# Error Detection and Correction

**Exercise 1**: Compute the modulo-4 checksum, $C$, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the sum at the receiver if there were no errors? Determine the sum if the received frame was: 3, 1, 1, $C$? 3, 1, 2, 0, $C$? 1, 2, 3, $C$?

$3+1+2=6$ $\quad C=2$ because $2+2 \bmod 4 = 0$

$6 \bmod 4 = 2$

transmit 3, 1, 2, 2

if no errors, sum = 0

if receive 3, 1, 1, 2 $\Rightarrow$ $3+1+1+2 \bmod 4 = 7 \bmod 4 = 3$

$\Rightarrow \neq 0 \Rightarrow$ error

if 3, 1, 2, 0, 2 $\Rightarrow$ $3+1+2+0+2 \bmod 4 = 8 \bmod 4 = 0$

$\Rightarrow$ no error.

if 1, 2, 3, 2 $\Rightarrow$ $1+2+3+2 \bmod 4 = 8 \bmod 4 = 0$

$\Rightarrow$ no error.

**Exercise 2**: What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number?

modulo-2 means remainder of sum after diving by 2

$(1+0+1) \bmod 2 = 2 \bmod 2 = 0$

even nombers are multiples of 2 so remainder is 0.

**Exercise 3**: A (5,3) code computes the two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where $d_i$ is the $i$'th data bit. What codeword is transmitted when the data bits are $(d_0, d_1, d_2) = (0, 0, 1)$? How many different codewords are there in the code? What are the first four codewords? In general, how many codewords are there for an $(n, k)$ code?

$(5,3)$
$(n,k)$

$K=3 \quad n-K$

$$\underbrace{\boxed{d_0 \; d_1 \; d_2} \boxed{p_0 \; p_1}}_{n=5}$$

$p_0 = d_0 \oplus d_1$

$p_1 = d_1 \oplus d_2$

if data: $(0,0,1)$

$p_0 = d_0 \oplus d_1 = 0 \oplus 0 = 0$

$p_1 = d_1 \oplus d_2 = 0 \oplus 1 = 1$

then $p = (0,1)$

transmitted codeword $= (0, 0, 1, 0, 1)$

how many codewords?

$2 \times 2 \times 2 = 2^3 = 8$ possible values for the data $(d_0, d_1, d_2)$

| $d_0$ $d_1$ $d_2$ | $p_0$ | $p_1$ |
|---|---|---|
| 0 0 0 | 0 | 0 |
| 0 0 1 | 0 | 1 |
| 0 1 0 | 1 | 1 |
| 0 1 1 | 1 | 0 |
| 1 0 0 | 1 | 0 |
| 1 0 1 | 1 | 1 |
| 1 1 0 | 0 | 1 |
| 1 1 1 | 0 | 0 |

for an $(n,k)$ code there are:

$2^n$ possible received c/words

$2^k$ possible valid c/words

**Exercise 4**: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

$$
\begin{array}{r}
1\,1\,1\,0\,0 \\
\oplus\ 1\,1\,0\,1\,1 \\
\hline
0+0+1+1+1=3
\end{array}
$$

Hamming distance = 3

$D_{min} = \min(2,2,2,2,2,2)$
$= 2$

| | 0111 | 1011 | 1101 | 1110 |
|---|---|---|---|---|
| 0111 | 0 | 2 | 1010 2 | 1001 2 |
| 1011 | | 0 | 0110 2 | 2 |
| 1101 | | | 0 | 2 |
| 1110 | | | | 0 |

**Exercise 5**: What is the code rate of a code with 4 codewords each of which is 4 bits long? *Hint: If a code has* $2^k$ *codewords, what is* $k$*?*.

00 | 0
01 | 1
10 | 2
11 | 3

$n=4$

A B C D

if $2^k = 4$ $\underline{k=2}$

$\#\text{ bits} = \log_2(2^k) = k$

$2^k$ → # messages or codewords

$(n, k)$

$k$ → # data bits.

$n$ → # bits

$$\text{code rate} = \frac{k}{n} = \frac{2}{4} = \frac{1}{2}$$

higher code rates more efficient
less able to detect/correct errors.

**Exercise 6**: The data rate over the channel is 50 Mb/s; a rate 1/2 code is used. What is the throughput?

50 Mb/s

rate $= \frac{1}{2}$

Throughput $= \frac{1}{2} \cdot 50 = 25$ Mb/s.

**Exercise 7**: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

code: $\left\{ \begin{matrix} 101 \\ 010 \end{matrix} \right.$ $(n,k) = (3,1)$

receive: 110

distances

| | 110 ← received. |
|---|---|
| 101 | 2 |
| 010 | 1 |

receiver chooses:
010 (min. distance).

most likely first bit was in error

yes, it's possible 101 was sent & channel introduced 2 errors (much less likely).

**Exercise 8**: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are $n$, $k$, and the code rate ($k/n$)?

| | 010 | 101 |
|---|---|---|
| 010 | 0 | $\boxed{3}$ |
| 101 | 3 | 0 |

$d = 3$ guaranteed to detect $d - 1 = 2$ errors

guaranteed to correct $\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$

**Exercise 9**: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Energy : Joules (J)

Power : Watts (W) ( J/s )

Bit Period : seconds (s).

