z^{-9} & -3z^{-10} & \\
\cline{6-9}
 & & & & & & -3z^{-8} & +1z^{-9} & +3z^{-10} & \\
 & & & & & & -3z^{-8} & -3z^{-9} & +3z^{-10} & +3z^{-11} \\
\cline{7-10}
 & & & & & & & +4z^{-9} & +0z^{-10} & -3z^{-11}
\end{array}
$$

Figure 1: Impulse response computed by polynomial division.

```
>> conv([1 (-0.5+j/2)],[1 (-0.5-j/2)]);
>> conv(ans,ans);
>> conv(ans,ans);
```

We then multiply by z^{-8}/z^{-8} to express the transfer function as the required ratio of two polynomials in powers of z^{-1}:

$$
\begin{aligned}
H(z) &= \frac{z^{-7}}{1z^{-0} - 4z^{-1} + 8z^{-2}} \\
&\times \frac{1}{-10z^{-3} + 8.5z^{-4} - 5z^{-5}} \\
&\times \frac{1}{+2z^{-6} - 0.5z^{-7} + 0.0625z^{-8}}
\end{aligned}
$$

The values of $b(i)$ are thus $\{0, 0, 0, 0, 0, 0, 0, 1\}$and the values of $a(i)$ are $\{1, -4, 8, -10, 8.5, -5, 2, -0.5, 0.0625\}$. The impulse response is computed by the following Matlab commands:

```
>> a=[1,-4,8,-10,8.5,-5,2,-0.5,0.0625];
>> b=[0,0,0,0,0,0,0,1];
>> filter(b,a,[1 zeros(1,19)])
```

and the output of the filter (the impulse response) is: $\{0, 0, 0, 0, 0, 0, 0, 1, 4, 8, 10, 7.5, 1, -6, -9.5, -7.8125, -2.5, 3, 5.75, 4.9375\}$.

Question 2

(a) Since the denominator has two distinct real roots the form of the partial-fraction expansion will be:

$$
H(z) = \frac{z}{(z-2)(z-3)} = \frac{A_1}{z-2} + \frac{A_2}{z-3}
$$

To solve for A_1 and A_2 set $z = A_1(z-3) + A_2(z-2)$. Then for $z = 2$ we find $A_1 = -2$ and for $z = 3$ we find $A_2 = 3$. The partial fraction expansion is therefore:

$$
H(z) = \frac{-2}{z-2} + \frac{3}{z-3}
$$

(b) The command `help residue` explains that the residue function expects two polynomial coefficient vectors and returns:

- a vector r containing the residues (the coefficients of the terms in the partial fraction expansion),
- a vector p containing the values of the poles,

- a vector k which will be null since the order of the numerator is less than the order of the denominator, and

The Matlab commands to compute the partial fraction expansion are:

```
>> b=[1 0];
>> a=[1 -5 6];
>> [r p k]=residue(b,a)
```

Which gives $r = \{3,-2\}$ and $p = \{3,2\}$. As expected, the results match those obtained above.

(c) The question gives the transfer function as a ratio of polynomials in powers of z so we can simply read the coefficients of the numerator and denominator polynomials and use the `residue()` function to find the partial fraction expansion:

```
>> b=[1 0];
>> a=[1 -7 16 -12];
>> [r p k] = residue(b,a)
```

This gives $r = \{3,-3,-2\}$, $p = \{3,2,2\}$, indicating one pole at $z = 3$ and one repeated pole at $z = 2$. The partial fraction expansion is thus:

$$H(z) = \frac{3}{z-3} + \frac{-3}{z-2} + \frac{-2}{(z-2)^2}$$

Question 3

(a) The transfer function is:

$$H(z) = \frac{1 - 1.094621z^{-1} - 1z^{-2}}{1 - 1.340228z^{-1} + 0.796831z^{-2}}$$

$$\times \frac{1 - 0.350754z^{-1} + 1z^{-2}}{1 - 0.5z^{-1} - 0.5z^{-2}}$$

To find the poles and zeros we first convert the numerator and denominator polynomials to powers of z by multiplying by z^4/z^4 and then find the roots of the two polynomials. We can solve for the roots using Matlab's `roots()` function:

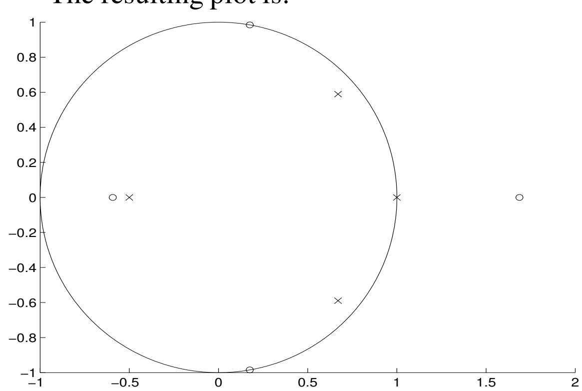
```
>> z1=roots([1 -1.094621 -1])
>> z2=roots([1 -0.350754 1])
>> p1=roots([1 -1.340228 0.796831])
>> p2=roots([1 -0.5 -0.5])
```

The zeros are at: 1.68729, -0.59267, and $0.17538 \pm 0.98450j$ and the poles are at: $0.67011 \pm 0.58973j, 1$ and -0.5.

