Solutions to Assignment 3

Question 1

The output of the solution for Question 5 of Assignment 2 is:

```
13784 frames had 3 or more errors
```

As explained in the solution, on average there should be 13817 frames with 3 or more errors. This result is about 0.2% less than the expected value.

Question 2

The distribution of childrens' heights is Gaussian and has a mean of 1.3 m and a standard deviation of 0.3 m. The fraction of children that can be admitted to the ball room is the fraction that are less than 59 inches (= 1.5 m; tall enough) minus the fraction of children that are less than 34 inches (= 0.86 m; too short).

The two normalized thresholds are $t_1 = \frac{1.50-1.3}{0.3} = 0.66$ and $t_2 = \frac{0.86-1.3}{0.3} = -1.47$ and the probability (the shaded area in the curve below) is $P(0.66) - P(-1.47) = 0.75 - 0.07 \approx 0.67$. So about 67% of the children can be admitted to the ball room.
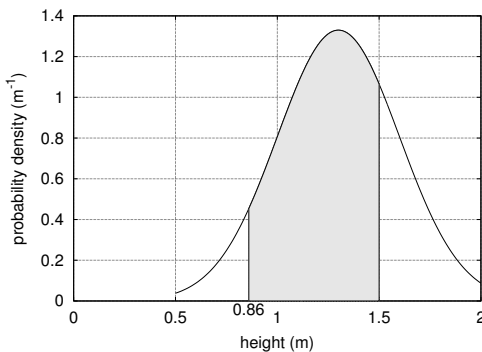
Question 3

Encoding data using 64 levels allows transmission of $\log_2(64) = 6$ bits per symbol.

If the bit rate is 1.544 Mb/s then the symbol rate must be $1.544/6 = 257.3$ kb/s.

To avoid ISI the the Nyquist no-ISI condition requires that the channel gain be 0.5 (-6 dB) at half of the symbol rate: $257.3/2 \approx 129$ kHz.

The channel could have different frequency responses but the gain must be 1 (0 dB) at DC ($f = 0$). It must also be zero at the symbol rate ($f \approx 257$ kHz).

Question 4

The gain of a raised-cosine channel is 0.5 at $f = 0.5$ (relative to f_s, the symbol rate), 0 above $0.5+\alpha/2$, and 1 below $0.5 - \alpha/2$. Between the two limits it has the shape of one period of a cosine function:
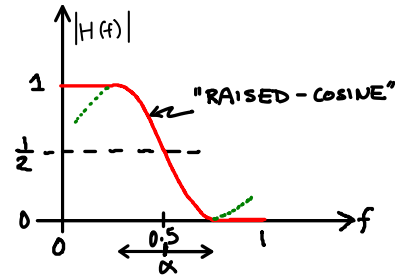
which can be written as:

$$H(f) = \frac{1}{2}\left(1 + \cos(af + b)\right)$$

To find a and b we evaluate $H(f)$ at two frequencies:

1. at $f = 1/2$, $H(f) = 1/2(1 + \cos(a/2 + b)) = 1/2$ so $\cos(a/2 + b) = 0$ and $a/2 + b = \pi/2$.

2. at $f = (1 + \alpha)/2$, $H(f) = 1/2(1 + \cos(a(1+\alpha)/2 + b)) = 0$ so $\cos(a(1+\alpha)/2 + b) = -1$ and $a(1 + \alpha)/2 + b = \pi$.

Subtracting the first equation from the second we get: $a(1+\alpha)/2+b-a/2-b = \pi-\pi/2$, or $a\alpha/2 = \pi/2$ or $a = \pi/\alpha$.

Substituting for a in the first equation we find $\pi/2/\alpha + b = \pi/2$ or $b = \pi/2(1 - 1/\alpha) = \pi/2\frac{\alpha-1}{\alpha} = \pi/\alpha\frac{\alpha-1}{2}$.

Thus the equation of the raised-cosine portion of the transfer function is:

	A	B	C
1	alpha	0.300	
2	f	**RC**	**H(f)**
3		=0.5*(1+COS(PI()/B$1*(A3+(B$1-1)/2)))	=IF(A3<0.5-B$1/2,1,IF(A3>0.5+B$1/2,0,B3))
4	=A3+0.02	=0.5*(1+COS(PI()/B$1*(A4+(B$1-1)/2)))	=IF(A4<0.5-B$1/2,1,IF(A4>0.5+B$1/2,0,B4))
5	=A4+0.02	=0.5*(1+COS(PI()/B$1*(A5+(B$1-1)/2)))	=IF(A5<0.5-B$1/2,1,IF(A5>0.5+B$1/2,0,B5))

Figure 1: Formulas for Question 4.

$$H(f) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi}{\alpha}(f + \frac{\alpha-1}{2})\right)\right]$$

The spreadsheet formulas could be set up in various ways. In this solution column A is for frequency, B is for the value of the raised-cosine function and C is for $H(f)$. Two nested `if()` functions in column C determine whether to use the value 0, 1 or the raised-cosine formula for $H(f)$. The equations are shown in Figure 1 and the resulting plot is:

The spreadsheet formulas could be set up in various ways. In this solution column A is for frequency,
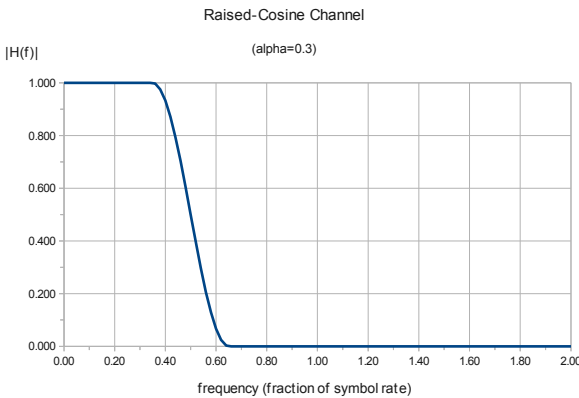
Question 5

To allow the ISI to die down in-between OFDM symbols the guard time must be at least as long at the duration of the channel's impulse response. This is 20 μs which corresponds to $20^{-6} \times 2 \times 10^6 = 40$ samples.

For this to be less than 10% of the symbol period the symbol period must be greater than $40/0.1 = 400$ samples. The next larger power of two is $2^9 = 512$. Thus the symbol length is 512 samples which is $\frac{512}{2\times10^6} = 256$ μs.

Question 6

The capacity of a BSC, C, is the information rate which can be communicated with an arbitrarily low error rate. For the BSC capacity is specified in information bits per transmitted bit ("channel use") which in this case must be $56/70 = 0.8$ (bits/bit).

The equation for the capacity of the BSC is:

$$C = 1 + (p\log_2 p + (1-p)\log_2(1-p))$$

We wish to find value of p for which C exceeds 0.8. This equation has a minimum (0) at $p = 0.5$ and increases to 1 for $p = 0$ and also for $p = 1$. Thus there will be two values, one at low BERs and for at high BERs where the capacity is 0.8.

There is no closed-form solution but a numerical answer can be obtained using a calculator, spreadsheet, Matlab, or graphically as described below. To find the allowed values of p, we can re-write the equation as:

$$1 + (p\ln p + (1-p)\ln(1-p))/\ln(2) - 0.8 = 0$$

and solve for p. The results show that we achieve this information rate for $p < 0.0311$ or $p > 0.9689$.

From this we can also conclude that there should be a rate 56/70 code that will correct practically all errors if the channel has a BER of 3% or less.

Calculator

To solve for p using the Sharp EL-W516 calculator[1]: enter the equation using the variable x, press Math -> Solver, enter an approximate solution (e.g. 0.25 or 0.75 for this example), '=', an error tolerance, and '=' to find the solution. The results are:

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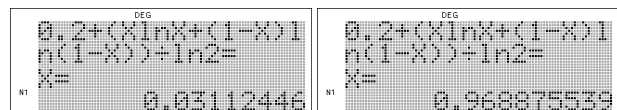
[1]Many programmable scientific calculators have a similar function.

Spreadsheet

Many spreadsheets have a "Solver" tool (under Tools -> Solver) that searches for the maximum, minimum or specific value of a function. The equation of the function to be optimized is entered in one cell as a function of a value in another cell. The solver can then be used to find the value of this parameter that maximizes, minimizes or gives the desired value of the function.

In the example below (using LibreOffice), p is in cell A1 and the equation above is in cell A2:

	A
1	0.001
2	=0.2+(A1*LN(A1)+(1-A1)*LN(1-A1))/LN(2)

By specifying the following solver parameters:

Target cell	A2	
Optimize result to	○ Maximum	
	○ Minimum	
	◉ Value of	0
By changing cells	A1	

Limiting Conditions

Cell reference		Operator	Value
A1	🔲	=>	0.001
A1	🔲	<=	0.5

we obtain the solution 0.0311244603047893 between 0.001 and 0.5 and 0.968875539695211 as the solution between 0.5 and 0.999.