$J = W \cdot s$ ← watts, ← seconds

$E_b = S \cdot T_b$ ← bit duration

$S$ ← sig. power

$N_0$ : W / Hz $\qquad \frac{E_b}{N_0} : \frac{W \cdot s}{W \cdot s} \qquad \frac{S}{N} = \frac{W}{W}$

**Exercise 10**: A system needs to operate at an error rate of $10^{-3}$. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is $E_b$ in each case? What is the coding gain?

$\underbrace{\boxed{1}\,\boxed{P}}_{1\mu s}$

| | w/o FEC | w/ FEC |
|---|---|---|
| channel bit rate | 1 Mb/s | 2 Mb/S |
| information (data) rate | 1 Mb/S | 1 Mb/s. |
| power | 1 W | 0.5 W |
| $E_b$. | 1 J/s · 1μs = 1μJ | 0.5 J/s · 1μs = 0.5 μJ |

coding gain $= \frac{1\,\mu J}{0.5\,\mu J} = 2 = 3\,dB$

**Exercise 11**: Assuming one bit at a time is input into the encoder in the diagram above, what are $k$, $n$, $K$ and the code rate?

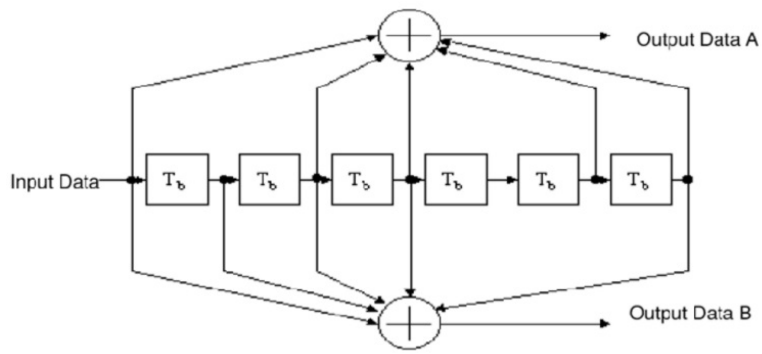



**Figure 18-8—Convolutional encoder (K = 7)**

$k = $ input bit (s) its
per output

$n = $ # output bits

$K = $ constraint length.

$k : 1$

$n : 2$

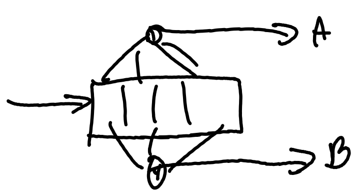
$K : 7$

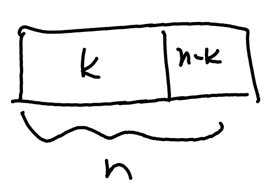
rate $\left(\frac{k}{n}\right) : \frac{1}{2}$

**Exercise 12**: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

| bit : | 1 | 2 | 3 |
|---|---|---|---|
| | A | A | ~~A~~ |
| | ~~B~~ | B | B |

$$\frac{k}{n} = \frac{\text{\# data bits}}{\text{\# trans. bits}} = \frac{3}{4}$$



**Exercise 13**: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.

| | ABC | DAB | CDA | BCD |
|---|---|---|---|---|
| ABC | | 3 | 3 | 3 |
| DAB | | | 3 | 3 |
| CDA | | | | 3 |
| BCD | | | | |

$t = 3$

min detected $= t-1 = 3-1 = 2$

corrected $: \left\lfloor \frac{t-1}{2} \right\rfloor = \left\lfloor \frac{3-1}{2} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$

if receive:

ADA yes error introduced by channel since ADA is not valid c/word.

| ADA | ABC | DAB | CDA | BCD |
|---|---|---|---|---|
| | 2 | 3 | 1 | 3 |

receiver chooses CDA

e.g. if A = 00
B 01
C = 11
D = 10

corrected A → C
00 → 10
00 → 11

| AAA | ABC | DAB | CDA | BCD |
|---|---|---|---|---|
| | 2 | 2 | 2 | 3 |

**Exercise 14**: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

| | | | |
|---|---|---|---|
| 1 | 2 | ~~3~~ | 4 |
| 5 | 6 | ~~7~~ | 8 |
| 9 | 10 | 11 | 12 |
| 13 | ~~14~~ | 15 | 16 |

read out by column $\Longrightarrow$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 9 | 13 | 2 | 6 | 10 | ~~14~~ | ~~3~~ | ~~7~~ | 11 | 15 | 4 | 8 | 12 | 16 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

- error appear in bits 14, 3 and 7 of the original (de-interleaved) sequence.
- without interleaving a burst of 3 errors would cause one codeword to have 2 or 3 errors & the errors in that codeword (row) could not be corrected.
- with interleaving there will be a maximum of one error per row (code word) so all of the errors could be corrected.

**Exercise 15**: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

interleave bytes because all errors in one byte will be corrected.