The z-Transform

In the same way that the Laplace Transform is useful in the analysis of linear continuous-time systems, the z-transform is a useful tool for the analysis of linear discrete-time *systems. In particular, we will use the z-transform to find the frequency response and evaluate the stability of linear discrete-time systems.*
We start by defining the class of systems to which the z-transform can be applied. Then we define the z-transform and some of its properties. Finally we obtain the z-transform of an an important class of LTI discrete-time systems.
After this lecture you should be able to: (1) determine if a given system is linear, time-invariant, and/or causal from a description of its time-domain behaviour, (2) find the z-transform of a finite-duration sequence, (3) apply the time-shift and convolution properties of the z-transform, and (4) write out the z-domain transfer function for an FIR or IIR system from its difference equation.
This material is covered in the textbook in sections 3.1, 3.2, 3.4, and 3.5.8.

LTI Systems

Linearity

A *linear system* is one which obeys the principle of superposition. If $y_1(n)$ is the response of the system to input $x_1(n)$ and $y_2(n)$ is its response to input $x_2(n)$ then the output of a linear system for an input of $ax_1(n) + bx_2(n)$ will be $ay_1(n) + by_2(n)$.

Exercise: Is the system defined by $y(n) = x^2(n)$ linear? What about $y(n) = x(n)x(n-1)$?

All real-world systems are non-linear for some inputs. For example, when the input levels exceed the linear operating range of a circuit. But normally we are only concerned with how the system will behave over a known range of input values.

Time (shift) Invariance

A system is *time-invariant* if delaying an input causes the output to be delayed by the same amount. If $y(n)$ is the output for an input of $x(n)$ then the output to an input $x(n-m)$ is $y(n-m)$.

Again, real-world systems eventually change and so no system is truly time-invariant. But normally we are only concerned with the behaviour of the system during a pre-determined time period (e.g. while the input is present).

Exercise: Is the system $y(n) = nx(n)$ time-invariant? Is it linear?

Causal Systems

A system is *causal* if its output is zero before an input is applied. Practical systems are causal.

Impulse Response

The impulse response (sometimes called the "unit sample" response) of a system is the output of the system for an input of a single non-zero sample with value 1 at $n = 0$ (i.e. $\delta(n)$).

Using the linearity and time invariance properties we can show that the output of an LTI system is the convolution of the input and the impulse ("unit sample") response, $h(n)$, of the system.

Exercise: What is the impulse response of a system whose output is defined by the equation $y(n) = x(n) + x(n-2)$?

Stable Systems

A system is said to be *stable* if its impulse response, $h(n)$ is such that ($\sum |h(n)| < \infty$).

Exercise: Is a system with an impulse response of $h(n) = -1^n$ stable? What about $h(n) = 0.9^n$? Hint: $\sum_{n=-\infty}^{\infty} x^n = 1/(1-x)$ for $|x| < 1$.

The z-Transform

The DFT of the sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable.

Exercise: What is the z-transform of {1,2,1} ?

Note that since z-transform is a complex function of a complex variable (z), it is defined over a plane (the "z-plane") and plotting the z-transform would require a plot of the magnitude surface and a plot of the phase surface.

Properties of the z-Transform

The z-transform has a number of useful properties. We will cover only a few of them here.

If $X(z)$ is the z-transform of $x(n)$, then we can show that the following properties apply:

Delay

Delaying $x(n)$ by m samples is equivalent to multiplying the z-transform of X by z^{-m}:

$$x(n-m) \rightleftharpoons z^{-m}X(z)$$

Exercise: What is the z-transform of {0,0,1,2,1} ?

Linearity

The z-transform is a linear operator, that is if x_1 and x_2 are two sequences then

$$ax_1(n) + bx_2(n) \rightleftharpoons aX_1(z) + bX_2(z)$$

Convolution

We have already stated that for a LTI system the response to an input $x(n)$ is given by the convolution of the impulse response and the input:

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

and not surprisingly the z-transform of the output, $Y(z)$, is given by the product of the the z-transform of the input, $X(z)$, and the z-transform of the impulse response, $H(z)$:

$$Y(z) = H(z)X(z)$$

$H(z)$ is known as the system *transfer function*.

Exercise: A system has a transfer function $H(z)$ of z^{-2}. A sequence whose z-transform is $1+2z^{-1}+z^{-2}$ is applied to the input. What is the z-transform of the output? What is the output sequence?

