# Lecture 13 - Error Detection and Correction

**Exercise 1:** Compute the modulo-4 checksum, $C$, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the checksum at the receiver if there were no errors? Determine the checksum if the received frame was: 3, 1, 1, $C$? 3, 1, 2, 0, $C$? 1, 2, 3, $C$?

$$3 + 1 + 2 = 6 \qquad 6 \bmod 4 = 2$$

additive complement =
$$C = 4 - 2 = 2$$

If transmit $\quad 3 + 1 + 2 + 2 = 8 \qquad 8 \bmod 4 = 0$

if send $\quad 3 + 1 + 1 + 2 = 7 \qquad 7 \bmod 4 = 3 \rightarrow$ error

$$3 + 1 + 2 + 0 + 2 = 8 \qquad 8 \bmod 4 = 0 \rightarrow \text{no error}$$
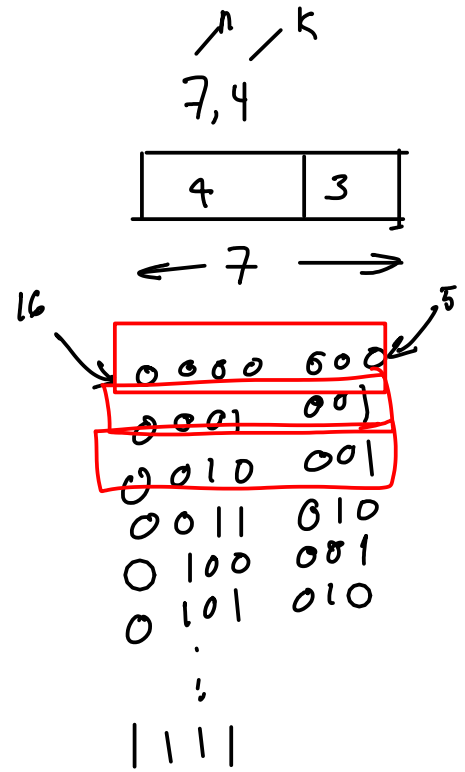
$$1 + 2 + 3 + 2 = 8 \qquad \rightarrow \text{no error}$$

**Exercise 2:** What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number? $\longrightarrow$ 0

$$1 + 0 + 1 = 2 \qquad 2 \bmod 2 = 0$$

**Exercise 3**: How many possible valid code words are there for an $(n, k)$ code? How many possible parity bit patterns are possible for each code word?

$n/k$

$7, 4$

| 4 | 3 |
|---|---|

$\leftarrow 7 \rightarrow$

$- 2^k$ possible valid codewords

$-$ only one possible parity pattern for a given codeword.

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$0\ 0\ 0\ 0 \quad 0\ 0\ 0$
$0\ 0\ 0\ 1 \quad 0\ 0\ 1$
$0\ 0\ 1\ 0 \quad 0\ 0\ 1$
$0\ 0\ 1\ 1 \quad 0\ 1\ 0$
$0\ 1\ 0\ 0 \quad 0\ 0\ 1$
$0\ 1\ 0\ 1 \quad 0\ 1\ 0$
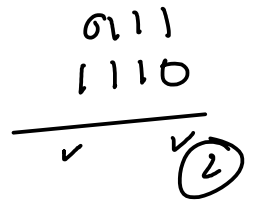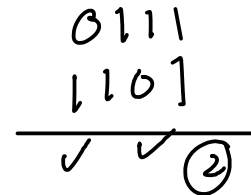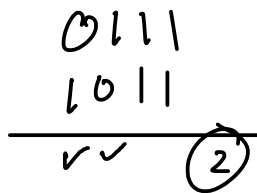$\vdots$
$1\ 1\ 1\ 1$

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**Exercise 4**: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

$1\ 1\ 1\ 0\ 0$

$\oplus\ 1\ 1\ 0\ 1\ 1$

$\overline{\text{sum}(0, 0, 1, 1, 1) = 3}$

$D_{min}$

$4 \times 3$ pairs

|      | 0111 | 1011 | 1101 | 1110 |
|------|------|------|------|------|
| 0111 | 0    | 2    | 2    | 2    |
| 1011 |      | 0    | 2    | 2    |
| 1101 |      |      | 0    | 2    |
| 1110 |      |      |      | 0    |

$0111$
$1011$
$\overline{\quad\checkmark\checkmark\quad} \;\textcircled{2}$

$0111$
$1101$
$\overline{\checkmark\;\;\checkmark\quad}\;\textcircled{2}$

$0111$
$1110$
$\overline{\;\checkmark\quad\quad\checkmark\;}\;\textcircled{2}$

**Exercise 5:** A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

| valid | received | distance |
|-------|----------|----------|
| 101 | 110 | 2 |
| 010 | 110 | 1 |

$\leftarrow$ most likely transmitted

choose this one

**Exercise 6:** What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are $n$, $k$, and the code rate $(k/n)$?

101
010

$d_{min} = 3$

101 | 3

010
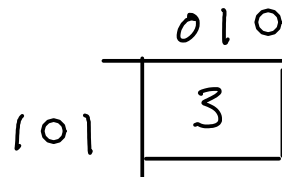
guaranteed to detect: $d-1 = 2$
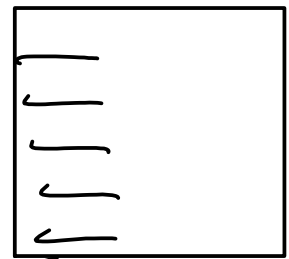
correct: $\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = \lfloor 1 \rfloor = 1$

$n = 3$

$k = 1$ (2 choices $\Rightarrow$ 1 bit).

$2^4 = 16$

$\log_2 16 = 4$

4 bits $\Rightarrow$

code rate $= \frac{k}{n} = \frac{1}{3}$

**Exercise 8:** A system needs to operate at an error rate of $10^{-3}$. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is $E_b$ in each case? What is the coding gain?

*channel bit or error rate*

| | W/o FEC | w/ FEC | | |
|---|---|---|---|---|
| ch. bit rate | $1 \times 10^6$ | $2 \times 10^6$ | b/s | |
| inf bit rate | $1 \times 10^6$ | $1 \times 10^6$ | b/s | |
| Tx Power | 1 | 0.5 | W | |
| $E_b = P \cdot T_b$ | 1 | 0.5 | $\times 10^{-6}$ J | |

*FEC* — *data or information bit & error rates*

$$\text{coding gain} = \frac{E_b (w/o)}{E_b (w)} = \frac{1}{0.5} = 2 = 3 \text{ dB}$$

**Exercise 8:** What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

- Energy: Joules       (mA·H @V     W·H)       J  (W·s)

  — Power: Watts       W

  — Bit Period: Seconds       S

  — $E_b$ = Power · Bit Period.

**Exercise 10:** Assuming one bit at a time is input into the en-
coder in the diagram above, what are $k$, $n$, $K$ and the code rate?

$$k = 1$$

$$n = 2$$

$$\text{code rate} = \frac{k}{n} = \frac{1}{2}$$

$$K = 7 \quad (6 + \text{input})$$

**Exercise 11:** Consider the encoder above. If the only the bits
corresponding to the outputs A, A and B, and B are transmitted
corresponding to every three input bits, what is the code rate
of this punctured code?



$$\text{new code rate} = \frac{\# \text{ data bits}}{\# \text{ coded bits}} = \frac{3}{4}$$

**Exercise 11:** Consider the encoder above. If the only the bits
corresponding to the outputs A, A and B, and B are transmitted
corresponding to every three input bits, what is the code rate
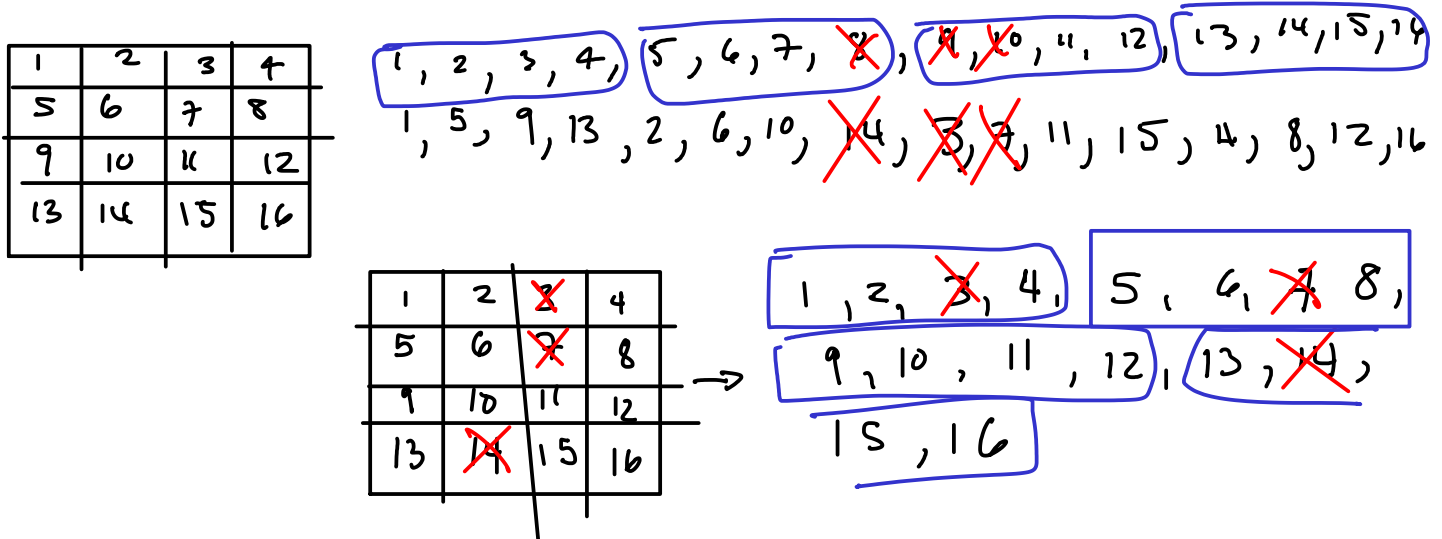of this punctured code?
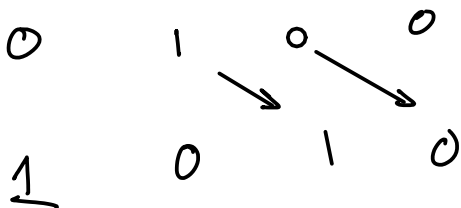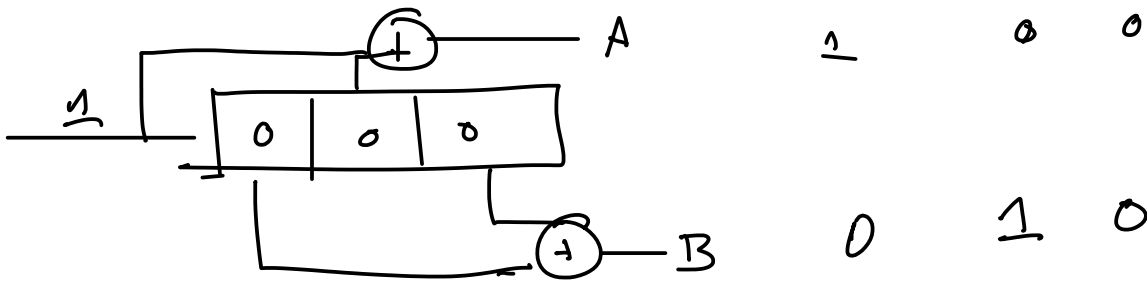
**Exercise 12:** Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

| 1  | 2  | 3  | 4  |
|----|----|----|----|
| 5  | 6  | 7  | 8  |
| 9  | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

(1, 2, 3, 4) (5, 6, 7, 8̶) (9̶, 1̶0̶, 11, 12) (13, 14, 15, 16)

1, 5, 9, 13, 2, 6, 10, 1̶4̶, 3̶, 7̶, 11, 15, 4, 8, 12, 16

| 1  | 2  | 8̶ | 4  |
|----|----|----|----|
| 5  | 6  | 9̶ | 8  |
| 9  | 10 | 11 | 12 |
| 13 | 1̶0̶ | 15 | 16 |

→

(1, 2, 3̶, 4) (5, 6, 7̶, 8)

(9, 10, 11, 12) (13, 1̶4̶)

15, 16

**Exercise 13:** If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

GF(256)

byte interleaving.