The Inverse z-Transform and Applications

This lecture describes two techniques that can be used to find the inverse z-transform of a rational z-domain transfer function and describes two important applications of the z-transform.
After this lecture you should be able to: (1) obtain the inverse z-transform of a rational z-transform by polynomial division, (2) re-write a rational z-transform as a sum of a partial fractions, and (3) evaluate the stability and frequency response of an FIR or IIR system from its transfer function.
This material is covered in the textbook in sections 3.3 and 3.5.1 through 3.5.7.

The Inverse z-Transform

If the function $X(z)$ is available as a power series in z^{-n} then its inverse z-transform can be found by inspection.

Exercise: If $X(z) = 1 + 2z^{-1} - 1z^{-2} + 3z^{-3}$ what is $x(n)$?

In many cases of interest the function $X(z) = N(z)/D(z)$ is a ratio of two polynomials in z^{-n}. In this case we first have to convert $X(z)$ into a sum of simpler functions whose transforms we can then find more easily.

Polynomial Division

The first method for doing this is to divide $N(z)$ by $D(z)$ using long division. The quotient is a polynomial in powers of z^{-n} whose inverse z-transform can then be found by inspection.

Exercise: Find the impulse response of the system described by the transfer function

$$H(z) = \frac{z^{-1}}{z^{-2} - 2z^{-1} + 1}$$

for values of n from 0 to 3.

The text describes an equivalent recursive numerical method to find the coefficients of z^{-n}.

Partial Fraction Expansion

The second method is to expand $X(z)$ into partial fractions. If the degree of $N(z)$ is greater than $D(z)$ we must first divide the two polynomials first until the orders are equal and then find the partial fraction expansion of the remainder.

The first step is to factor $D(z)$. If the coefficients of $D(z)$ are real, then the factors of $D(z)$ will be of the form $(z-p_k)^m$ (an m-fold real pole at p_k) or $((z-p_k)(z-p_k^*))^m$ (an m-fold complex conjugate pole pair at p_k and p_k^*).

For each factor of $D(z)$ of the form

$$(z - p_k)^m$$

the partial fraction expansion will have terms

$$\frac{D_1}{(z-p_1)} + \frac{D_2}{(z-p_2)^2} + \cdots \frac{D_m}{(z-p_m)^m}$$

For each factor of $D(z)$ of the form

$$((z-p_k)(z-p_k^*))^m$$

the partial fraction expansion will have terms of the form

$$\frac{D_1 z + E_1}{(z-p_1)(z-p_1^*)} + \frac{D_2 z + E_2}{((z-p_2)(z-p_2^*))^2} + \cdots$$

$$\cdots \frac{D_m z + E_m}{((z-p_m)(z-p_m^*))^m}$$

The last step is to solve for the unknown coefficients, the Ds and Es. In principle it's straightforward to solve for them by substitution or other techniques. However, in most practical cases the number of terms will be large and a solution is obtained using a computer (e.g. the Matlab `residuez` function).

Exercise: Find the partial fraction expansion of

$$H(z) = \frac{z}{(z-2)(z-1)}$$

Exercise: How many terms will there be in the partial fraction expansion of

$$H(z) = \frac{(z-1)^5}{(z-2)^3(z^2-1)^2}$$

Pole - Zero Representation

If we factor the numerator and denominator of the transfer function $H(z)$ we can re-write it as:

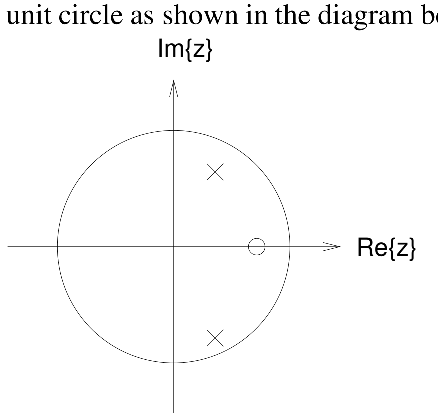
$$H(z) = \frac{(z-z_1)(z-z_2)\cdots}{(z-p_1)(z-p_2)\cdots}$$

where the z_i are the zeroes and the p_i are the poles.

