Error Detection and Correction

Exercise 1: Compute the modulo-4 checksum, C, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the sum at the receiver if there were no errors? Determine the sum if the received frame was: 3, 1, 1, C? 3, 1, 2, 0, C? 1, 2, 3, C?

$3+1+2 \bmod 4 = 6 \bmod 4 = 2$

$C = 2$

would transmit $4-2 = 2$ as checksum:

3, 1, 2, 2

at receiver:

$3+1+2+2 \bmod 4 = 8 \bmod 4 = 0$. (=0 ok)

for 3, 1, 1, 2: $3+1+1+2 \bmod 4 = 7 \bmod 4 = 3$ ($\neq 0$ error)

for 3, 1, 2, 0, 2: $3+1+2+0+2 \bmod 4 = 8 \bmod 4 = 0$ (=0 → no error)

for 1, 2, 3, C: $1+2+3+2 \bmod 4 = 8 \bmod 4 = 0$ (=0 → no error).

∴ checksum did not detect additional zeros or different order.

Exercise 2: What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number?

mod 2 sum = remainder after dividing by 2

1+0+1 mod 2 = 2 mod = 0

for even number of 1's mod 2 sum is 0.

Exercise 3: A (5,3) code computes the two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where d_i is the i'th data bit. What codeword is transmitted when the data bits are $(d_0, d_1, d_2) = (0, 0, 1)$? How many different codewords are there in the code? What are the first four codewords? In general, how many codewords are there for an (n, k) code?

$p_0 = d_0 \oplus d_1 = 0 \oplus 0 = 0$

$p_1 = d_1 \oplus d_2 = 0 + 1 = 1$

transmit d_0, d_1, d_2, p_0, p_1 = 0 0 1 0 1

there are $2^3 = 8$ possible codewords

first four codewords:

000 00
001 01
010 11
011 10

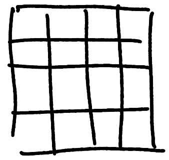
an (n,k) code has 2^k valid codewords.

Exercise 4: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

$$\begin{array}{r} 11100 \\ \oplus\ 11011 \\ \hline \underbrace{00111}_{3\ 1\text{'s}\ :\ 3\ \text{bits are different}} \end{array}$$

	0111	1011	1101	1110
0111	0	2	2	2
1011		0	2	2
1101			0	2
1110				0

Exercise 5: What is the code rate of a code with 4 codewords each of which is 4 bits long? *Hint: If a code has 2^k codewords, what is k?.*



$N = 2^k$

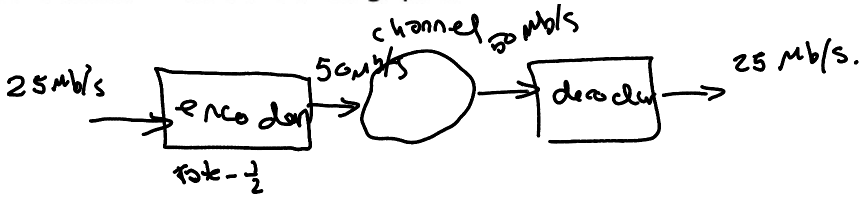
$k = \log_2(N)$.

in this case $N=4$ $k = \log_2(4) = 2$

n in this case $= 4$

code rate $= \frac{k}{n} = \frac{2}{4} = \frac{1}{2}$.

Exercise 6: The data rate over the channel is 50 Mb/s; a rate 1/2 code is used. What is the throughput?



$k \longrightarrow n$

$1 \longrightarrow 2$

Throughput = 25 Mb/S

Exercise 7: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

received → 110   110

transmitted → 101   or   010

d: 2   d = 1 ← decoder chooses codeword 010

- first bit most likely in error.
- yes, possible but not as likely.

Exercise 8: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?

010
101

$D_{min} = 3$

can detect $d-1 = 3-1 = 2$ bit errors

can correct $\lfloor \frac{3-1}{2} \rfloor = \lfloor 1 \rfloor = 1$ bit errors.

