Solutions to Assignment 2

Question 1

From the phasor diagram the horizontal component of the sum of the two vectors is $1 + x\cos(\theta)$, the vertical component is $x\sin(\theta)$ and the amplitude of the sum is:

$$\sqrt{(1 + x\cos(\theta))^2 + (x\sin(\theta))^2}$$

The delayed component has a phase shift of $\theta = -2\pi f\tau$ due to the delay τ. The minimum output will happen when the angle of the delayed component relative to the direct component is π plus any multiple of 2π. Thus $\pi + 2n\pi = -2\pi f\tau$ where n is any integer (positive or negative). The frequencies of the minima are thus $f = \frac{1}{2\tau} + \frac{n}{\tau}$.

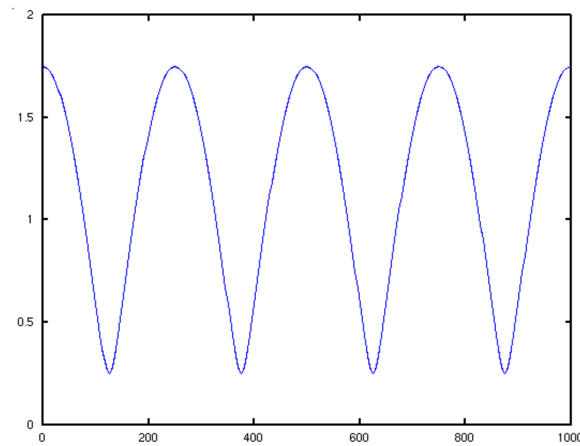
Similarly, the maxima are found when the relative phase is zero which happens when $2n\pi = -2\pi f\tau$ or $f = \frac{n}{\tau}$.

To plot the graph we can compute the value of the equation above for frequencies from 0 to 1 kHz making sure to include the the maxima and minima frequencies by choosing frequencies that are multiples of a fraction of $\frac{1}{2\tau}$.

For example if $x = 0.75$ and $\tau = 4 \times 10^{-3}$ we can compute 30 frequencies between each minimum and maximum and compute and plot the amplitude of the result using octave:

```
x=0.75;
tau=4e-3;
f=[0:1/(30*2*tau):1e3];
a=sqrt( (1+x*cos(-2*pi*f*tau)).^2 + ...
  (x*sin(-2*pi*f*tau)).^2 ) ;
plot(f,a)
```

to get the following plot:



Question 2

Since the phase shift due to a delay τ is given by $\theta = -2\pi f\tau$, the slope of the phase versus frequency curve is $-2\pi\tau$ or -360τ if the phase is measured in degrees. A slope of -90 degrees per MHz thus corresponds to a delay of $\tau = -90/-360 = 0.25$ μs and a slope of -270 degrees per MHz corresponds to a delay of $\tau = -270/-360 = 0.75$ μs. Thus for this channel a signal at 51 MHz will be delayed $0.75 - 0.25 = 0.5$ μs more than one at 50 MHz.

Unfortunately, most students tried to compute the delay through the channel at the two frequencies (using $\theta = -2\pi f\tau$). But the phase shifts (θ) were not given in the question, only the slopes ($\frac{\Delta\theta}{\Delta f}$). Even if the total phase shift were known, this approach would require that the delay be independent of frequency (which it cannot be since the slope is different at the two frequencies).

Some students computed the total phase shift through the channel assuming it was zero degrees at DC (incorrect as explained above) and then gave the difference in degrees (as if the delay had been converted to a lead or lag). But a phase difference can only represent a delay if the frequency were known (i.e. if there was only one frequency).

Question 3

The power spectral densities in linear units are 1×10^{-6} for the central 1 MHz portion and 1×10^{-8} W/Hz for the "sidebands".

The total power is $10^6 \times 1 \times 10^{-6} + 2 \times 10^6 \times 10^{-8} = 1 + 0.02 = 1.02$ W. 99% of this is 1.0098 W.

The amount of power falling in a bandwidth B for $3 > B > 1$ MHz is $1 + (B - 10^6) \times 10^{-8} = 1.0098$ so the 99% power bandwidth is $B = 1.98$ MHz.

Question 4

We can compute the THD using the formula in the lectures notes:

$$\text{THD} = \sqrt{\frac{P_1 + P_2 + P3 + \ldots}{P_0}}$$

using the following spreadsheet to convert the dBV values to power and perform the calculation:

	A	B	C	D
1	**frequency**	**dBV**	**voltage**	**power**
2	1	-10	0.3162	0.1000
3	2	-20	0.1000	0.0100
4	3	-16	0.1585	0.0251
5	4	-30	0.0316	0.0010
6	5	-22	0.0794	0.0063
7			sum (2 to 5 kHz)	0.0424
8			THD	65.14%

Question 5

A possible solution is:

```
#include <stdlib.h>
#include <stdio.h>

void main()
{

   int i, j, nerr, nframe ;

   nframe = 0 ;

   for ( i=0 ; i<1000000 ; i++ ) {

      nerr = 0 ;

      for ( j=0 ; j<50 ; j++ ) {
         if ( rand() < 0.01*RAND_MAX ) nerr++ ;
      }

      if ( nerr >=3 ) nframe++ ;

   }

   printf ( "%d frames had 3 or more errors\n", nframe ) ;

}
```

In this case an exact solution can be found. If the probability of any one bit being in error is p, then the probability of receiving a specific N-bit frame that has k bits in errors and $N - k$ bits not in error is $(1 - p)^{(N-k)} p^k$. But there are $\binom{N}{k}$ different frames with k errors. Thus the probability of having k errors in a frame is $\binom{N}{k}(1-p)^{(N-k)}p^k$. The probability of having 3 or more errors is 1 minus the probability of having 0 to 2 errors and is:

$$1 - \sum_{k=0}^{2} \binom{N}{k} (1-p)^{(N-k)} p^k$$

which for $N = 50$ and $p = 1 \times 10^{-2}$ can be computed using the following Matlab/octave code:

```
N=50; p=1e-2; sum=0;
for k=0:2; sum+=nchoosek(N,k)*(1-p)^(N-k)*p^k ; end
1-sum
```

as: 1.3817% so a simulation of 1 million frames should result in about 13,817 frames with 3 or more errors.

Question 6

The question asks for the probability that a normally distributed random variable with mean of $\mu = 1.3$ and standard deviation of $\sigma = 0.3$ will exceed a threshold of 1. The normalized threshold is $t = \frac{1-1.3}{0.3} = -1$. The probability of being less than the threshold is $P(-1) \approx 16\%$ and the probability of being larger than the threshold is $1 - P(-1) \approx 84\%$.