Error Detection and Correction

Exercise 1: Compute the modulo-4 checksum, C, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the sum at the receiver if there were no errors? Determine the sum if the received frame was: 3, 1, 1, C? 3, 1, 2, 0, C? 1, 2, 3, C?

$3 + 1 + 2 = 6$

$6 \bmod 4 = \underline{2 = C}$ $\qquad 2+2 \bmod 4 = 0$

addite complent is 2 because

3, 1, 2, 2 $\qquad$ sum mod 4 = 0 ✓

If receive:

3, 1, 1, 2 = 7 $\qquad$ 7 mod 4 = 3 ⇒ error

3, 1, 2, 0, 2 = 8 $\qquad$ 8 mod 4 = 0 ⇒ added 0 not detected

1, 2, 3, 2 = 8 $\qquad$ 8 mod = 0 $\qquad$ word swap not detected.

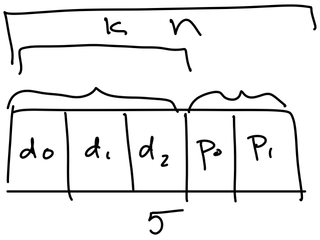
Exercise 2: What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number?

modulo-2 sum: remainder after dividing sum by 2

1+0+1 = 2 $\qquad$ 2 mod 2 = 0.

even number of 1's: 2, 4, 6, ... ⇒ modulo 2 sum is 0.

Exercise 3: A (5,3) code computes the two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where d_i is the i'th data bit. What codeword is transmitted when the data bits are $(d_0, d_1, d_2) = (0, 0, 1)$? How many different codewords are there in the code? What are the first four codewords? In general, how many codewords are there for an (n, k) code?



0 0 1

$p_0 = 0 \oplus 0 = 0$

$p_1 = 0 \oplus 1 = 1$

0 0 1 0 1

There are only 8 valid codewords!

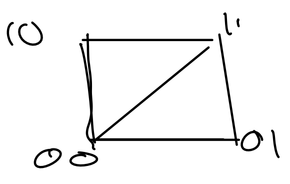
There are 32 possible 5-bit values

			p_0	p_1
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	0

first 4 codewords.

In general there are 2^k valid codewords

but 2^n possible received n-bit values.



Exercise 4: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

11100
⊕ 11011

$0+0+1+1+1 = 3.$

	0111	1011	1101	1110
0111	0	2	2	2
1011	2	0	~~3~~ 2	2
1101	2	2	0	2
1110	2	2	2	0

minimum distance is 2.

0111
1011
1100

0111
1101
1010

0111
1110
1001

1011
1101
0110

1011
1110
0101

1101
1110
0011

Exercise 5: What is the code rate of a code with 4 codewords each of which is 4 bits long? *Hint: If a code has* 2^k *codewords, what is* k?.

$2^k=4$

0	0	1	0
0	1	1	1
1	0	1	0
1	1	0	0

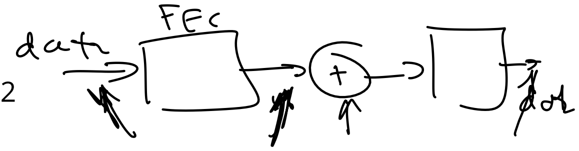
n

$2^k = 4$ $\quad k = \log_2 4 = 2.$

code rate $= \frac{k}{n} = \frac{2}{4} = \frac{1}{2}$

bit rate

Exercise 6: The data rate over the channel is 50 Mb/s; a rate 1/2 code is used. What is the throughput?



throughput $= \frac{1}{2} \cdot 50$ Mb/s. $= 25$ Mb/s

(25 Mb/s is data
25 Mb/s is parity).

$n,k = (3,1)$ $R = \frac{k}{n} = \frac{1}{3}$

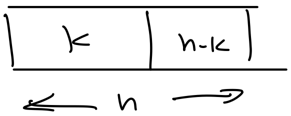
Exercise 7: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

	110 ← received
valid 101	2
010	1 ← decide on this codeword.

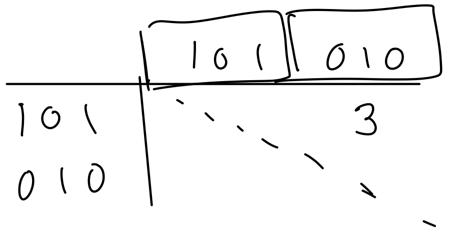
$$\begin{array}{r} 110 \\ 101 \\ \hline 011 \end{array} \qquad \begin{array}{r} 110 \\ 010 \\ \hline 100 \end{array}$$

first bit most likely in error.

- most likely first bit $(1 \Rightarrow 0)$.
- yes its possible 101 was transmitted & 2 errors were introduced. (2 last bits).



Exercise 8: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?



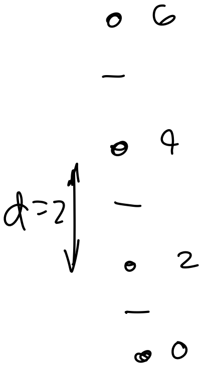
$d = 3$

can detect $3-1 = 2$ errors

can correct:

$$\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor$$

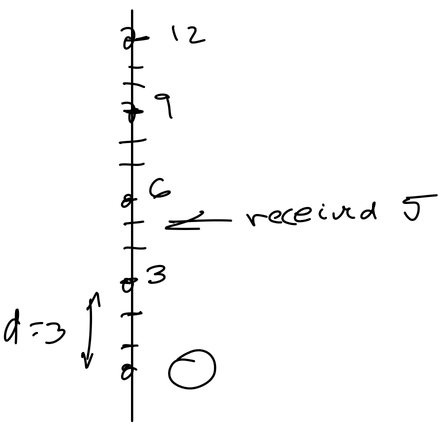
$\lfloor 1 \rfloor = 1$ error.



can detect
1 error

$$\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor$$

$= 0$



$$\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor$$

$\lfloor 1 \rfloor = 1$

$K = 1 = \log_2(2^K)$ $(2^K = 2)$

$n = 3$ bits

$\frac{k}{n} = \frac{1}{3}$

Exercise 9: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Energy : J(oules)

Power : W (J/s)

Time : s.

