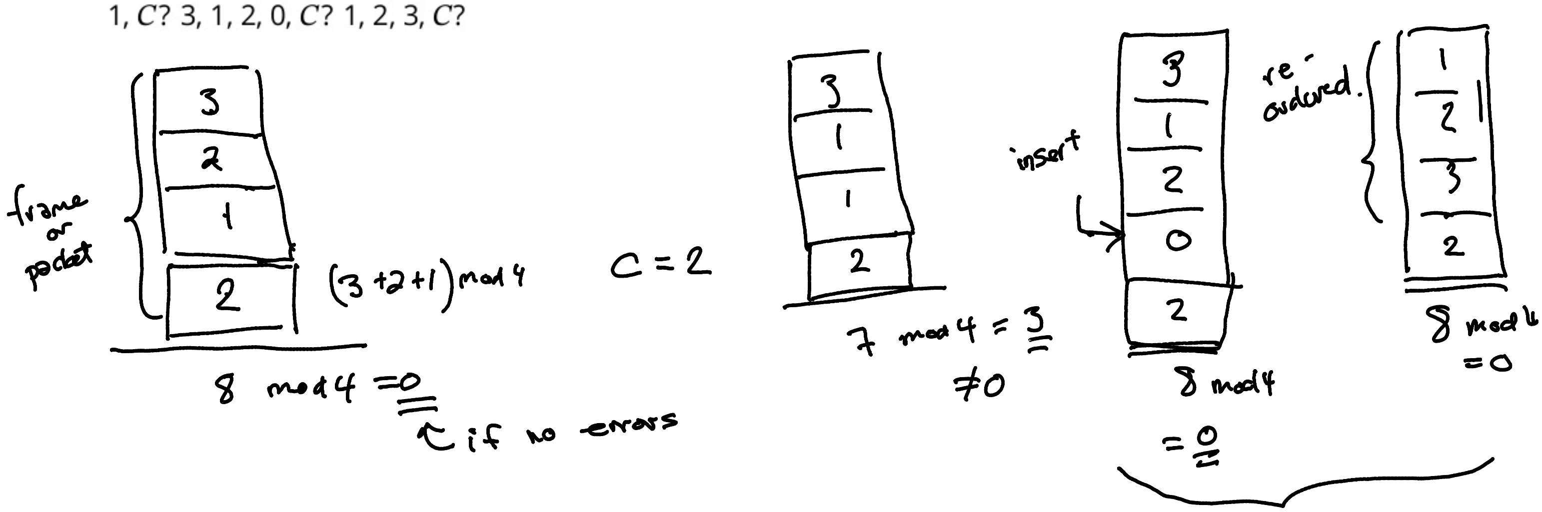
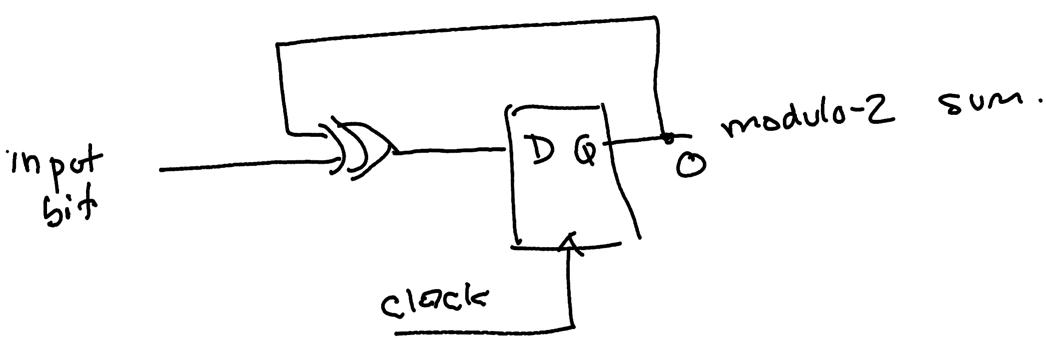
Error Detection and Correction

Exercise 1: Compute the modulo-4 checksum, C, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the sum at the receiver if there were no errors? Determine the sum if the received frame was: 3, 1, 1, C? 3, 1, 2, 0, C? 1, 2, 3, C?

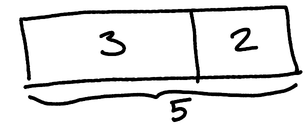


Exercise 2: What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number?

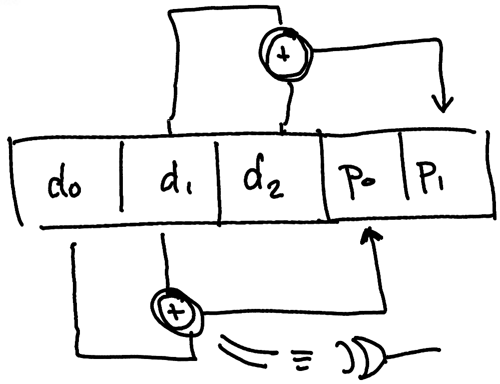
- modulo-2 sum is remainder after dividing the sum by 2
- 1+0+1 = 2 mod 2 = 0
  - = 10_2 ← l.s. bit
- if it's even the modulo-2 sum = 0 & l.s. bit = 0

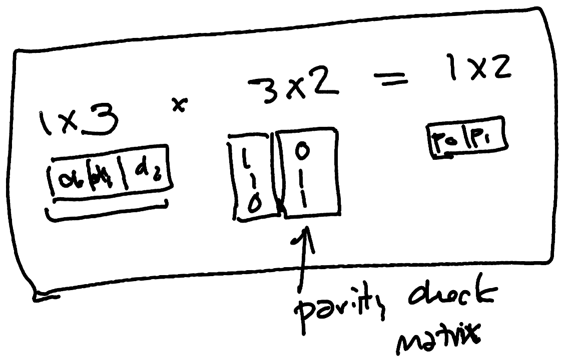


n=5 k=3



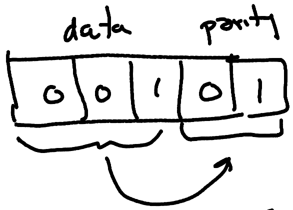
Exercise 3: A (5,3) code computes the two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where d_i is the i'th data bit. What codeword is transmitted when the data bits are $(d_0, d_1, d_2) = (0,0,1)$? How many different codewords are there in the code? What are the first four codewords? In general, how many codewords are there for an (n,k) code?





$d_0 = 0,\ d_1 = 0,\ d_2 = 1 \Rightarrow p_0 = d_0 \oplus d_1 = 0 \oplus 0 = 0$

$p_1 = d_1 \oplus d_2 = 0 \oplus 1 = 1$



k data bits $\Rightarrow 2^k$

$2^3 = 8$

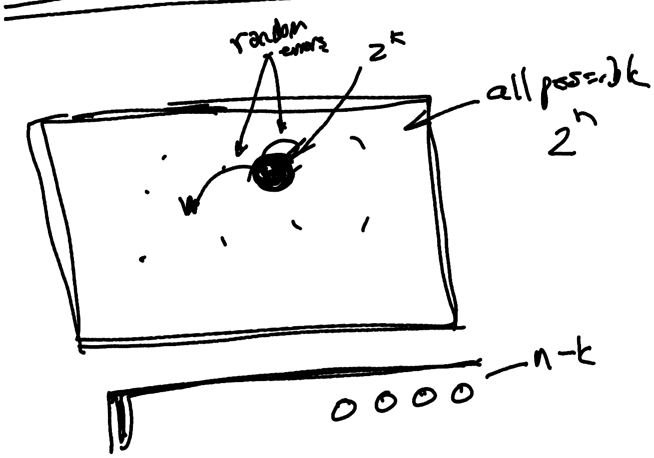
0 0 0	0 0 ← only correct codeword for data = 000
0 0 1	0 1
0 1 0	1 1
0 1 1	1 0
⋮	
1 1 1	1 0

for (n,k) code only have 2^k correct codewords

but if channel introduces errors, we can receive.

2^n different possible received

$2^5 = 32$ possible received



2^k possible transmitted

2^n possible received.

Exercise 4: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

11100
11011

0+0+1+1+1 = 3 bits are different ⇒ Hamming distance

0111
1011
1100
d=2

0111
1101
1010
d=2

0111
1110
1001
d=2

there should be 3 + 2 + 1

n-1, n-2 ... 1

1011
1101
0110
d=2

1011
1110
0101
d=2

	0111	1011	1101	1110
0111	0	2	2	2
1011		0	2	2
1101			0	2
1110				0

1101
1110
0011
d=2

minimum Hamming distance = 2

Exercise 5: What is the code rate of a code with 4 codewords each of which is 4 bits long? *Hint: If a code has 2^k codewords, what is k?*

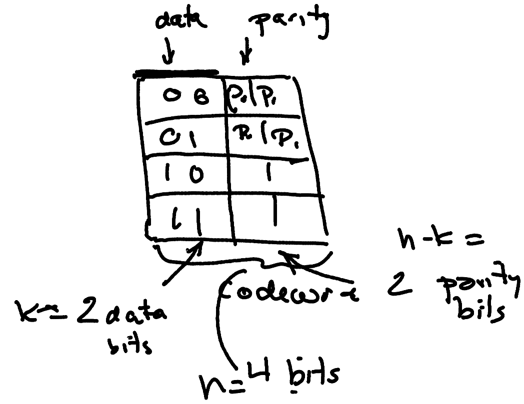
4 codewords $\Rightarrow 2^k = 4$

$k = 2$

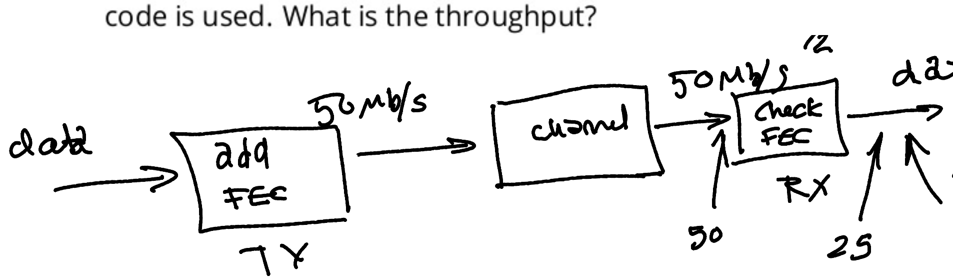
$n = 4$

$k = 2$

$R = \frac{k}{n} = \frac{2}{4} = \frac{1}{2}$



Exercise 6: The data rate over the channel is 50 Mb/s; a rate 1/2 code is used. What is the throughput?



Mb/s = 25 Mb/s.

throughput

Exercise 7: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

	101	010
received →	110	110
	011	100
	d=2	d=1

← choose this as most likely transmitted codeword.

first bit was probably in error.

Yes it's possible the first word was sent, but it's less likely

Exercise 8: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?

x y z
010
101
―――
111

$d = d_{min} = 3$

$k = 1$
$n = 3$

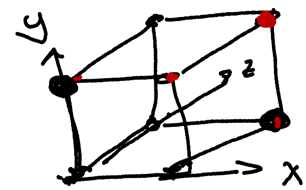
rate $= \frac{1}{3}$

guaranteed to detect $d-1 = 3-1 = 2$ errors.

$$\left\lfloor \frac{d_{min}-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$$

floor = round down

guaranteed to correct 1 error

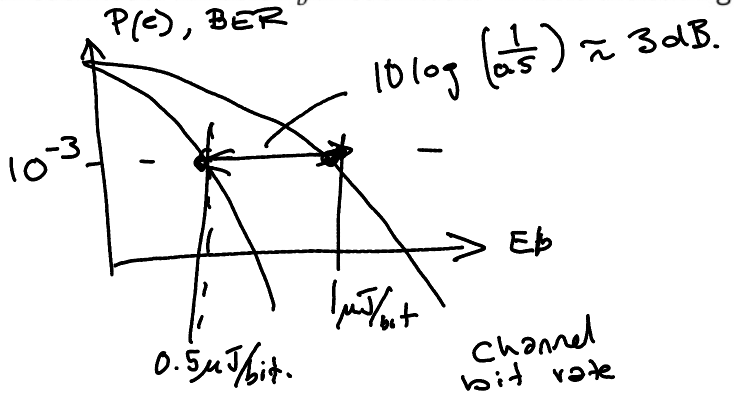


Exercise 9: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Energy is in Joules (J)
Power watts (W) = J/s. S
Bit Period seconds (s). T_b

$E_b = S \cdot T_b = S \frac{1}{f_b}$

Exercise 10: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the ~~information~~ useful data rate in each case? What is E_b in each case? What is the coding gain?