We can superimpose four plots showing the locations of the poles using Matlab's `plot` function. A format string can be used to specify a scatter plot using $\circ$ and $\times$ symbols. The following commands also generate a unit circle for reference:

```
>> hold on
>> plot(real(z1),imag(z1),'o')
>> plot(real(z2),imag(z2),'o')
>> plot(real(p1),imag(p1),'x')
>> plot(real(p2),imag(p2),'x')
>> t=[0:0.01:2*pi];
>> plot(cos(t),sin(t)) ;
>> print -deps sol4.q3.eps
```

The resulting plot is:



(b) This system is not stable because one of its poles (the one at $1 + 0j$) lies on the unit circle.

Question 4

To obtain a passband centered about $F_s/4$ the angles of the three poles should be centered about $\pi/2$. Because the passband is symmetrical about this frequency, one pole will be at $\pi/2$ and the other two should be arranged with the same angular displacement on either size of $\pi/2$. Since the peaks of the response will occur at the pole angles, we can read the angles from the plot of the desired response (at approximately $\pi/2$ and $\pi/2 \pm 0.2\pi$.

The magnitudes of the poles are adjusted manually to obtain the desired passband response shape. Note that the closer a pole gets to the unit circle the more pronounced the peak of the frequency response at that frequency.

A reasonable approximation to the response was obtained using poles at $0 \pm 0.694j$, $0.431 \pm 0.674j$, and $-0.431 \pm 0.674j$.

A plot of the pole/zero locations, the impulse response and phase/amplitude responses is shown in Figure 2 on the next page.

As in Question 1 we can use the Matlab `conv` function to expand the denominator from factored form to powers of z:

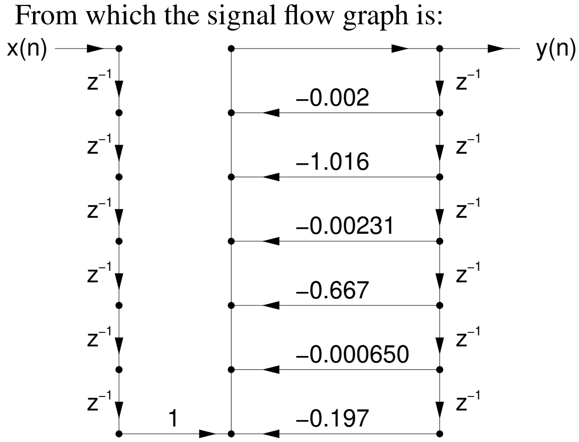
```
>> conv([1],[1  ( 0.4322-0.6732*j)]);
>> conv(ans,[1  ( 0.4322+0.6732*j)]);
>> conv(ans,[1  (-0.4312+0.6739*j)]);
>> conv(ans,[1  (-0.4312-0.6739*j)]);
>> conv(ans,[1  (      0-0.6938*j)]);
>> conv(ans,[1  (      0+0.6938*j)]);
```

and convert the result to a ratio of two polynomials in powers of z^{-1}:

$$H(z) = \frac{z^{-6}}{1z^{-0} + 0.0020z^{-1} + 1.0160z^{-2}} \times \frac{1}{+0.0023z^{-3} + 0.667z^{-4} + 0.0006z^{-5}} \times \frac{1}{+0.1972z^{-6}}$$

This gives us the coefficients b_i and a_i of a direct form implementation: $b = \{0, 0, 0, 0, 0, 1\}$, and $a = \{1, 0.002, 1.016, 0.0023, 0.667, 0.0006, 0.1972\}$.