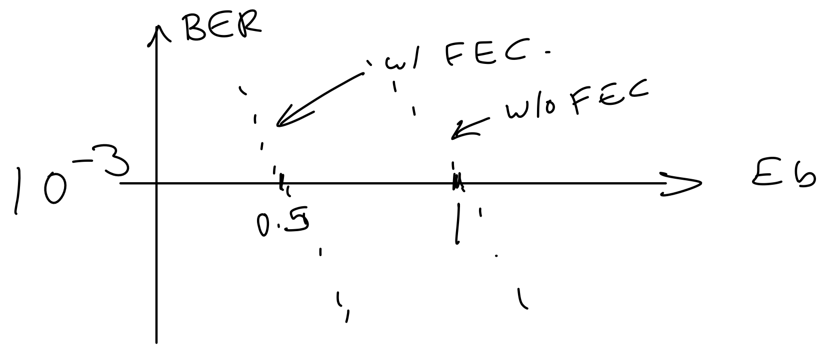
P = power

T_b = time for 1 bit.

$(J)\ E_b = P\ (J/s) \cdot T_b\ (s)$.

Exercise 10: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the ~~information~~ rate in each case? What is E_b in each case? What is the coding gain?

	w/o FEC	with FEC
channel bit rate	1 Mb/s	2 Mb/s.
~~information~~/ usable bit rate	1 Mb/s	1 Mb/s
Eb = power · usable-bit duration	1W · 1μs/bit 1 μJ/bit	0.5W · 1μs/bit = 0.5μJ/bit



$$\text{coding gain} = \frac{Eb\,(w/o\,FEC)}{E\,(w\ FEC)}$$

$$= \frac{1}{0.5} = 2$$

$$= 3\text{ dB}.$$

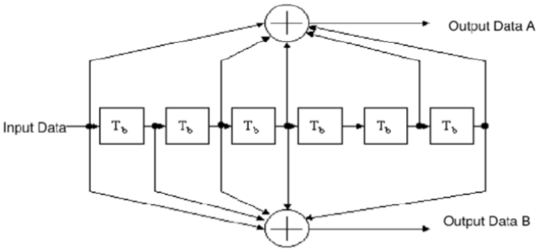
Exercise 11: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?

k = # input bits = 1

n = # o/p bits = 2

K = "constraint length" (memory) = 7

code rate = $\frac{k}{n} = \frac{1}{2}$.



Exercise 12: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

A_0 A_1 $\cancel{B_2}$

$\cancel{B_0}$ B_1 B_2

code rate = ? $\frac{3 \text{ input bits}}{4 \text{ output bits}}$

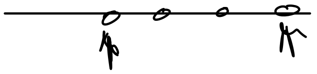
Exercise 13: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.



	ABC	DAB	CDA	BCD
ABC		3	3	3
DAB			3	3
CDA				3
BCD				

$d_{min} = 3$

can detect $d_{min} - 1$ errors
3 - 1 = 2 error

can correct $\left\lfloor \frac{d_{min} - 1}{2} \right\rfloor$ = 1 error.

if receive ADA then error was made (not a valid codeword).

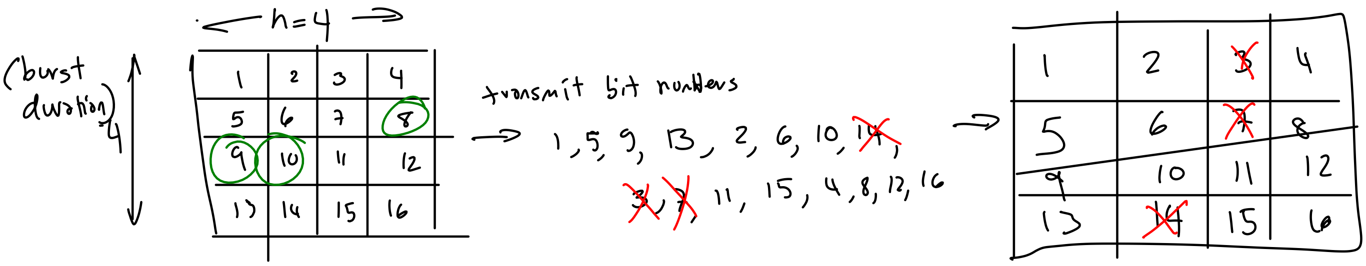
	ADA
ABC	2
DAB	3
CDA	1 → choose this as most likely transmitted
BCD	3

if correct 1 symbol could have corrected 1 or 2 bit errors.

	AAA
ABC	2
DAB	2
CDA	2
BCD	3

an error detected choose at random from the first 3.

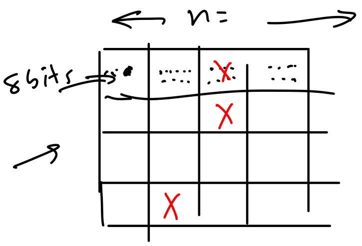
Exercise 14: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?



with interleaving can correct all errors

otherwise can't correct third c/w.

Exercise 15: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?



for a R.S code w/ symbols from GF(256) the code can correct any number of errors within one symbol (8 bits).