Error Detection and Correction

Exercise 1: Compute the modulo-4 checksum, C, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the sum at the receiver if there were no errors? Determine the sum if the received frame was: 3, 1, 1, C? 3, 1, 2, 0, C? 1, 2, 3, C?

$$\begin{array}{rl} 011 & 3 \\ 001 & 1 \\ 010 & 2 \\ \hline 110 & 6 \bmod 4 = 2 \end{array}$$

transmit: 3, 1, 2, 2

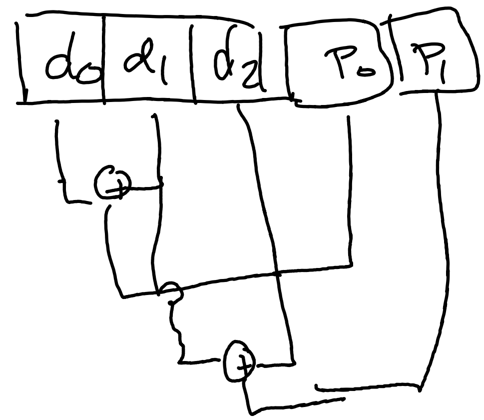
if no errors: sum = 8 mod 4 = 0

$\sum 3,1,1,2 = 7$, 7 mod 4 = 3 → error

$\left.\begin{array}{l} 3,1,2,0,2 = 8 \\ 1,2,3,2 = 8 \end{array}\right\}$ no error detected.

n K

Exercise 3: A (5,3) code computes the two parity bits as: $p_0 = d_0 \oplus d_1$ and $p_1 = d_1 \oplus d_2$ where d_i is the i'th data bit. What codeword is transmitted when the data bits are $(d_0, d_1, d_2) = (0, 0, 1)$? How many different codewords are there in the code? What are the first four codewords? In general, how many codewords are there for an (n, k) code?



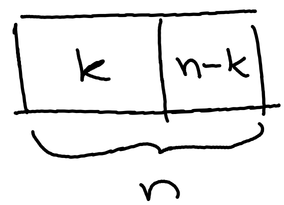
$2^5 = 32$

000 00
000 01
⋮
111 11

00000
00101 ✓
01011
01110
10010
10111
11001
11100

01010

8 codewords

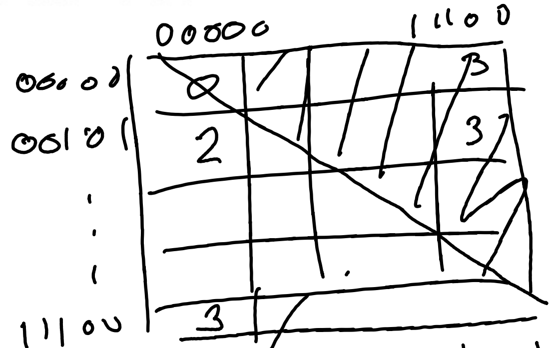


n, k

2^k possible valid codewords.

Exercise 4: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

$$
\begin{array}{r}
1\,1\,1\,0\,0 \\
\oplus\ 1\,1\,0\,1\,1 \\
\hline
0+0+1+1+1 = 3
\end{array}
$$



smallest value is minimum distance of the code

	0111	1011	1101	1110
0111	0			
1011	2	0		
1101	2	2	0	
1110	2	2	2	0

Exercise 5: What is the code rate of a code with 4 valid codewords each of which is 4 bits long? *Hint: If a code has 2^k codewords, what is k?*.

$2^k = 4 \quad \therefore k = 2$

$n = 4 \qquad \frac{k}{n} = \frac{2}{4} = \frac{1}{2}$

Exercise 6: The data rate over the channel is 50 Mb/s; a rate 1/2 code is used. What is the throughput?

throughput = code rate · data rate

$= \frac{1}{2} \cdot 50$ Mb/s $= 25$ Mb/s

Exercise 7: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the receiver decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

$\left.\begin{matrix} 101 \\ 010 \end{matrix}\right\} 2^k$

$n = 3$

$k = 1$

$n - k = 2$

receive 110 → 101 ⇒ $d = 2$

→ 010 ⇒ $d = 1$

receiver should select 010 as transmitted codeword.

1st bit most likely in error.

yes, but less likely.

Exercise 8: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?

	010	101
010		
101	3	

$d_{min} = 3$

this code can detect 2 errors

& correct $\left\lfloor \frac{3-1}{2} \right\rfloor$

$= \lfloor 1 \rfloor = 1$ error

Exercise 9: What are the units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

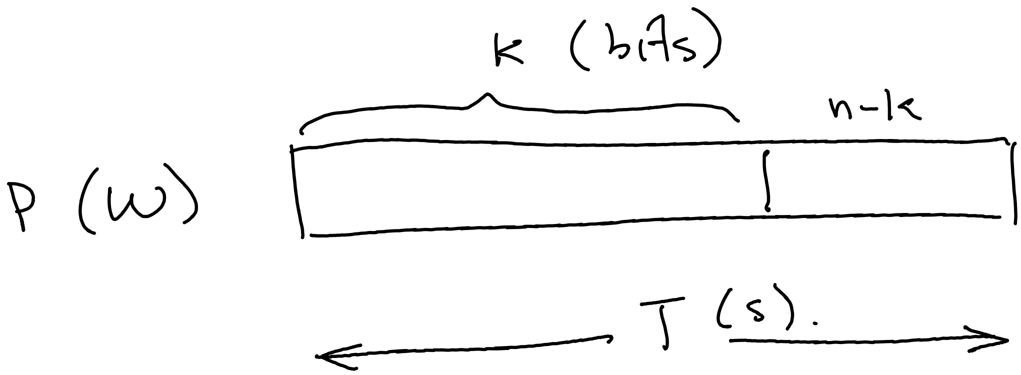
Energy : Joules = W · s

Power : Watts

Period : s

Energy = f(power, duration)

$$\boxed{\text{Energy} = \text{power} \times \text{duration}} \quad (J)\ (W \cdot s).$$



$$E_b = \frac{PT}{k}.$$

Exercise 10: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is E_b in each case? What is the coding gain?

