The Discrete Fourier Transform

The DFT is the discrete-time version of the Fourier Transform (FT). This lecture explains how the DFT and inverse DFT are computed, how they are related to the FT and several of their properties.
After this lecture you should be able to compute the DFT and inverse DFT of a given sequence by hand and be able to use the properties of the DFT to verify the correctness of various DFTs and their inverses.
This material is covered in Sections 1.3.4, 2.2 and 2.3 of the text. You should already be familiar with Fourier Series and the Fourier Transform. These are reviewed in Section 2.1.

The DFT

Given N samples $x(n)$, $n = 0, 1, ..., N-1$, the DFT, $X(k)$ of $x(n)$ is:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

or more compactly,

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

where

$$W_N = e^{-j2\pi/N}$$

Exercise: Find the DFT of the 4-point sequence $x(n) = 1, 0, 1, 0$ at $k = 0$ and $k = 2$.

If $x(n)$ represents samples taken every T seconds then the values of $X(k)$ will be at frequencies spaced at multiples of $\Omega = 2\pi/(NT)$ radians/second[1].

The Inverse DFT

We can convert from $X(k)$ back to $x(n)$ using the inverse DFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

Note that W_N is periodic with period N, and thus both the DFT, $X(k)$, and its inverse, $x(n)$, are periodic with period N.

[1] *not* Hz as in the text.

The DFT and the FT

The Fourier Transform is normally used to find the (continuous) frequency spectrum of an aperiodic signal $x(t)$. The FT is defined as:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

If we replace $x(t)$ by a sum of delta functions having values $x(t = nT)$ and compute the FT at frequencies $\omega = k\Omega$ we see that the FT reduces to the expression for the DFT. The DFT can thus be used to compute the FT at frequencies $k\Omega$ of a sequence of N impulses $x(nT)\delta(nT)$.

Recall that both $X(k)$ and its inverse, $x(n)$, are periodic. This implies that the N samples represent one period of a periodic signal. This is an extremely important point and ignoring it will cause much confusion:

> The DFT *always* computes the FT of a *periodic* signal. The N values supplied to the DFT represent one period of this signal.

When using the DFT to compute an approximation to the FT of an aperiodic function it is usually necessary to modify the original signal by windowing and/or zero-padding. These techniques will be discussed later.

Properties of the DFT

Many of these properties have counterparts that apply to Fourier Series and Fourier Transforms.

Periodicity

Because of the periodicity of W_N, both the DFT and its inverse are periodic:

$$x(n) = x(n+N)$$

and

$$X(k) = X(k+N).$$

Linearity

The DFT and inverse DFT are linear operators:

$$ax_1(n) + bx_2(n) \rightleftharpoons aX_1(k) + bX_2(k)$$

Exercise: What is the DFT of $\{2,0,2,0\}$?

Time Shift

A delay of m samples in the time domain corresponds to multiplying the transform by W_N^{km}.

$$x(n-m) \rightleftharpoons X(k)W_N^{km}$$

Since the sequence is periodic the delayed sequence still consists of the original N values which are shifted in a circular fashion (e.g. $x(n-1) = \{x_3, x_0, x_1, x_2\}$).

Exercise: What is the DFT of $\{0,1,0,1\}$?

Circular Convolution

Convolving two time sequences is equivalent to multiplying their DFTs. If:

$$x_1(n) \rightleftharpoons X_1(k) \text{ and } x_2(n) \rightleftharpoons X_2(k)$$

then

$$x_1(n) \circledast x_2(n) \rightleftharpoons X_1(k)X_2(k).$$

Note that since the time sequences are assumed to be periodic with period N, the convolution is "circular."

Modulation

Conversely, multiplying two sequences in time is equivalent to convolving their DFTs:

$$x_1(n)x_2(n) \rightleftharpoons X_1(k) \circledast X_2(k).$$

Transforms of Real Sequences

As with Fourier Series, the magnitude (or real part) of the DFT of a real sequence is even and the phase (or imaginary part) is odd.

Transforms of Even or Odd Sequences

The transform of an even sequence is real and the transform of an odd sequence is imaginary.

Preservation of "Degrees of Freedom"

If there are N "degrees of freedom" (samples that are independent of others) in $x(n)$, there will be N "degrees of freedom" in $X(k)$. This statement follows from the two previous properties. For example, forcing $x(n)$ to be real and even means that $X(k)$ will have only $N/4$ "independent" values since the DFT will be even and real.

Parseval's Theorem

The total power in the time and frequency domains is equal:

$$\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N} \sum_{n=0}^{N-1} |X(k)|^2$$