We specify range limits because the arguments of the log function are undefined for $p = 0$ and $p = 1$.

Matlab

Using Octave (or Matlab) we can use the `fzero()` function to find the zeros of a function. We can create a function handle[2] and pass it to the `fzero()` function with the appropriate search limits as shown below.

```
octave:1> fun = @(p) 1+(p*log2(p)+(1-p)*log2(1-p))-0.8;
octave:2> fzero(fun,[0.01,0.5])
ans =  0.031124
octave:3> fzero(fun,[0.5,0.99])
ans =  0.96888
```

---

[2]Similar to a pointer to a function in C.

Web Site

On the Wolfram Alpha web site we can enter the query:

1+(p*log2(p)+(1-p)log2(1-p)) = 56/70

and get the results:

Numerical solutions:

$p \approx 0.0311244603047894...$

$p \approx 0.968875539695211...$

Graphical Method

Graphing a function shows it's overall behaviour. Software (or calculator) that graphs functions, rather than computed values, allows us to "zoom in" to a precise solution. The graph below, produced by the Octave/Matlab `fplot()` function, shows an example.
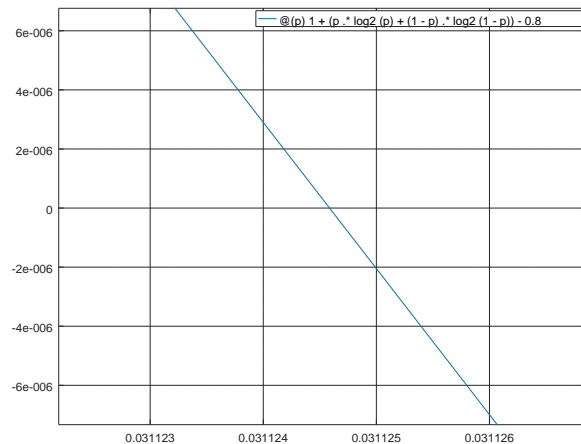
Question 7

(a) The capacity or maximum error-free information rate for an AWGN[3] channel is given by:

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

The bandwidth is given as $B = 1$ MHz and the SNR is $10 - (-3) = 13$ dB which is a power ratio

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[3]We assume a Gaussian noise distribution; this was not specified in the question.

of $10^{\frac{13}{10}} \approx 20$. Thus the capacity is $1 \times 10^6 \log_2(1 + 20) \approx 4.4$ Mb/s.

(b) A brick-wall channel is an example of a channel that meets the Nyquist non-ISI criteria. The channel gain is 0.5 at $f_S/2 = 1$ MHz so the maximum symbol rate that can be transmitted without ISI is $f_S = 2$ MHz.

(c) Bipolar NRZ transmission encodes one bit per symbol so the bit rate is 2 Mb/s. The error rate is the probability that the (Gaussian) noise crosses a threshold at 0 V. The mean of the signal plus noise is the mean signal voltage, $\mu = 10^{\frac{10}{20}} \approx 3.16$ mV and the standard deviation is $\sigma = 10^{\frac{-3}{20}} \approx 0.707$ mV. The normalized threshold is $t = \frac{0-\mu}{\sigma} = \frac{-3.16}{0.707} \approx -4.47$. Although this is outside the range of values covered by the graph in Lecture 4, the probability of error can be found with a calculator as $P(-4.47) \approx 4 \times 10^{-6}$:
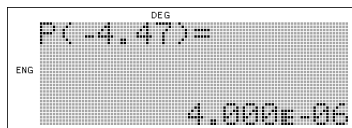
using the Matlab function `normcdf()`:

```
octave:4> normcdf(-4.47)
ans =  3.9110e-06
```

or evaluating $\frac{1}{2} \operatorname{erfc}(-(-4.47)/\sqrt{2})$ with a spreadsheet as in Lab 8.

Question 8

The power spectrum of a random bit stream using Delay Modulation (or "Miller") line coding has been derived [1], [2], [3, p. 21] as[4]:

$$S(f) = \frac{A^2 T_b}{2\theta^2 (17 + 8\cos 8\theta)} (23 - 2\cos\theta - 22\cos 2\theta \\ - 12\cos 3\theta + 5\cos 4\theta + 12\cos 5\theta + 2\cos 6\theta \\ - 8\cos 7\theta + 2\cos 8\theta)$$

_____

[4]This is the two-sided spectrum (at both positive and negative frequencies). The PSD of the one-sided spectrum computed by the `pwelch()` function is twice this.

where $\theta = \pi f T_b$, T_b is the bit period and the signal levels are $\pm A$.

We can also run a Monte-Carlo simulation and compute the power spectrum of encoded random data. The code below does this and plots both the theoretical spectrum and the spectrum of the simulated signal.

```
% Delay Modulation Line Code Spectrum
% Ed. Casas, 2016-11-19

n=1E6 ;                     % number of bits
ns=20 ;                     % samples/bit
fs=2e6 ;                    % sampling rate
fb=fs/ns ;                  % bit rate
nfft=2048 ;                 % fft size

% generate bits (0 or 1)

x = rand(1,n) > 0.5 ;

% find '1' bits (should have a transition in the middle
% of the bit period) (ignore the first bit):

x1=x(2:n);

% find '0' bits preceeded by another 0 bit (should have
% a transition at the start of a bit period):

x0=x(2:n)==0 & x(1:n-1)==0 ;

% insert 1's where the transitions should happen into
% an array with ns samples/bit

z=zeros(ns,length(x0));
z(1,:)=x0;
z(ns/2,:)=x1;

% encode so each '1' causes a transition between 0 and
% 1 then offset to a bipolar signal

x=mod(cumsum(z(:)),2) ;
x=2*x-1 ;

% compute the Power Spectral Density of the signal

[psd,f]=pwelch(x,hamming(nfft),[],nfft,fs,'onesided');

% plot frequencies from DC to 2fb

nf=ceil((2*fb)/(fs/2)*length(f)) ;
f=f(1:nf) ;

% compute theoretical (two-sided) spectrum

t=pi*f/fb ;

s=(1/fb)./(2*t.^2.*(17+8*cos(8*t))).* ...
        ( 23 - 2*cos(t) - 22*cos(2*t) ...
    - 12*cos(3*t) + 5*cos(4*t) + 12*cos(5*t) + ...
          2*cos(6*t) ...
    - 8*cos(7*t) + 2*cos(8*t) );

% plot spectrum of signal and the (one-sided)
% theoretical PSD
```
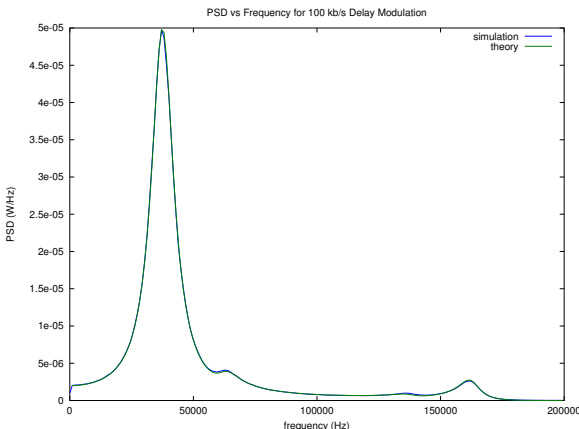
4

```
plot(f(1:nf),[psd(1:nf), 2*s(1:nf)])
```

```
xlabel('frequency (Hz)');
ylabel('PSD (W/Hz)');
legend('simulation','theory');
title('PSD vs Frequency for 100 kb/s Delay Modulation');
```

The theoretical and simulation spectra agree closely. The units of "W/Hz" are for a unit impedance (otherwise V^2/Hz) .

References

[1] M. Hecht and A. Guida, "Delay modulation," *Proceedings of the IEEE*, vol. 57, pp. 1314–1316, July 1969. url: http://ieeexplore.ieee.org/document/11449179/.

[2] A. Guida, "Delay modulation," *Proceedings of the IEEE*, vol. 58, pp. 182–182, Jan 1970. Correction to earlier paper. url: http://ieeexplore.ieee.org/document/1449516/.

[3] W. Lindsey and M. Simon, *Telecommunication Systems Engineering*. Dover Books on Electrical Engineering, Dover Publications, 1973.

Question 9

As an example, if the least-significant 3 digits of the ID were 508, the corresponding binary value would be 01 1111 1100. To apply HDLC framing we add flag sequences (0111 1110) before and after the frame and zero-stuff after every 5 consecutive 1 bits in the data. In this case the result is:

purpose	bits
start flag	0111 1110
data bits	01 1111
stuffed zero bit	0
data bits	1100
end flag	0111 1110

The frame would consist of the sequence of all of the bits in the second column:

0111 1110 01 1111 0 1100 0111 1110

Question 10

When the voltages are +5 and 0, the differential voltage is the difference between them, 5 V, and the common-mode voltage is the average, 2.5V. When the voltages are 0 and +5 the differential voltage is -5 V and the common-mode voltage remains 2.5 V.