Error Detection and Correction

Exercise 1: Compute the modulo-4 checksum, C, of a frame with byte values 3, 1, and 2. What values would be transmitted in the packet? What would be the value of the checksum at the receiver if there were no errors? Determine the checksum if the received frame was: 3, 1, 1, C? 3, 1, 2, 0, C? 1, 2, 3, C?

mod in C: 4 % 3 = 1

3, 1, 2, C

3 + 1 + 2 = 6   6 mod 4 = 2

C = 4 − 2   = 2

3 + 1 + 2 + 2 = 8 mod 4 = 0

error

3, 1, 1, 2 $\xrightarrow{\text{sum}}$ 7 mod 4 = 3 ! error!

0, 2, 3, 3 = 8 m 4 = 0

3, 1, 2, 0, 2 → extra 0 values not detected !

1, 2, 3, 2 $\xrightarrow{= 8 \bmod 4 = 0}$ reordering not detected !

Exercise 2: What is a modulo-2 sum? What is the modulo-2 sum of 1, 0 and 1? What is the modulo-2 sum if the number of 1's is an even number?

modulo2 sum is remainder after ÷ 2

1 + 0 + 1 = 2 , 2 mod 2 = 0

if 1's is even   2n mod 2 = 0

Exercise 3: How many possible valid code words are there for an (n, k) code? How many possible parity bit patterns are possible for each code word?

$$\underbrace{\boxed{k \mid n-k}}_{n}$$

2^k valid codewords

could receive any of 2^n codewords $\begin{cases} 2^k \text{ are valid} \\ 2^n - 2^k \text{ are not (have errors)} \end{cases}$

for a given k data bits

1 correct set of n-k parity bits

$2^{n-k} - 1$ wrong set of parity bits.

Exercise 4: What is the Hamming distance between the codewords 11100 and 11011? What is the minimum distance of a code with the four codewords 0111, 1011, 1101, 1110?

$$\begin{array}{r} 11100 \\ \oplus\ 11011 \\ \hline 0+0+1+1+1 \end{array}$$

$$\begin{array}{r} 0111 \\ 1011 \\ \hline 11 \end{array} \quad \begin{array}{r} 0111 \\ 1101 \\ \hline 1\ 1 \end{array} \quad \begin{array}{r} 0111 \\ 1110 \\ \hline 1\ \ 1 \end{array}$$

$$\begin{array}{r} 1011 \\ 1101 \\ \hline 11 \end{array} \quad \begin{array}{r} 1011 \\ 1110 \\ \hline 1\ 1 \end{array}$$

$$\begin{array}{r} 1101 \\ 1110 \\ \hline 11 \end{array}$$

minimum distance is $\underline{2}$

d_{min}

0000 0001

0010 0006

0000 0000

1111 1111

how many comparisons?

there are $\binom{2^k}{2}$ pairs

$\binom{a}{b} = \dfrac{a!}{b!(a-b)!}$

e.g. $\binom{4}{2}$

$= \dfrac{4!}{2!(4-2)!}$

$= \dfrac{4 \times 3 \times 2}{2 \cdot 2}$

$= 6$

Exercise 5: A block code has two valid codewords, 101 and 010. The receiver receives the codeword 110. What is the Hamming distance between the received codeword and each of the valid codewords? What codeword should the received decide was sent? What bit was most likely in error? Is it possible that the other codeword was sent?

010 → 110
101 → 110

$$\begin{array}{r} 010 \\ 110 \\ \hline 1 \end{array} = 1 \qquad \begin{array}{r} 101 \\ 110 \\ \hline 1+1 \end{array} = 2$$

should decide 010 was sent

1st bit was most likely wrong

yes, possible but less likely.

Exercise 6: What is the minimum distance for the code in the previous exercise? How many errors can be detected if you use this code? How many can be corrected? What are n, k, and the code rate (k/n)?

$d_{min} = 3$

correct:

$$\left\lfloor \frac{3-1}{2} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$$

detect

$3 - 1 = 2$

$k = 1$
$n = 3$

$\frac{k}{n} = \frac{1}{3}$

Exercise 7: What are the SI units of Energy? Power? Bit Period? How can we compute the energy transmitted during one bit period from the transmit power and bit duration?

Joules, W, seconds. 1 W = 1 J/s.

E_b = Power · time.

$E_b = S \cdot T_b$

Exercise 8: A system needs to operate at an error rate of 10^{-3}. Without FEC it is necessary to transmit at 1W at a rate of 1 Mb/s. When a rate-1/2 code is used together with a data rate of 2 Mb/s the power required to achieve the target BER decreases to 500mW. What is the channel bit rate in each case? What is the information rate in each case? What is E_b in each case? What is the coding gain?

@ 10^{-3}

	rate	S	chanel bitrate	inf. rate	E_b (J)
w/o FEC	1	1W	1 Mb/s	1 Mb/s	1×10^{-6}
w/ FEC	$\frac{1}{2}$	0.5W	2 Mb/s	1 Mb/s	0.5×10^{-6}
					ratio = $\frac{1}{2}$

3dB

coding gain = 3dB

Exercise 9: Assuming one bit at a time is input into the encoder in the diagram above, what are k, n, K and the code rate?

$k = 1$ bit in, $n = 2$ bits out

$$\text{rate} = \frac{k}{n} = \frac{1}{2}$$

K = constraint length = 7 (6 S/R bits + 1 input bit)

Exercise 10: Consider the encoder above. If the only the bits corresponding to the outputs A, A and B, and B are transmitted corresponding to every three input bits, what is the code rate of this punctured code?

Bit →	1	2	3
	A	A	A
	B	B	B

$$\frac{\text{inf. bits in}}{\text{total bits out}} = \frac{3}{4}$$

Exercise 11: Give the numbering of the bits coming out of a 4x4 interleaver. If bits 8, 9 and 10 of the interleaved sequence have errors, where would the errors appear in the de-interleaved sequence? If the receiver could correct up to one error per 4-bit word, would it be able to correct all the errors without interleaving? With interleaving?

1, 2, 3, 4 $\longrightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\longrightarrow$ 1, 5, 9, 13, 2, 6, 10, ~~14~~, ~~3~~, ~~7~~, 11, 15, 4, 8, 12, 16

1	2	~~3~~	4
5	6	~~7~~	8
9	10	11	12
13	~~14~~	15	16

$\longrightarrow$

Exercise 12: If errors on the channel happened in bursts and you were using a RS code using 8-bit words, would you want to interleave bits or bytes? bytes

0, 01 0 . . .

k k

α α α α | α α α α

α α α

← one RS symbol = several bits

$\alpha \in GF(64)$

$k \quad n-k$

$\underbrace{\alpha_0 \ \alpha_1 \ \alpha_0,}_{n} \longleftarrow \alpha_0 \ \alpha_1 \ \cancel{\alpha_3}, \alpha_0, \ldots$

α	α	α
α	α	α
α	α	α

bit 0, 1 ← information

[RS symbols] ← packed into

codeword ← after adding parity symbols

code ← all valid codewords

$\alpha_0 \ \alpha_1 \ \alpha_2 \ \alpha_3$

0 1 0

$\| \alpha_0 \ \alpha_1 \ \alpha_3 \|$

$| \alpha_1 \ \alpha_0 \ \alpha_3 |$