Because we are considering only real coefficients then if any of the roots of the numerator or denominator are complex then they always occur as complex conjugate pairs (e.g. the roots of (z^2+1) are $\pm j$). Thus complex poles and zeroes will always occur in complex conjugate pairs.

The poles and zeroes therefore completely specify the transfer function and are an alternative representation of it. The pole and zero representation is useful for checking stability and as a way to visualize the frequency response of the system.

The convention is to plot the locations of the poles (as "X"s) and zeros (as "O"s) on the z-plane along with the unit circle as shown in the diagram below.



Stability

A necessary condition for a system to be stable is that all of the *poles* of its transfer function must lie inside the unit circle. We can therefore verify that a system is stable by finding the poles of its transfer function and checking that they all have magnitude less than 1.

Exercise: For what values of a is the system described by the transfer function

$$\frac{z}{(z-a)^2}$$

stable?

Exercise: Is a system with a transfer function

$$H(z) = \frac{z^{-1}}{z^{-2}-1}$$

stable?

An alternative method to check for stability, described in section 3.5.7, is to use the system's difference equation to compute its impulse response. If the impulse response decays over time then the system can be considered stable.

Geometric Evaluation of Frequency Response

An approach that helps to visualize the frequency response of the transfer function is to consider a geometric interpretation of the substitution of $e^{j\omega}$ for z in the transfer function.

We've seen that a transfer function with zeroes $z_1 z_2 \ldots$ and zeroes $z_1 z_2 \ldots$ can be written in factored form as

$$H(z) = \frac{(z-z_1)(z-z_2)\cdots}{(z-p_1)(z-p_2)\cdots}$$

Each factor of the form $(z-z_i)$ or $(z-p_i)$ is a vector whose length is the distance between z and the particular pole or zero.

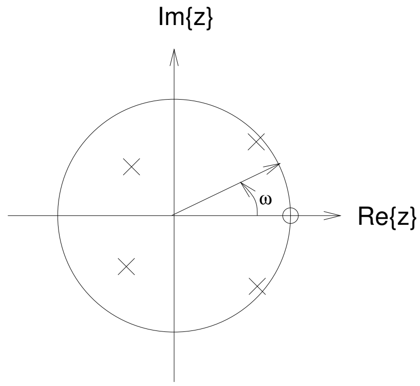
The magnitude of the transfer function for a particular value of z will be the ratio of the product of the lengths in the numerator and the product of the lengths in the denominator:

$$|H(z)| = \frac{|z-z_1||z-z_2|\ldots}{|z-p_1||z-p_2|\ldots}$$

To evaluate the amplitude frequency response of the system we set $z = e^{j\omega}$ and let ω vary over $\pm\pi$ corresponding to frequencies $\pm F_s/2$. This corresponds to evaluating $|H(z)|$ as z moves around the unit circle.

The effect of a pole on the frequency response will depend on its position. If pole i is close to the unit circle the effect will be very pronounced because the distance $|z-p_i|$ will become small as z approaches p_i and so $|H(z)|$ will become large. Conversely, a pole at $z = 0$ will not have any effect on the frequency response because the distance $|z - p_i| = |z| = 1$ will be independent of the angle (frequency) ω.

Exercise: Sketch the amplitude frequency response of the system whose transfer function has the following poles:



In a similar fashion the effect of a zero on the frequency response will depend on its position. As z approaches the location of the zero the magnitude of the transfer function decreases.

Simple filters (consisting of a few poles and zeroes) can often be designed by trial-and-error positioning of the pole and zeros.

Numerical Evaluation of Frequency Response

Although the geometric interpretation of the frequency response is useful, accurate evaluation of the frequency response is usually done by numerically evaluating the transfer function at $z = e^{j\omega}$.

If the values $x(n)$ correspond to samples taken at intervals T_s, then the frequency values of ω will be scaled by $1/T_s$. As ω varies from $-\pi$ to π the "real world" frequency will vary between $-F_s/2$ and $F_s/2$.

Exercise: A system has a sampling frequency of 8 kHz. We would like to design a filter that has high gain at 2 kHz and high attenuation at DC. Approximately where would the poles and zeroes of this filter's transfer function appear?