	w/o FEC	w/ FEC
ch. rate (including parity)	1 Mb/s	2 Mb/s
power	1 W	0.5 W
"information" rate	1 Mb/s	1 Mb/s
Eb	1w · 1μs = 1μJ/bit	0.5w · 1μs = 0.5μJ/bit

coding gain: $\frac{E_b(w/o)}{E_b(w)} = \frac{1}{0.5} = 2$

3 dB

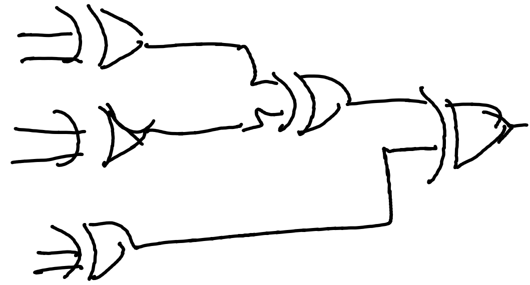
$a \oplus (b \oplus c) \equiv (a \oplus b) \oplus c$

$((((a \oplus b) \oplus c) \oplus d) \oplus e)$

$((a \oplus b) \oplus (b \oplus c)) \oplus (c \oplus d))$

associative

commutati $a \oplus b \equiv b \oplus a$



Exercise 11: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?

$k = 1$

$n = 2$

$K = 7$

rate $(R) = \frac{k}{n} = \frac{1}{2}$

Exercise 12: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

input: 0, 1, 0

A :	1	1	~~0~~	0
B :	~~0~~	1	0	~~1~~

output: 1, 1, 1, 0

rate $= \frac{3}{4}$ $\begin{matrix} \rightarrow 3\text{ Mb/s} \\ \rightarrow 4\text{ Mb/s} \end{matrix}$

rate $= \frac{1}{2}$ $\begin{matrix} \rightarrow 2\text{ Mb/s} \\ \rightarrow 4\text{ Mb/s} \end{matrix}$

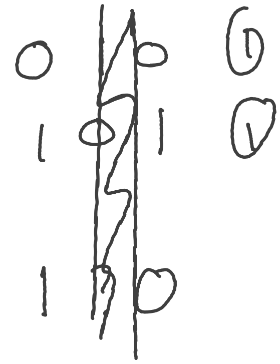
Exercise 13: A block FEC code uses values from GF(4). The 4 possible elements are represented using the letters A through D. The valid code words are: ABC, DAB, CDA, and BCD.

What is the minimum distance of this code? How many errors can be detected? Corrected?

If the codeword ADA is received, was an error made? Can it be corrected? If so, what codeword should the decoder decide was transmitted?

If each codeword represents two bits, how many bit errors were corrected?

Repeat if the codeword received was AAA.



$d_{min} = 3$

can detect $3-1=2$

can correct $\left\lfloor \frac{3-1}{2} \right\rfloor = 1$ symbol

ADA		ABC	DAB	CDA	BCD
2	ABC	0			
3	DAB	3	0		
1	CDA	3	3	0	
3	BCD	3	3	3	0

ADA - yes error

- would pic CDA - distance of 1

- could correct 1 or 2 errors by correcting on symbol

A 00
B 01
C 10
D 11

- if use GF(2^7) can correct up to 7 bit errors

Exercise 14: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

1	2	~~3~~	4
5	6	~~7~~	8
9	10	11	12
13	~~14~~	15	16

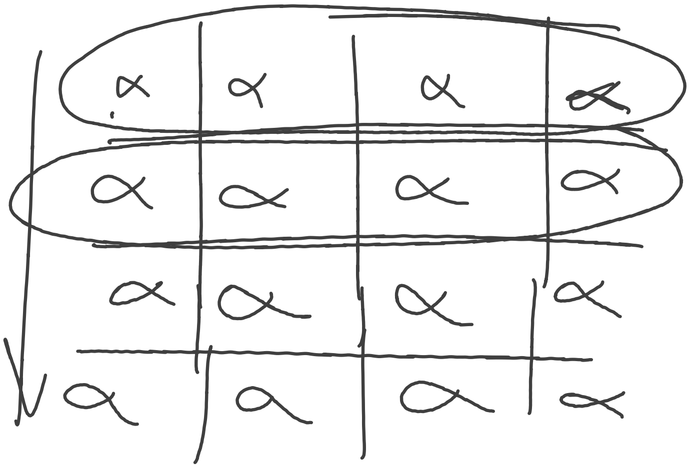
1, 5, 9, 13, 2, 6, 10, ~~14~~, ~~3~~, ~~7~~, 11, 15, 4, 8, 12, 16

1	2	~~3~~	4	← 1
5	6	~~7~~	8	← 1
9	10	11	12	← 10
13	~~14~~	15	16	← 1

$\underbrace{1\ 2,\ 3,\ 4}_{✓}\quad \underbrace{5\ 6\ 7\ \cancel{8}}_{✓}\quad \underbrace{\cancel{9}\ \cancel{10}\ 11\ 12}_{✗}\quad \underbrace{13\ 14\ 15\ 16}_{✓}$

Exercise 15: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes?

GF(256)



$\alpha_0 \to 00000\cdot$

α_1

$\vdots$

$\alpha_{127} \to 0111\ 1111$

α_{255} 1111 1111