$n = 3$
$k = \log_2(\# \text{ codewords}) = \log_2(2) = 1$
$k/n = 1/3$

Exercise 9: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Joules
Watts = J/s
Seconds

bit energy = Watts · second = Joules
= transmit power · bit duration.

Exercise 10: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is E_b in each case? What is the coding gain?

data rate before/after FEC

Channel rate	1 Mb/s	2 Mb/s.
before/after FEC	1 Mb/s	1 Mb/s (= 2 · ½).
Eb	$1 \cdot 1\times10^{-6} = 1\mu J/\text{bit}$	$0.500 \cdot 1\times10^{-6} = 0.5\mu J/\text{bit}$

coding gain =

$$\text{coding gain} = \frac{E_b\ (\text{without coding})}{E_b\ (\text{with coding})}$$

$$\frac{1}{0.5} = 2 \quad (3dB).$$

Exercise 11: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?

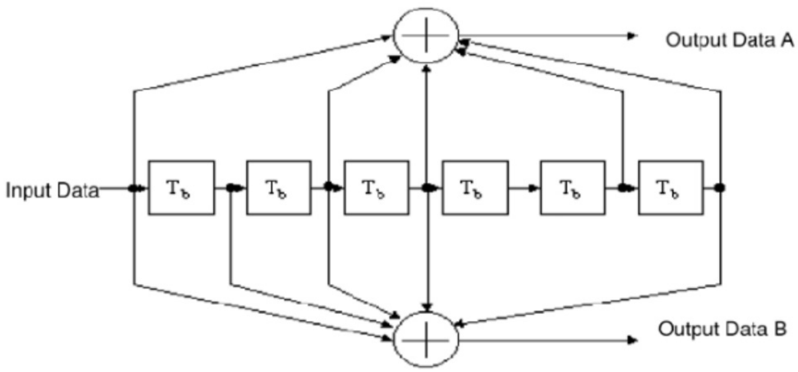


Figure 18-8—Convolutional encoder (k = 7)

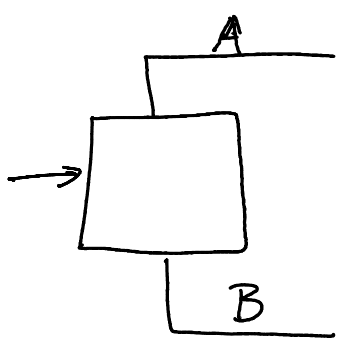
$k = 1$

$n = 2$

$K = 7$

$\text{code rate} = \frac{k}{n} = \frac{1}{\;}$

Exercise 12: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?



1	2	3
A	A	~~A~~
~~B~~	B	B

$$\text{rate} = \frac{\text{input bits}}{\text{coded bits}} = \frac{3}{4}$$

Exercise 13: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.

	ABC	DAB	CDA	BCD
ABC		3	3	3
DAB			3	3
CDA				3
BCD				

$$d_{min} = 3$$

can detect $= d_{min} - 1 = 3 - 1 = 2$

correct $= \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = \underline{1}$

ADA : not valid codeword → error introduced by channel

	ABC	DAB	CDA	BCD
ADA	2	3	1	3
ADC?	1	3	2	3

decoder should choose CDA (distance 1) as most likely codeword transmitted.

ADA → CDA

corrected A to C

e.g. if A = 00 then corrected 1 bit
C = 10

if A = 00 then corrected 2 bits
C = 11

up to 2 bit errors could have been corrected.

Exercise 14: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

errors

error burst

1, 5, 9, 13, 2, 6, 10, ~~14~~, ~~3~~, ~~7~~, 11, 15, 4, 8, 12, 16

8 9 10

1	2	~~3~~	4
5	6	~~7~~	8
9	10	11	12
13	~~14~~	15	16

1, 2, ~~3~~, 4, 5, 6, ~~7~~, 8, 9, 10, 11, 12, 13, ~~14~~, 15, 16

with interleaving could correct all the errors.

w/o interleaving. could not correct all of the errors (could not correct 2 errors in the 3rd codeword).

Exercise 15: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

interleave <u>bytes</u> to concentrate errors into as few 8-bit RS symbols as possible.