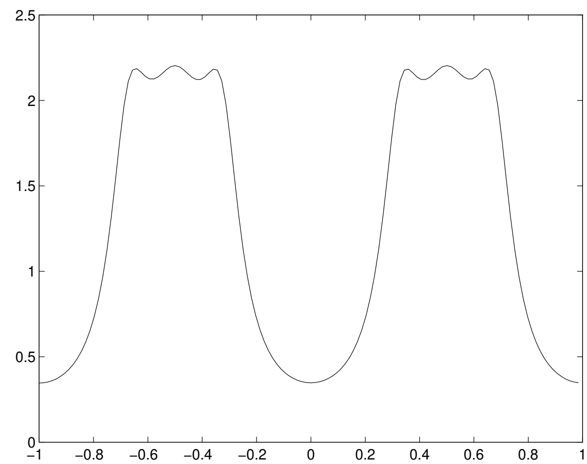
From which the signal flow graph is:



The multiple zero at the origin has no effect on the amplitude response – it only introduces delay – and could be removed. We can check the result by looking at the DFT of the impulse response:

```
>> b=[0 0 0 0 0 1];
>> a=[1 0.002 1.016 0.0023 0.667 0.0006 0.1972];
>> f=[-64:63]/64;
>> h=fft(filter(b,a,[1 zeros(1,127)]));
>> plot(f,abs(h));
>> print -deps sol4.q4b.eps
```

which has the desired amplitude response:



Question 5

From the signal flow graph,

$$w(n) = 0.496w(n-1) - 0.762w(n-2) + 0.1367x(n)$$

or

$$x(n) = \frac{w(n) - 0.496w(n-1) + 0.762w(n-2)}{0.1367}$$

and

$$X(z) = W(z)\frac{1 - 0.496z^{-1} + 0.762z^{-2}}{0.1367}$$

Similarly

$$y(n) = w(n) + 0.676w(n-1) + w(n-2)$$

and

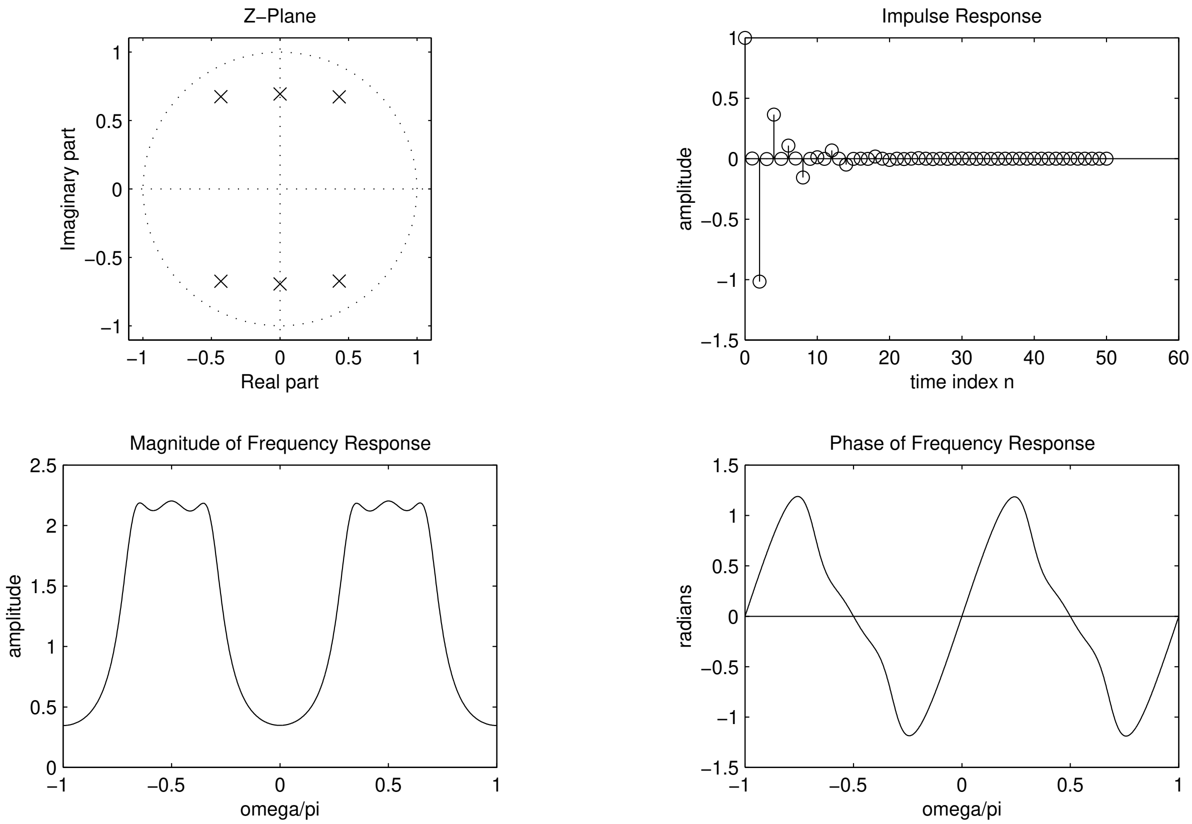


Figure 2: Filter pole/zero locations and responses.

$$Y(z) = W(z)(1 + 0.676z^{-1} + 1z^{-2})$$

The overall transfer function is thus

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{0.1367(1 + 0.676z^{-1} + 1z^{-2})}{1 - 0.496z^{-1} + 0.762z^{-2}}$$