Poles and Zeroes

The values of z for which the z-transform is zero are known, appropriately enough, as *zeroes*. The values of z for which the z-transform is infinite are known as *poles*.

Exercise: Find the poles and zeroes of the following function: $H(z) = \frac{(z-2)(z+1)}{z}$.

We will consider the significance of poles and zeroes and their use in predicting the stability of LTI systems in a subsequent lecture.

The z-transform and the FT

The variable z in the definition of the z-transform is a complex number. We can re-write z in magnitude-phase form as:

$$z = re^{j\omega}$$

If we substitute $re^{j\omega}$ into the z-transform and fix $r = 1$ we see that the DFT corresponds to the z-transform evaluated for $|z| = 1$.

Thus the values of the DFT correspond to the values of the z-transform for values of z located around the unit circle at angles $\omega = 2\pi k/N$, for $k = 0, 1, ...N-1$.

Exercise: What is the DFT of {1,2,1} for $k = 2$?

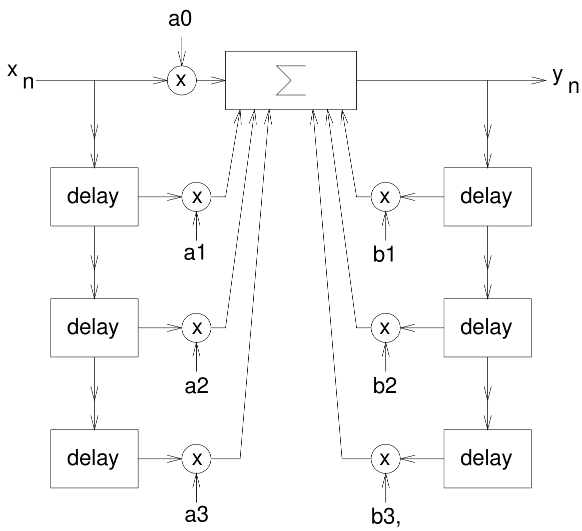
FIR and IIR Systems

An important class of DSP systems can be described by the *difference equation*:

$$y(n) = \sum_{k=0}^{N} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k)$$

where $x(n)$ is the input, $y(n)$ is the output and a_k's and b_k's are constants that define the behaviour of the system.

A diagram of a system with $N = 3$ and $M = 3$ is shown below:



The output of this system is a function of the current input, the N previous inputs and the M previous outputs.

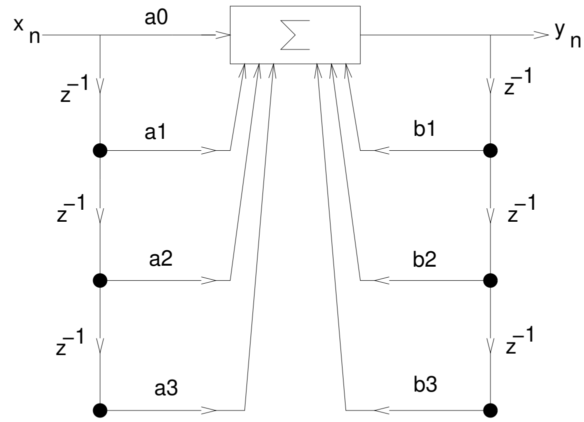
Systems of this type for which the b_k's are zero (i.e. the output does not depend on the previous outputs) are known as finite impulse response (FIR) systems because the impulse response has a finite duration. Systems where some b_k's are non-zero are known as IIR systems because the effect of an input impulse can last infinitely long.

Exercise: If the b_k's are zero (an an FIR system) what is the impulse response[1]? If the only non-zero coefficient is $b_1 = 1$, what will be the impulse response?

Using the time shift property of the z-transform we can write the z-transform of the above equation as:

$$Y(z) = \sum_{k=0}^{N} a_k z^{-k} X(z) - \sum_{k=1}^{M} b_k z^{-k} Y(z)$$

If we use the symbol z^{-1} to indicate a unit delay on a branch instead of a branch gain, we can draw the difference equation for the system as a signal flow graph:



With some algebraic manipulation we can also find the z-domain transfer function of the system:

$$Y(z)\left[1 + \sum_{k=1}^{M} b_k z^{-k}\right] = X(z) \sum_{k=0}^{N} a_k z^{-k}$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{k=1}^{M} b_k z^{-k}}$$

In next lecture we will study the behaviour of this system by finding the inverse z-transform (i.e. the impulse response) and the locations of the poles and zeroes of $H(z)$.

[1] assume the initial outputs of the delay elements are zero