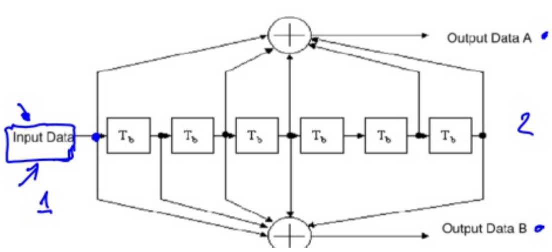
channel bit rate

for information bits only.

		channel bit rate	E_b
w/o FEC:	1W	@ 1 Mb/s ($T_b = 10^{-6}$)	1μJ/bit
w/ rate-½ FEC:	500mW	@ 2 Mb/s ($T_b = 1\times10^{-6}$)	0.5μJ/bit

- in both cases 1Mb/s.
- 3 dB coding gain

Exercise 11: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?



$k = 1$

$n = 2$

rate $= \frac{k}{n} = \frac{1}{2}$

$K = 7$ (contraint length)

starting state of S.R. is all zeros.

input is 101

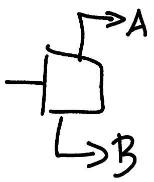
what is the output?

$A_0 = 1$ $A_1 = 0$ $A_2 = 0$

$B_0 = 1$ $B_1 = 1$ $A_2 = 0$

1, 1, 0, 1, 0, 0

Exercise 12: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?



$A_0, A_1, \cancel{A_2}, \quad A_3 \quad A_4 \quad \cancel{A_5}$

$\cancel{B_0}, B_1, B_2, \quad \cancel{B_3} \quad B_4 \quad B_5$

$\frac{3 \text{ bits in}}{4 \text{ bits out}}$ code rate = $\frac{3}{4}$

after puncturing

1, ~~0~~, 0, 1, ~~0~~, 0, 1 ⟶ 1, 0, 1, 0 ← transmitted.

code rate = $\frac{3}{4}$ (3 ← data; 4 ← transmitted.)

Exercise 13: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD. ←

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.

$4 \times 4 \times 4 = 4^3 = 2^{2^3} = 2^6 = 64$

	ABC	DAB	CDA	BCD
ABC	0	3	3	3
DAB		0	3	3
CDA			0	3
BCD				0

$d_{min} = 3$

guaranteed to correct $t = \left\lfloor \frac{3-1}{2} \right\rfloor = 1$ errors.

detect $3-1 = 2$ errors.

receive ADA → doesn't match any valid c/word so there must be an error

	ADA
ABC	2
DAB	3
CDA	1
BCD	3

← closest so choose this one.
∴ error is in first letter (symbol).
& correct c/word is CDA.

A = 00
B = 01
C = 10
D = 11

corrected 1 error.

A = 00
B = 01
C = 11
D = 10

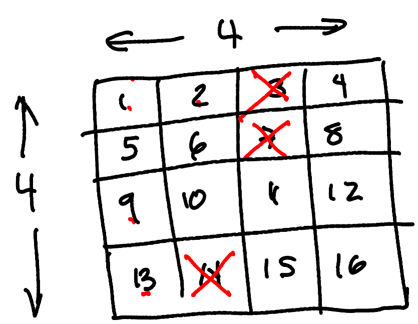
corrected 2 errors!

	AAA
ABC	2
DAB	2
CDA	2
BCD	3

} pick one of these but we know there were at least 2 errors made.

so less likely that we'll be succesful in correcting all the errors.

Exercise 14: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?



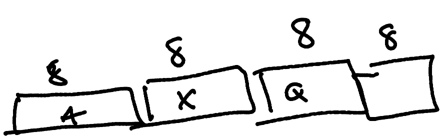
order transmitted over channel:

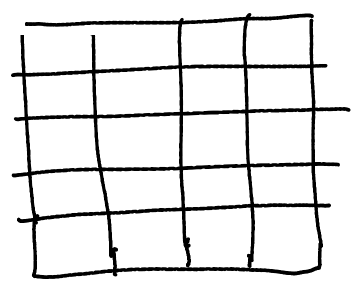
1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11, 15, 4, 8, 12, 16

no could not correct all errors w/o interleaving.

yes could correct all errors w/ interleaving.

Exercise 15: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?





interleave bytes — we want errors concentrated into a small number of bytes