Mid-Term Exam Solutions

Question 1 A sine wave with an amplitude of 3 volts has a power of $\frac{A^2}{2} = \frac{3^2}{2} = 9/2$ V^2. The quantization noise power is given by $q^2/12$. The quantization step size is given by $q = V_{\text{fs}}/(2^B - 1) = 12/255$. The SQNR is thus $\frac{9/2}{12/65025} = 43.9$ dB.

A simpler solution is to realize that the signal being quantized extends over half of the input range and the effective number of quantization bits is 7. Using the formula, the SQNR $\approx 6B + 2 = 42 + 2 = 44$.

Question 2 A 4-th order ($n = 4$) analog Butterworth filter with a 3 dB frequency (f_c) of 2 kHz has an amplitude response given by:

$$H(f) = \frac{1}{\sqrt{1 + (f/f_c)^{2n}}}$$

$$= \frac{1}{\sqrt{1 + (f/2)^8}}$$

This will also be the shape of the amplitude spectrum of the filtered signal because the original signal has a flat (constant) spectrum. When this filtered signal is sampled at a rate of 8 kHz the spectrum at the output of the filter is replicated every 8 kHz. The power [spectral density] of the signal at 2 kHz is:

$$|H(2)|^2 = \frac{1}{|1 + (2/2)^8|} = \frac{1}{2}$$

The aliasing at 2 kHz will be caused by copies of the filtered signal that are shifted to multiples of 8 kHz. The most significant of these will be the component from the signal shifted to 8 kHz. The power [spectral density] of the signal that falls at 2 kHz will be:

$$|H(8-2)|^2 = \frac{1}{|1 + (6/2)^8|} = \frac{1}{6562}$$

The next strongest aliasing component will be:

$$|H(-8-2)|^2 = \frac{1}{|1 + (-10/2)^8|} \approx 10^{-7}$$

which can be ignored compared to the aliasing component from 8 kHz.

The ratio of signal power to aliasing noise in dB is thus: $10\log(6562/2) = 35$ dB.

Question 3

(a) The z-transform of

$$y(n) - 0.8y(n-2) = x(n) + 2x(n-1) + x(n-2)$$

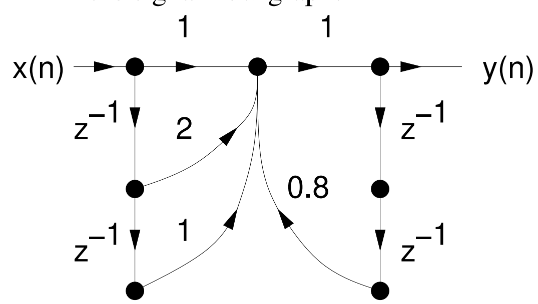
is:

$$Y(z)(1 - 0.8z^{-2}) = X(z)(1 + 2z^{-1} + z^{-2})$$

and the z-domain transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.8z^{-2}}$$

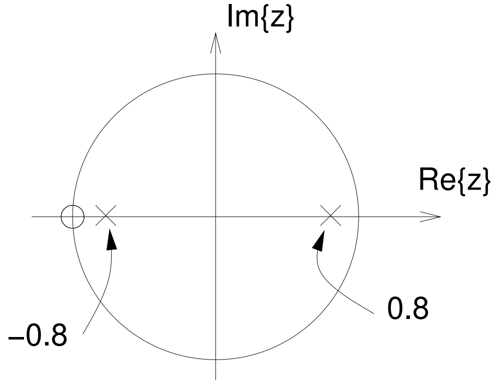
(b) From the difference equation we can draw the signal flow graph:



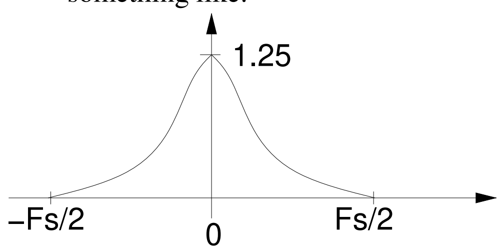
(c) We can convert the transfer function to positive powers of z:

$$H(z) = \frac{z^2 + 2z + 1}{z^2 - 0.8}$$

The zeros are the roots of the numerator, $z^2+2z+1 = (z+1)(z+1)$, which are $z = -1$ and $z = -1$. The poles are the roots of $z^2 - 0.8$ which are $z = \pm\sqrt{0.8}$. The pole and zero positions are shown below:



(d) The poles will cause peaks in the amplitude response of the system at angles $\omega = 0$ and $\omega = \pm\pi$ corresponding to frequencies of 0 and $\pm F_s/2$. The zeros will cause nulls at frequencies $\pm F_s/2$. Since the zeros are on the unit circle the response will be zero at $\pm F_s/2$. The response will look something like:



(e) This system is stable because all of the poles lie inside the unit circle.

Question 4 $X(k)$, the DFT of $x(n)$ is defined by:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

In this case $N = 3$ and we are asked to find $X(k = 5)$. The sum is:

$$X(5) = 1e^{-j2\pi\cdot 5\cdot 0/3}+0e^{-j2\pi\cdot 5\cdot 1/3}+2e^{-j2\pi\cdot 5\cdot 2/3}$$

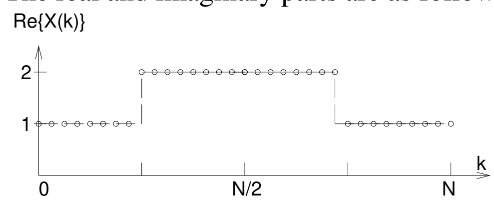
$$= 1 + 0 + 2e^{-j2\pi/3}$$

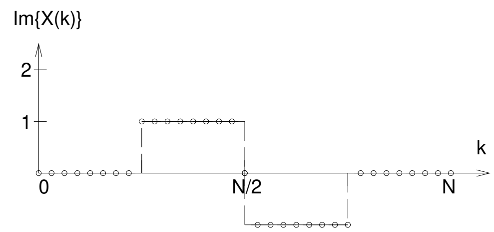
$$= 1 + 2(\cos(-2\pi/3) + j\sin(-2\pi/3))$$

$$= 1 + 2(-1/2 - j\sqrt{3}/2) = -j\sqrt{3}$$

Question 5 Since the signal is real the magnitude of the DFT must be even and the phase odd. This allows us to determine the values for $k = -N/2 \ldots 0$. Since the DFT is periodic with period N the values for $k = -N/2 \ldots 0$ will be the same as for $k = N/2 \ldots N$.

The real and imaginary parts are as follows:





The corresponding magnitude and phase are: