A Set-Theoretic Method for Parametric Uncertainty Analysis in Markov Reliability and Reward Models

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Abstract

This paper proposes a set-theoretic method to capture the effect of parametric uncertainty in reliability and performability indices obtained from Markov reliability and reward models. We assume that model parameters, i.e., component failure and repair rates, are not perfectly known, except for upper and lower bounds obtained from engineering judgment or field data. Thus, the values that these parameters can take are constrained to lie within a set. In our method, we first construct a minimum volume ellipsoid that upper bounds this set, and hence contains all possible values that the parameters can take. This ellipsoid is then propagated via set operations through a second-order Taylor series expansion of the Markov chain stationary distribution, resulting in a set that provides approximate bounds on reliability and performability indices of interest. Case studies pertaining to a two-component shared load system with common-cause failures, and preventative maintenance of an electric-power distribution transformer, are presented.

Index Terms

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Markov reliability models, Markov reward models, parametric uncertainty, unknown-but-bounded uncertainty models.

NOTATION

Λ Markov chain generator matrix

$\theta = [\theta_1, \ldots, \theta_m]^T$ Vector of model parameters

$\pi(\theta) = [\pi_0(\theta), \ldots, \pi_n(\theta)]^T$ Stationary distribution of the Markov chain

$\bar{\theta} = [\bar{\theta}_1, \ldots, \bar{\theta}_m]^T$ Vector of nominal model parameters

$\bar{\pi} = [\bar{\pi}_0, \ldots, \bar{\pi}_n]^T$ Stationary distribution corresponding to nominal model parameters

J Jacobian of $\pi(\theta)$ excluding $\pi_0(\theta)$

ξ Reward associated with a Markov reward model

$\mathcal{X}$ Parallelotope that describes uncertainty in model parameters

$\mathcal{E}$ Minimum-volume upper bounding ellipsoid to $\mathcal{X}$

$\mathcal{F}$ Ellipsoidal bound to the stationary distribution assuming first-order approximation

$\mathcal{S}$ Set that describes uncertainty in second-order variation of stationary distribution

$\mathcal{G}$ Minimum-volume upper bounding ellipsoid to $\mathcal{S}$

$\mathcal{H}$ Ellipsoidal bound to the stationary distribution assuming second-order approximation

e Column vector with all entries equal to 1

I. INTRODUCTION

The efficacy of reliability and performability indices obtained from Markov reliability and reward models depends on the accuracy of component failure and repair rate data. However, accurate values of failure and repair rates are seldom available [1]. Therefore, to quantify the impact of this inaccurate information, reliability models should incorporate the effect of parametric uncertainty [1]. The effect of parametric uncertainty in Markov models can be analyzed with probabilistic and set-theoretic methods [1], [2]. In the realm of probabilistic methods, parameters are modeled as random variables with known distributions which are propagated through the Markov model to obtain distributions of reliability indices; we proposed an analytical solution to this problem in [3]. In this paper, we explore the set-theoretic counterpart of [3]. Instead of assuming that parameter probability distributions are available (or can be obtained from field data), we assume that only upper and lower bounds around nominal parameter values are known. Thus, the values that these parameters can take are constrained within a set. Then, we obtain bounds on reliability indices by propagating the set that describes all possible values the parameters can take through to the stationary distribution of the Markov chain. This approach represents a worst-case uncertainty analysis as no assumptions are made on the failure rate and repair rate statistics. This method is more suited to reliability assessment when extensive failure rate and repair rate data, that would enable constructing probability distributions, are unavailable. In summary, probabilistic parametric uncertainty models can be used to obtain statistics of reliability and performability indices; i.e., instead of obtaining a single-point estimate, we can obtain the distribution of the relevant indices. On the other hand, unknown-but-bounded parametric uncertainty models do not yield any statistical information about reliability and performability indices. Instead, they yield a bounding set that contains all possible values the relevant indices can take, i.e., in an unknown-but-bounded model, there is no notion of a most likely value, but we know with certainty that the actual value is contained in this bounding set [4]. The method we propose has broad applicability to areas where failure and repair rate data are limited, and repeated computation of the reliability and reward metrics is computationally expensive due to the size of the models. Some applications that contend with these challenges include power systems, power electronic circuits, programmable electronic circuits, weather models, software systems, and communication networks.

In the proposed method, we assume that the uncertain parameters take values in a parallelotope, i.e., an extension of a parallelogram (in two dimensions) or a parallelepiped (in three dimensions) to any dimension [5]. This parallelotope-based description of uncertainty naturally arises from the fact that model parameters can vary around some nominal known value; these variations are upper and lower bounded by known limits. Thus, the center of the parallelotpe corresponds to the nominal values that the parameters can take, and its sides correspond to the bounds on the maximum and minimum parameter values. A minimum-volume ellipsoid is constructed to upper bound this parallelotope. Then, by using set operations, this ellipsoid is propagated through a second-order Taylor series expansion of the Markov chain stationary distribution.[1] This approach facilitates computation of approximate bounds on reliability and performability indices that arise from the Markov chain stationary distribution. The Taylor series coefficients are evaluated only once for the nominal values that the parameters take, and therefore the approach is computationally inexpensive compared to repeated simulations, i.e., computing the relevant indices for all possible parameter values. A significant contribution of this work is a numerical method to propagate ellipsoidal sets through second-order polynomial systems. Previous work in this area has been largely focused on linear systems [4]. This fact is relevant, as second-order Taylor-series expansions enable studies involving the impact of larger uncertainties in parameter values, and as demonstrated in the case studies and examples, provide more accurate bounds than those obtained with first-order approximations only.

Methods to assess the impact of unknown-but-bounded parametric uncertainty that exploit the sensitivity of the Markov chain transient solution to model parameters are proposed in [1], [2]. By contrast, we focus directly on the stationary distribution of ergodic Markov chains employed in modeling repairable systems [3]. It is worth noting that the sensitivities of the stationary distribution could be computed following the methods outlined in [1], [2], before applying the set-theoretic methods we propose based on propagating ellipsoidal-shaped sets to bound the reliability and reward metrics.

Techniques based on interval arithmetic (see, e.g., [6]) have been proposed in [7], [8]. In these methods, the unknown parameters are assumed to belong to an interval which is propagated through the Markov model using methods from interval arithmetics. Parameters can also be modeled as fuzzy sets to derive membership functions of the reliability and performability indices of interest (see, e.g., [9], and the references therein).

The main advantage of our method is that the Markov chain generator matrix is the only required input, i.e., closed-form expressions for the stationary distribution or performability indices

[1]Ellipsoids are preferred instead of the original parallelotope that they upper bound because set operations with ellipsoids involve simple matrix algebra. We will elaborate on this aspect later.

as a function of the model parameters are not required a priori. Based on these features, the proposed method is best suited to analyze parametric uncertainty in multi-state, multi-parameter Markov models, where closed-form expressions for the relevant indices are difficult to obtain, and exhaustive simulation of all possible parameter values is computationally expensive. We demonstrate the application of the proposed method with two case studies: i) a two-component shared-load system with common-cause failures, and ii) an electric-power distribution transformer with deterioration and preventative maintenance. In the first case study, we quantify the impact of parametric uncertainty on a notion of reward defined for the two-component system; and in the second, we explore the optimal preventative maintenance rate to maximize transformer availability, while taking into account the effect of parametric uncertainty. We also compare the execution time of the proposed method with exhaustively simulating all possible parameter values as the model order grows.

The remainder of this paper is organized as follows. In Section II, we describe basic notions of Markov reliability and reward models, and pose the problem we address in the paper. The method to propagate ellipsoidal sets through the Taylor-series expansion of the Markov chain stationary distribution is explained in Section III. The case studies described above are presented in Section IV. Concluding remarks and directions for future work are discussed in Section V.

II. Preliminaries and Problem Formulation

In this section, we provide a brief introduction to Markov reliability and reward models. For a detailed treatment of these topics, the reader is referred to [10]–[12]. We then pose the problem of concern in this paper, which is to capture the effect of unknown-but-bounded parametric uncertainty on the stationary distribution of a Markov chain.

A. Markov Reliability and Reward Model Fundamentals

Note that the discussion below follows from [3], [12]. Let $X = \{X(t), t \geq 0\}$ denote a stochastic process taking values in a countable set $\mathcal{M}$. The stochastic process X is called a continuous-time Markov chain if it satisfies the Markov property

$$\Pr\{X(t_n) = i | X(t_{n-1}) = j_{n-1}, \ldots, X(t_1) = j_1\} = \Pr\{X(t_n) = i | X(t_{n-1}) = j_{n-1}\}, \quad (1)$$

for $t_1 < \ldots < t_n$, and $i,\, j_1, \ldots,\, j_{n-1} \in \mathcal{M}$ [12]. The chain X is said to be *homogeneous* if it satisfies

$$\Pr\{X(t) = i | X(s) = j\} = \Pr\{X(t-s) = i | X(0) = j\}, \ \forall i,\, j \in \mathcal{M},\, 0 < s < t. \quad (2)$$

The homogeneity of X implies that the times between transitions are exponentially distributed. The chain X is said to be *irreducible* if every state in the chain is accessible from every other state.

In this paper, we consider the class of continuous-time Markov chains that are homogeneous, irreducible, and take values in a finite set $\mathcal{M} = \{0,\, 1,\, 2, \ldots,\, n\}$, where $0,\, 1,\, 2, \ldots,\, n-1$ index system configurations that arise due to component faults, and n indexes the nominal, non-faulty system configuration. Let X denote a chain belonging to this class; then, because the chain is irreducible and takes values in a finite set, it follows that X is *ergodic*, i.e., it has a unique stationary distribution statistically independent of initial conditions [11].

Define $\tilde{\pi}_i(t) \equiv \Pr\{X(t) = i\}$, and denote the vector of occupational probabilities by $\tilde{\pi}(t) = [\tilde{\pi}_0(t),\, \tilde{\pi}_1(t), \ldots,\, \tilde{\pi}_n(t)]^T$. From the Chapman-Kolmogorov equations (see, e.g., [12]), it follows that

$$\dot{\tilde{\pi}}(t) = \Lambda^T \tilde{\pi}(t), \quad (3)$$

with $\tilde{\pi}_n(0) = 1$, $\tilde{\pi}_i(0) = 0$, $i = 0,\, 1, \ldots,\, n-1$, and where $\Lambda \in \mathbb{R}^{n+1 \times n+1}$ is the Markov chain generator matrix whose entries are a function of component failure and repair rates [10]. The steady-state solution of (3) is referred to as the stationary distribution of the chain. It is denoted by π, and is obtained as the solution of

$$\Lambda^T \pi = 0, \quad \pi^T e = 1, \quad (4)$$

where $e \in \mathbb{R}^{n+1}$ is a column vector with all entries equal to one.

We assume that the generator matrix is a function of m model parameters denoted by $\theta_j,\ j = 1,\, 2, \ldots,\, m$. To explicitly represent parametric dependence, the generator matrix, and stationary distribution are expressed as $\Lambda(\theta)$, and $\pi(\theta) = [\pi_0(\theta), \ldots,\, \pi_n(\theta)]^T$, respectively, where $\theta = [\theta_1,\, \theta_2, \ldots, \theta_m]^T$.

We can quantify a notion of performance by defining a *reward function* $\varrho : \mathcal{M} \to \mathbb{R}$ that maps each state $i = 0,\, 1,\, 2, \ldots,\, n$ into a real-valued quantity ρ_i which quantifies system performance while in state i [11]. A long-term measure of system performance can be described by the *reward*

$$\xi(\theta) = \sum_{i=0}^{n} \rho_i \pi_i(\theta) = \pi(\theta)^T \rho, \tag{5}$$

where $\rho = [\rho_0,\, \rho_1, \ldots,\, \rho_n]^T$ is the *reward vector* [13]. In the case studies, we demonstrate how reliability and performability indices of interest can be recovered by appropriately formulating the reward vector.

B. Problem statement

We assume that the model parameters, $\theta_j,\ j = 1,\, 2, \ldots,\, m$, are not perfectly known, but are constrained to a range. The parameter vector can be expressed as $\theta = \bar{\theta} + \Delta\theta$ where $\bar{\theta}$ is the vector of nominal parameter values, and $\Delta\theta \in \mathcal{X} \subseteq \mathbb{R}^m$ where $\mathcal{X}$ is a parallelotope defined as

$$\mathcal{X} \equiv \left\{ \Delta\theta : \left| \kappa_i^T \Delta\theta \right| \leq 1,\, \forall i = 1, \ldots,\, m \right\}. \tag{6}$$

The vertices of $\mathcal{X}$ are determined by the parameter value ranges, while the vectors κ_i define the edges of $\mathcal{X}$ [14]. Given this unknown-but-bounded parametric uncertainty model, we are interested in characterizing the uncertainty in the stationary distribution $\pi(\theta)$. In particular, we are interested in determining the set $\mathcal{Y}$, such that $\Delta\pi = \pi - \bar{\pi} \in \mathcal{Y} \subseteq \mathbb{R}^{n+1}$, where $\bar{\pi} \equiv \pi(\bar{\theta}) = [\bar{\pi}_0,\, \bar{\pi}_1,\, \bar{\pi}_2,\, \ldots,\, \bar{\pi}_n]^T$ is the stationary distribution corresponding to the nominal parameter values. Notice that the reward $\xi(\theta) = \pi(\theta)^T \rho$ is a linear projection of the stationary distribution $\pi(\theta)$ onto the direction specified by the vector ρ. Therefore, to bound the values that the reward can take, we need to obtain the set $\mathcal{Y}$, and then apply a linear transformation to recover a set $\mathcal{C} \in \mathbb{R}$ such that $\xi(\theta) = \pi(\theta)^T \rho \in \mathcal{C}$. The brute-force solution to the problem discussed here is to repeatedly compute the stationary distribution and the associated reward (by solving (4), (5), respectively) for a range of parameter values in the set $\{\bar{\theta}\} \oplus \mathcal{X}$.[2] However, this approach is bound to be computationally expensive as the dimension of the state space n, or the number of model parameters m, increases. Therefore, in this work, we seek an analytical approach based on the Taylor-series expansion of the stationary distribution.

Example 1. Fig. 1 graphically describes the problem discussed above in the context of the two-state Markov reliability model for a single component with two operating modes: failed in state 0, and operational in state 1. The probability that the component is failed is given by $\pi_0(\lambda, \mu) = \lambda/(\lambda+\mu)$, and the probability that it is operational is given by $\pi_1(\lambda, \mu) = \mu/(\lambda+\mu)$, where λ is the component failure rate, and μ is the repair rate. The set $\mathcal{X}$ describes the values that the parameters λ and μ may take. We are interested in recovering the set $\mathcal{Y}$ that captures all values that the stationary distribution $\pi = [\pi_0, \pi_1]$ may take, due to uncertainty in the values of λ and μ. In the proposed method, we describe parametric uncertainty by an ellipsoidal set $\mathcal{E}$ (an upper bound to the set $\mathcal{X}$). Then, we recover the set $\mathcal{H}$ by propagating $\mathcal{E}$ through a second-order Taylor series expansion of $\pi_0(\lambda, \mu)$ and $\pi_1(\lambda, \mu)$ about the nominal values of λ, and μ ($\bar{\lambda}$, and $\bar{\mu}$, respectively). The set $\mathcal{H}$ captures variability in π_1 and π_0 up to second order. As hinted in the figure, if the parametric uncertainty is large, the set $\mathcal{H}$ may not upper bound the set $\mathcal{Y}$. In general, for multi-state, multi-parameter models, it is difficult to obtain closed-form expressions for the stationary distribution, let alone analytically probe the impact of parametric uncertainty on such expressions.[3] The proposed method addresses this problem by i) expressing the stationary distribution with a Taylor-series expansion as a function of the model parameters, and ii) providing a general method to propagate ellipsoidal-shaped sets through a second-order Taylor-series expansion.

[2] $\mathcal{A} \oplus \mathcal{B}$ denotes the vector (Minkowski) sum of the sets $\mathcal{A}$ and $\mathcal{B}$.

[3] To appreciate this aspect, readers are referred to (28)-(30) in Section IV-A for closed-form expressions for the stationary distribution of a two-component shared-load system with common-cause failures.

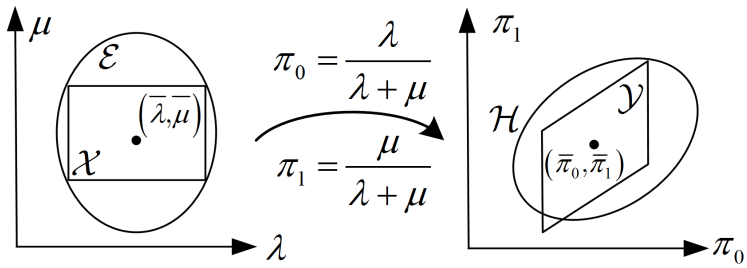


Figure 1. Illustrating the set-theoretic method for a two-state Markov reliability model.

III. Unknown-but-Bounded Uncertainty Modeling

In this section, we characterize the Markov-chain stationary-distribution Taylor-series expansion. Then, we propose methods to propagate ellipsoidal-shaped sets through first- and second-order Taylor series expansions of the Markov chain stationary distribution.

A. Taylor-Series Expansion of the Stationary Distribution

To characterize uncertainty in the entries of the stationary distribution $\pi(\theta) = [\pi_0(\theta),\, \pi_1(\theta), \dots,\, \pi_n(\theta)]^T$, we can omit $\pi_0(\theta)$, and only consider the other n (out of the $n+1$) entries of $\pi(\theta)$. We can do this because $\pi_0(\theta) = 1 - \sum_{i=1}^{n} \pi_i(\theta)$. For small perturbations around the nominal parameter values, $\pi_i(\theta),\, i = 1,\, 2, \dots,\, n$, can be approximated by a second-order Taylor series expansion

$$\pi_i(\theta) = \pi_i(\bar{\theta}) + \Delta\pi_i \approx \bar{\pi}_i + \nabla\pi_i(\bar{\theta})\Delta\theta + \frac{1}{2}\Delta\theta^T\nabla^2\pi_i(\bar{\theta})\Delta\theta, \tag{7}$$

where $\Delta\theta = [\Delta\theta_1,\, \Delta\theta_2, \dots,\, \Delta\theta_m]^T = [\theta_1 - \bar{\theta}_1,\, \theta_2 - \bar{\theta}_2, \dots,\, \theta_m - \bar{\theta}_m]^T \in \mathbb{R}^m$. In (7), the gradient $\nabla\pi_i(\bar{\theta})$, and Hessian $\nabla^2\pi_i(\bar{\theta})$, are given by

$$\nabla\pi_i(\bar{\theta}) = \left[\frac{\partial\pi_i(\bar{\theta})}{\partial\theta_1}, \frac{\partial\pi_i(\bar{\theta})}{\partial\theta_2}, \dots, \frac{\partial\pi_i(\bar{\theta})}{\partial\theta_m}\right], \tag{8}$$

$$\nabla^2\pi_i(\bar{\theta}) = \begin{bmatrix} \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_1^2} & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_1\partial\theta_2} & \dots & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_1\partial\theta_m} \\ \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_2\partial\theta_1} & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_2^2} & \dots & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_2\partial\theta_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_m\partial\theta_1} & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_m\partial\theta_2} & \dots & \frac{\partial^2\pi_i(\bar{\theta})}{\partial\theta_m^2} \end{bmatrix}. \tag{9}$$

Because the generator matrix Λ is singular, it is easy to verify that the entries of the matrices in (8)-(9) cannot be obtained by direct differentiation of (4). However, it has been shown in [15], [16] that the group inverse[4] of Λ, denoted by $\Lambda^{\#}$, is a powerful kernel to study the sensitivity of the stationary distribution to parameter variations. In particular, the sensitivities $\partial\pi_i(\bar{\theta})/\partial\theta_j$, $\partial^2\pi_i(\bar{\theta})/\partial\theta_j^2$, and $\partial^2\pi_i(\bar{\theta})/\partial\theta_j\partial\theta_k$, $\forall i = 0,\, 1, \ldots,\, n$, $\forall j,\, k = 1,\, 2, \ldots,\, m$, are given by

$$\frac{\partial\pi_i(\bar{\theta})}{\partial\theta_j} = -\pi(\bar{\theta})^T \frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#} e_i, \tag{10}$$

$$\frac{\partial\pi_i^2(\bar{\theta})}{\partial\theta_j^2} = 2\pi(\bar{\theta})^T \left(\frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#}\right)^2 e_i, \tag{11}$$

$$\frac{\partial\pi_i^2(\bar{\theta})}{\partial\theta_j\partial\theta_k} = \pi(\bar{\theta})^T \left(\frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#}\frac{\partial\Lambda}{\partial\theta_k}\Lambda^{\#} + \frac{\partial\Lambda}{\partial\theta_k}\Lambda^{\#}\frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#}\right) e_i, \tag{12}$$

where $\pi(\bar{\theta})$ is the stationary distribution evaluated at the nominal parameter values, and $e_i \in \mathbb{R}^{n+1}$ is a vector with 1 as the i entry and zero otherwise. The first, and second-order sensitivities have been derived in our previous work ([13], and [3], respectively). The expression for the mixed partial derivatives is a contribution of this paper; a short derivation of this result is included in the Appendix. Both the group inverse and the stationary distribution for the nominal parameter values can be obtained by a QR factorization of the generator matrix, Λ [13], [15]. A short discussion of this QR factorization method is included in the Appendix too. The nominal stationary distribution and the sensitivities have to be computed just once, which provides a complete characterization of the second-order Taylor-series expansion.

Recall from Section II-B that we assume the exact values the model parameters can take are unknown, but they lie within a parallelotope $\mathcal{X}$ centered around $\bar{\theta}$, i.e., $\theta = \bar{\theta} + \Delta\theta$, where $\Delta\theta \in \mathcal{X} \subseteq \mathbb{R}^m$. We are interested in propagating the set $\mathcal{X}$ through the system defined in (7) to obtain the set $\mathcal{Y}$ that contains all possible values that $\Delta\pi = [\Delta\pi_0, \ldots,\, \Delta\pi_n]^T$ can take. To address this problem, we build on results for unknown-but-bounded analysis in affine systems

[4]The group inverse of Λ is denoted by $\Lambda^{\#}$, and is given by the unique solution of i) $\Lambda\Lambda^{\#}\Lambda = \Lambda$, ii) $\Lambda^{\#}\Lambda\Lambda^{\#} = \Lambda^{\#}$, and iii) $\Lambda\Lambda^{\#} = \Lambda^{\#}\Lambda$, iff rank$(\Lambda)$ = rank(Λ^2), which is a condition that always holds for generator matrices of ergodic Markov chains [16].

[4], which provides a straightforward solution to the problem when (7) is truncated after the first-order term. A major contribution of our work is to extend these results and include the effect of unknown-but-bounded input uncertainty in second-order polynomial systems, i.e., it addresses the case where (7) is not truncated. This result is relevant because we cannot guarantee linear (or almost linear) behavior of $\pi(\theta)$ with variations of θ around the nominal value $\bar{\theta}$. As demonstrated in the case studies, second-order approximations are far more accurate.

The proposed method performs best when the stationary distribution is approximately a second-order function of the model parameters. Otherwise, the higher-order sensitivities (closed-form expressions are derived in [3]) can yield further insight into the impact of higher-order terms. Expectedly, the percentage uncertainty of the model parameters around the nominal value also has an impact on the accuracy of the results. Particularly, when $\mathcal{X}$ is small, the Taylor-series approximation is more accurate.

Note that the proposed method is modular, in the sense that one can choose to conduct the analysis by only using the first-order term, or include both the first- and second-order terms. Particularly, if the second-order sensitivities are small, one can choose to conduct the analysis by only using the first-order term. On the other hand, if the second-order sensitivities are large, one can also include the second-order term in the analysis.

B. First-Order Approximation

Consider (7) truncated after the first-order term, and express it in matrix form as

$$\pi(\theta) = \pi(\bar{\theta}) + \Delta\pi \approx \pi(\bar{\theta}) + J\Delta\theta, \tag{13}$$

where $J = \left[\partial\pi_i(\bar{\theta})/\partial\theta_j\right] \in \mathbb{R}^{n\times m},\ i = 1,\dots,n,\ j = 1,\dots,m$. Note that following the discussion in Section III-A, the first entry of the stationary distribution is omitted, but we persist with the same notation in the remainder of the manuscript. Given the uncertainty in the values that θ can take, we are interested in determining the values that π can take, i.e., we wish to characterize the set $\mathcal{Y}$ such that $\Delta\pi = [\Delta\pi_1,\dots,\Delta\pi_n]^T = \pi - \bar{\pi} \in \mathcal{Y} \subseteq \mathbb{R}^n$.

Assume that each entry in $\Delta\theta = [\Delta\theta_1, \Delta\theta_2,\dots,\Delta\theta_m]^T$ is constrained to a symmetric interval (around 0), which implies that the set $\mathcal{X}$ which contains all possible values of $\Delta\theta$ is a parallelotope. A parallelotope can be tightly bound by the intersection of a family of ellipsoids that satisfy some criteria, e.g., minimum volume or tightness along a given direction in the input space [17]. We bound the uncertain parameters by a single minimum-volume ellipsoid $\mathcal{E}$ as

$$\Delta\theta \in \mathcal{X} \subseteq \mathcal{E} = \left\{\Delta\theta : \Delta\theta^T \Psi^{-1} \Delta\theta \leq 1\right\}, \tag{14}$$

where Ψ is a positive definite matrix that determines the shape of $\mathcal{E}$. In particular, the eigenvectors of Ψ determine the orientation of $\mathcal{E}$, while the eigenvalues of Ψ determine the lengths of the semimajor axes of $\mathcal{E}$ in the direction of the corresponding eigenvectors [4]. The volume of the ellipsoid is proportional to $(\det \Psi)^{1/2}$; therefore, Ψ can be determined by solving the optimization program [18]

$$\begin{aligned} \min \quad & (\det \Psi)^{1/2} \\ \text{s.t.} \quad & v^T \Psi^{-1} v \leq 1,\ \forall v \in \mathcal{V}, \end{aligned} \tag{15}$$

where $\mathcal{V}$ is the set of vertices that define $\mathcal{X}$. The program in (15) can be efficiently solved using convex optimization techniques [18].

Define $\mathcal{F}$ as the set that bounds $\Delta\pi = J\Delta\theta$ resulting from variations in $\Delta\theta$ as described in (14), i.e., $\Delta\pi \in \mathcal{Y} \subseteq \mathcal{F}$. Then, it follows that $\mathcal{F}$ is an ellipsoid (see e.g., [4]) described by

$$\mathcal{F} = \left\{\Delta\pi : \Delta\pi^T \Gamma^{-1} \Delta\pi \leq 1\right\}, \tag{16}$$

where the shape matrix Γ is given by

$$\Gamma = J\Psi J^T. \tag{17}$$

C. Second-Order Approximation

Here, we extend the ideas presented in Section III-B to the second-order Taylor-series approximation in (7). As before, the set $\mathcal{X}$ that contains all possible values that $\Delta\theta$ can take is bounded

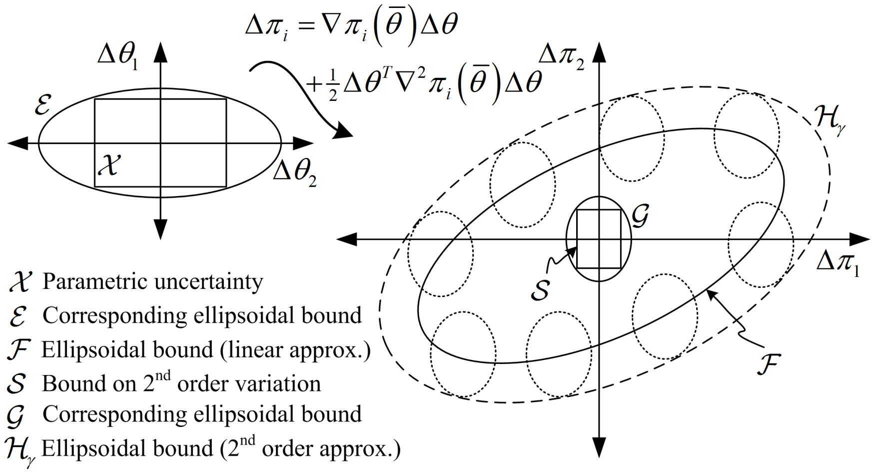


Figure 2. Ellipsoidal bounds for the Markov chain stationary distribution.

by a minimum-volume ellipsoid $\mathcal{E}$ as described in (14). Following the method in Section III-B, the linear component of (7), i.e., $J\Delta\theta$, is bounded by the ellipsoid $\mathcal{F}$ defined in (16). We handle the second-order term, i.e., $(1/2)\Delta\theta^T\nabla^2\pi_i(\bar{\theta})\Delta\theta$, as follows. First, for each i, we solve the optimization problems

$$\begin{aligned} \Delta\pi_i^{min} \;&=\; \min \;\; \frac{1}{2}\Delta\theta^T\nabla^2\pi_i(\bar{\theta})\Delta\theta \\ &\quad\;\; \text{s.t.} \quad \Delta\theta \in \mathcal{X}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \Delta\pi_i^{max} \;&=\; \max \;\; \frac{1}{2}\Delta\theta^T\nabla^2\pi_i(\bar{\theta})\Delta\theta \\ &\quad\;\; \text{s.t.} \quad \Delta\theta \in \mathcal{X}. \end{aligned} \tag{19}$$

Because we are interested in a worst-case bound, we can guarantee that $(1/2)\Delta\theta^T\nabla^2\pi_i(\bar{\theta})\Delta\theta \in \mathcal{S}_i$, where $\mathcal{S}_i = [-s_i,\, s_i]$, and $s_i = \max\{|\Delta\pi_i^{max}|\,,\, |\Delta\pi_i^{min}|\}$. Repeating this procedure for $i = 1, 2, \ldots, n$, we obtain a set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_n \subseteq \mathbb{R}^n$ that bounds the second-order term. Then, we can obtain a minimum-volume ellipsoid $\mathcal{G}$ that contains $\mathcal{S}$. The set $\mathcal{Y} \in \mathbb{R}^n$ (which contains $\Delta\pi$) can be upper bounded by the Minkowski sum of the ellipsoids $\mathcal{F}$ and $\mathcal{G}$, i.e., $\mathcal{Y} \subseteq \mathcal{F} \oplus \mathcal{G}$. In general, $\mathcal{F} \oplus \mathcal{G}$ is not an ellipsoid, but we can obtain a family of ellipsoids $\mathcal{H}_\gamma = \left\{\Delta\pi : \Delta\pi^T \Phi_\gamma^{-1} \Delta\pi \leq 1\right\}$ that upper bounds $\mathcal{F} \oplus \mathcal{G}$ by choosing

$$\Phi_\gamma = (1-\gamma)^{-1}\Sigma + \gamma^{-1}\Gamma, \quad 0 < \gamma < 1, \tag{20}$$

which ensures that $\mathcal{Y} \subseteq \mathcal{F} \oplus \mathcal{G} \subseteq \bigcap \mathcal{H}_\gamma$. The result in (20) follows from a special type of Holder's inequality as discussed in [4]. Fig. 2 illustrates the concepts introduced so far: a single ellipsoid $\mathcal{H}_\gamma$ bounding $\mathcal{Y}$ is depicted in the figure, whereas a family of ellipsoids bounding $\mathcal{Y}$ (note that $\mathcal{Y}$ is not depicted in the figure) may be obtained by varying γ between 0 and 1. The intersection of the ellipsoids in this family would yield a tighter upper bound to the set $\mathcal{Y}$.

D. Capturing Uncertainty with a Family of Tight Upper-Bounding Ellipsoids

So far, we have exclusively considered minimum-volume ellipsoids to bound the first- and second-order terms. Less conservative results can be obtained by considering a family of tight upper-bounding ellipsoids as described next.

First, consider the set $\mathcal{X}$ that captures all values the parameters can take. In addition to a minimum volume ellipsoid $\mathcal{E}$, define a family of ellipsoids $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_E\}$, where each $\mathcal{E}_i$ is tight to $\mathcal{X}$ along a direction specified by a unitary vector η_i.[5] The shape matrix Ψ_i of $\mathcal{E}_i$ can be obtained through the solution of the following optimization program

$$\begin{aligned} \min \quad & \sqrt{\eta_i^{\mathrm{T}} \Psi_i \eta_i} \\ \text{s.t.} \quad & v^T \Psi^{-1} v \leq 1, \, \forall v \in \mathcal{V}, \\ & \sqrt{\eta_j^{\mathrm{T}} \Psi_i \eta_j} \leq k_j, \, \forall j \neq i, \end{aligned} \tag{21}$$

where $\mathcal{V}$ is the set of vertices that enclose $\mathcal{X}$, and k_j is the maximum length of the semi-axis in the η_j direction [19]. The objective function is the projection of the ellipsoid $\mathcal{E}_i$ onto the η_i direction. The first inequality constraint enforces the set containment requirement $\mathcal{X} \subseteq \mathcal{E}_i$, while

[5]Notice that the set $\mathcal{E} \cap (\cap_i \mathcal{E}_i)$ is a tighter upper bound to $\mathcal{X}$ compared to just the minimum-volume ellipsoid, $\mathcal{E}$.

the second inequality ensures solvability of (21) by constraining the projection of $\mathcal{E}_i$ in all other directions (except that specified by η_i).

Following (16), we propagate each element in the family $\{\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_E\}$ through the first-order term. This results in a family of ellipsoids $\{\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_E\}$; the intersection of the ellipsoids in this family captures the output uncertainty up to first order, and it is less conservative that any single element in the family $\{\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_E\}$. Now, focused on the second-order term, recall that the set $\mathcal{S}$ describes the bounds on the second-order variations. In addition to the minimum volume upper-bounding ellipsoid $\mathcal{G}$, define a family of ellipsoids $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_G\}$, where each $\mathcal{G}_i$ is tight to $\mathcal{S}$ along a direction specified by a unitary vector ζ_i. Note that the $\mathcal{G}_i$ can be constructed by solving an optimization program similar to the one in (21). Finally, for all $\mathcal{F}_i \in \{\mathcal{F}_1, \ldots, \mathcal{F}_E\}$, $\mathcal{G}_j \in \{\mathcal{G}_1, \ldots, \mathcal{G}_G\}$, and appropriate values of γ, we obtain ellipsoidal upper bounds on $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}_j$, $\mathcal{F}_i \oplus \mathcal{G}$, and $\mathcal{F}_i \oplus \mathcal{G}_j$. These ellipsoidal upper bounds define a family of ellipsoids that we denote by $\{\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_H\}$, and their intersection, $\cap_i \mathcal{H}_i$, provides a tighter upper bound to the output uncertainty, compared to any individual $\mathcal{H}_i$ (which may include, e.g., the ellipsoid corresponding to the sum of the minimum-volume ellipsoids, i.e., $\mathcal{F} \oplus \mathcal{G}$).

We now illustrate the ideas presented in the sections above with two simple numerical examples. The first example examines a second-order system (for which the Taylor-series expression is exact), and therefore the method proposed in Section III-C provides an upper bound on all possible values that the output can take. Also, following the approach outlined in Section III-D, we demonstrate how less conservative results can be obtained by defining a family of ellipsoids to bound the parametric uncertainty and the second-order terms. The second example investigates a third-order system, in which case a second-order Taylor-series expansion is an approximation. Therefore, while the method outlined in Section III-C improves the linear approximation, the intersection of the $\mathcal{H}_\gamma$ does not provide, in general, an upper bound for all possible output variations. Note that the examples do not correspond to the stationary distribution of actual Markov chains, but are constructed primarily to illustrate the notation and the concepts introduced so far.

Example 2. Consider the second-order system

$$\begin{cases} \pi_1(\theta_1,\, \theta_2) = 2\theta_1^2 + 3\theta_1\theta_2 + \theta_2^2, \\ \pi_2(\theta_1,\, \theta_2) = \theta_1 + \theta_2 - 9\theta_1^2 - 9\theta_2^2. \end{cases} \tag{22}$$

Suppose the nominal parameter values are given by $\bar{\theta} = [\bar{\theta}_1,\, \bar{\theta}_2]^{\mathrm{T}} = [5,\, 1]^T$; then from (22), it follows that $\bar{\pi} = [\bar{\pi}_1,\, \bar{\pi}_2]^{\mathrm{T}} = [66,\, -228]^{\mathrm{T}}$. The Taylor-series expansion for the system in (22) is given by

$$\begin{cases} \Delta\pi_1 = \nabla\pi_1(\bar{\theta})\Delta\theta + \frac{1}{2}\Delta\theta^T\nabla^2\pi_1(\bar{\theta})\Delta\theta, \\ \Delta\pi_2 = \nabla\pi_2(\bar{\theta})\Delta\theta + \frac{1}{2}\Delta\theta^T\nabla^2\pi_2(\bar{\theta})\Delta\theta, \end{cases} \tag{23}$$

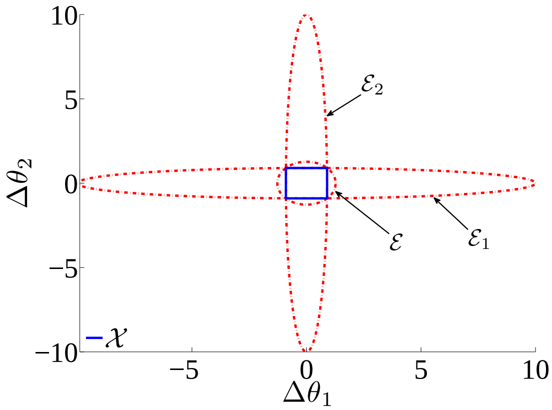
where $\nabla\pi_1(\bar{\theta})$, and $\nabla\pi_2(\bar{\theta})$ are given by

$$\begin{cases} \nabla\pi_1(\bar{\theta}) = [4\bar{\theta}_1 + 3\bar{\theta}_2,\, 3\bar{\theta}_1 + 2\bar{\theta}_2] = [23,\, 17], \\ \nabla\pi_2(\bar{\theta}) = [1 - 18\bar{\theta}_1,\, 1 - 18\bar{\theta}_2] = [-89,\, -17]; \end{cases} \tag{24}$$

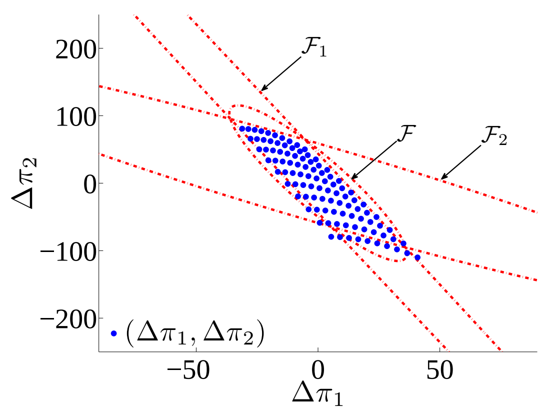
and $\nabla^2\pi_1(\bar{\theta})$, and $\nabla^2\pi_2(\bar{\theta})$ are given by

$$\nabla^2\pi_1(\bar{\theta}) = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix},\, \nabla^2\pi_2(\bar{\theta}) = \begin{bmatrix} -18 & 0 \\ 0 & -18 \end{bmatrix}. \tag{25}$$

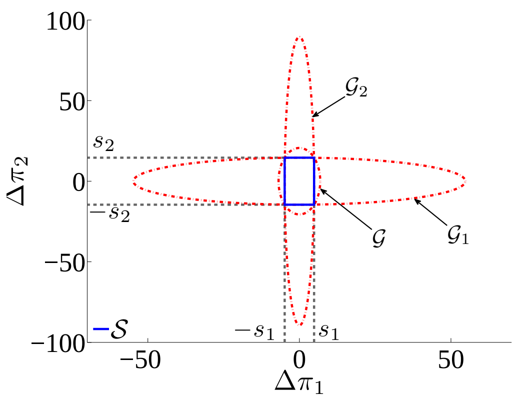
Suppose $\Delta\theta_1 = \Delta\theta_2 \in [-0.9,\, 0.9]$. The set $\mathcal{X}$ depicted in Fig. 3a is tightly bound by the intersection of the minimum-volume ellipsoid $\mathcal{E}$, and ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ that are tight to $\mathcal{X}$ along the directions specified by $\eta_1 = [1, 0]^{\mathrm{T}}$ and $\eta_2 = [0, 1]^{\mathrm{T}}$, respectively (these are also depicted in Fig. 3a). Corresponding to $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$, and assuming a linear approximation as in Section III-B, we obtain the bounding ellipsoids $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ that are depicted in Fig. 3b. The set $\mathcal{F} \cap \mathcal{F}_1 \cap \mathcal{F}_2$ captures the effect of the first order uncertainty in (23). In Fig. 3b, we also plot a cloud of points that results from solving (23) for all possible values of θ_1 and θ_2. As expected, because the system is second order, this bound fails to capture all possible values that $\Delta\pi$ can take. To improve the linear bound, we follow the procedure in Section III-D to determine an ellipsoidal bound for the second-order term in (23). First, the solution to (18)-(19) yields the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, and a corresponding minimum-volume ellipsoidal bound $\mathcal{G}$, all illustrated in



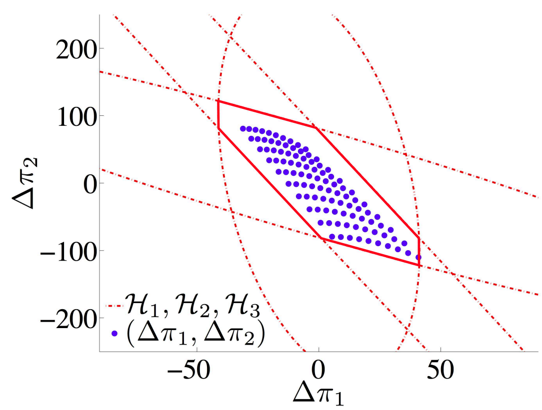
(a) Set that describes parametric uncertainty $\mathcal{X}$, and ellipsoidal bounds $\mathcal{E}$, $\mathcal{E}_1$, and $\mathcal{E}_2$.



(b) Exact solution to output uncertainty and ellipsoidal bounds obtained from linear approximation.



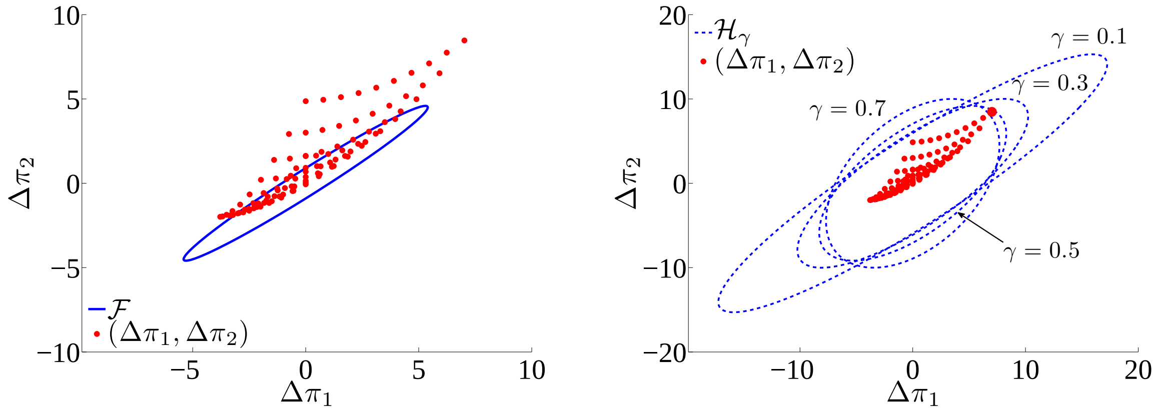
(c) Set that describes second-order uncertainty $\mathcal{S}$, and ellipsoidal bounds $\mathcal{G}$, $\mathcal{G}_1$, and $\mathcal{G}_2$.



(d) Exact solution to output uncertainty, and ellipsoidal bounds that capture second-order variations; note that $\cap_{i=1}^{3}\mathcal{H}_i \subset \mathcal{H}_i, \forall i = 1, 2, 3$.

Figure 3. Results for second-order system studied in Example 2.

Fig. 3c. In addition, we also depict the ellipsoids $\mathcal{G}_1$, and $\mathcal{G}_2$ that are tight to $\mathcal{S}$ in the directions specified by $\zeta_1 = [1, 0]^{\mathrm{T}}$, and $\zeta_2 = [0, 1]^{\mathrm{T}}$, respectively. Finally, for appropriate values of γ, we obtain a family of ellipsoids, $\{\mathcal{H}_i\}$, the elements of which upper bound each combination that results from the Minkowski sum of the $\mathcal{F}$, and the $\mathcal{G}$. Three of the ellipsoids in this set, i.e., $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$, and their intersection in a bold red line, are shown in Fig. 3d, superimposed to the exact values that $\pi(\theta)$ can take. Because the function is a second order polynomial, (23) is exact. Therefore, each $\mathcal{H}_i$ captures the entire range of values that $\pi(\theta)$ can take, and the intersection provides a tighter bound.



(a) Exact solution, and ellipsoidal bound obtained from linear approximation.

(b) Exact solution, and ellipsoidal bounds that capture second-order variations.

Figure 4. Results for the third-order system studied in Example 3.

Example 3. Consider the third-order system

$$\begin{cases} \pi_1(\theta_1,\, \theta_2) = 2\theta_1 + \theta_2^2 + \theta_1\theta_2, \\ \pi_2(\theta_1,\, \theta_2) = \theta_1^2 + \theta_2^3. \end{cases} \tag{26}$$

Suppose θ_1 and θ_2 vary by up to 90% around their nominal values, $\bar{\theta}_1 = 1$, $\bar{\theta}_2 = 1$. Fig. 4a plots a cloud of points obtained by solving (23) for all possible values of θ_1 and θ_2, as well as the linear bound $\mathcal{F}$. As expected, this bound does not capture all possible values that $\pi(\theta)$ can take because the system is of third order. To improve the linear bound, we determine the ellipsoids $\mathcal{H}_\gamma$ for a set of values γ chosen from the range (0, 1). These are plotted superimposed in Fig. 4b. Because the system is of third order, we see that the intersection of the $\mathcal{H}_\gamma$, while significantly better than the first-order bound, fails to capture all possible values of $\pi(\theta)$; in particular, there is a single point (emphasized for clarity in Fig. 4b) that lies outside the intersection of the $\mathcal{H}_\gamma$. This example demonstrates that the validity of the proposed approach is contingent on how closely a second-order Taylor-series expansion approximates the original function.

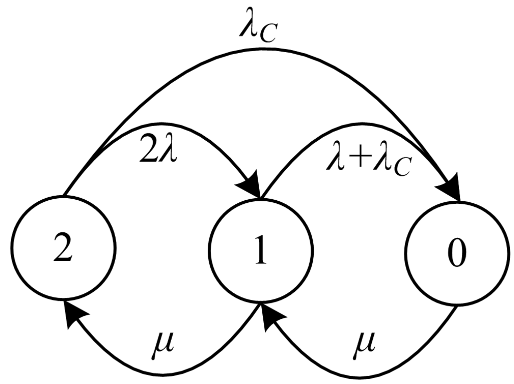


Figure 5. System of two identical components with shared load, and common-cause failures.

IV. CASE STUDIES

In this section, we examine the impact of parametric uncertainty in two reliability models described in the Introduction. In all the case studies, we compare the results obtained using the ellipsoidal-shaped sets with repeated simulations. The repeated simulations are performed as follows. We first create vectors with evenly spaced values for each parameter θ_j (about the nominal values $\bar{\theta}_j$) $\forall\ j = 1, \ldots, m$. The set $\{\bar{\theta}\} \oplus \mathcal{X}$, where $\bar{\theta} = [\bar{\theta}_1, \ldots, \bar{\theta}_m]$, describes the Cartesian product of all the θ_j. For each $\hat{\theta} \in \{\bar{\theta}\} \oplus \mathcal{X}$, we obtain the corresponding generator matrix $\Lambda(\hat{\theta})$ by substituting the corresponding values of the parameters. Then, through a QR factorization of $\Lambda(\hat{\theta})$, we obtain the stationary distribution of the chain $\pi(\hat{\theta})$ without having to solve the Chapman-Kolmogorov equations (for the specific $\Lambda(\hat{\theta})$ as $t \to \infty$). This approach is repeated for all elements in $\{\bar{\theta}\} \oplus \mathcal{X}$. For a large number of parameters m, or as the number of values in each θ_j is increased (to increase the accuracy of the results), this process can get computationally expensive, as demonstrated in the second case study.

A. Two Components with Shared Load

This example, adapted from [10], explores the Markov reliability model for a system of two identical components that share a common load. The failure rate, and repair rate of the components are denoted by λ, and μ, respectively. Additionally, the system is susceptible to common-cause failures which cause both components to fail at a rate λ_C. The state-transition diagram of the Markov chain describing the availability of this system is depicted in Fig. 5. Both components are operational in state 2, a single component is operational in state 1, and in state 0 both components are failed. Repairs restore the operation of one component at a time. From the state-transition diagram, the Markov chain generator matrix can be derived as

$$\Lambda = \begin{bmatrix} -\mu & \mu & 0 \\ \lambda + \lambda_C & -(\lambda + \lambda_C + \mu) & \mu \\ \lambda_C & 2\lambda & -(2\lambda + \lambda_C) \end{bmatrix}. \tag{27}$$

Solving (4) with Λ given in (27), it can be shown that [10]

$$\pi_0 = \frac{(\lambda + \lambda_C)(2\lambda + \lambda_C) + \lambda_C \mu}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}, \tag{28}$$

$$\pi_1 = \frac{(2\lambda + \lambda_C)\mu}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}, \tag{29}$$

$$\pi_2 = \frac{\mu^2}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}. \tag{30}$$

This result illustrates a major advantage of our proposed framework in that closed-form expressions of the sort in (28)-(30), which are difficult to obtain in general, are not required a priori. Additionally, even if the expressions are available, given the information that the parameters λ, μ, and λ_c belong to some set, it is difficult to compute bounds on the stationary distribution without repeatedly solving (4) for all possible parameter values in the set.

The nominal values of the failure rate, repair rate, and common-cause failure rate are given by $\bar{\lambda} = 1.6 \times 10^{-4}$ hr^{-1}, $\bar{\mu} = 1.25 \times 10^{-1}$ hr^{-1}, and $\bar{\lambda}_c = 2 \times 10^{-5}$ hr^{-1}, respectively [10]. The nominal steady-state probabilities are given by $\bar{\pi}_0 = 1.63 \times 10^{-4}$, $\bar{\pi}_1 = 0.0027$, and $\bar{\pi}_2 = 0.9971$. Let $\theta_1 = \lambda$, $\theta_2 = \mu$, $\theta_3 = \lambda_c$, and consider that each parameter $\theta_i \in [\bar{\theta}_i - (p/100)\bar{\theta}_i, \bar{\theta}_i + (p/100)\bar{\theta}_i]$, $i = 1, 2, 3$, where $\bar{\theta}_i$ is the nominal value of the i parameter, and p describes the % variation in the value that the i parameter θ_i can take. Figs. 6a, and 6b depict the sets $\mathcal{X}$ and corresponding upper-bounding minimum-volume ellipsoids $\mathcal{E}$ that the parameters are constrained to, assuming uncertainty $p = 20\%$, and $p = 30\%$, respectively. Figs. 7a, and 7b depict linear, and

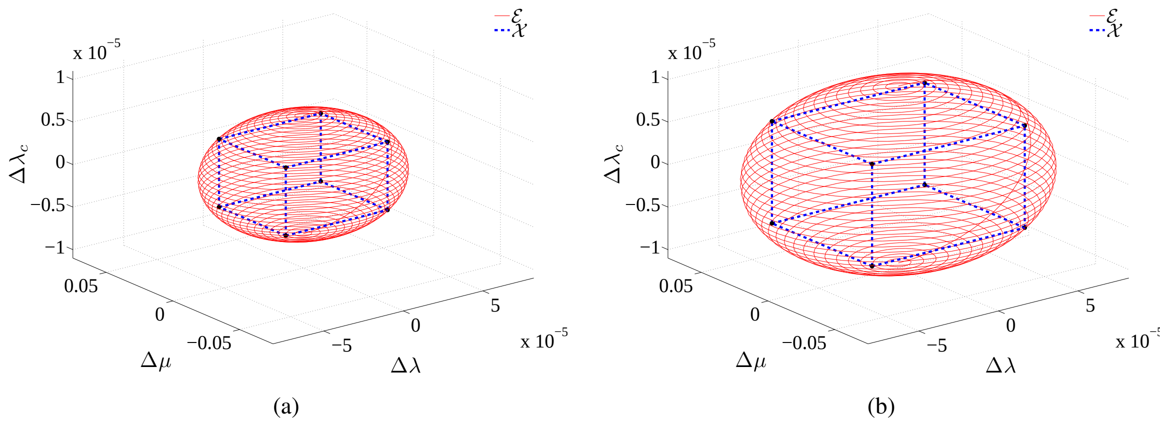


Figure 6. Ellipsoidal upper bounds to parameters with (a) 20%, and (b) 30% uncertainty.

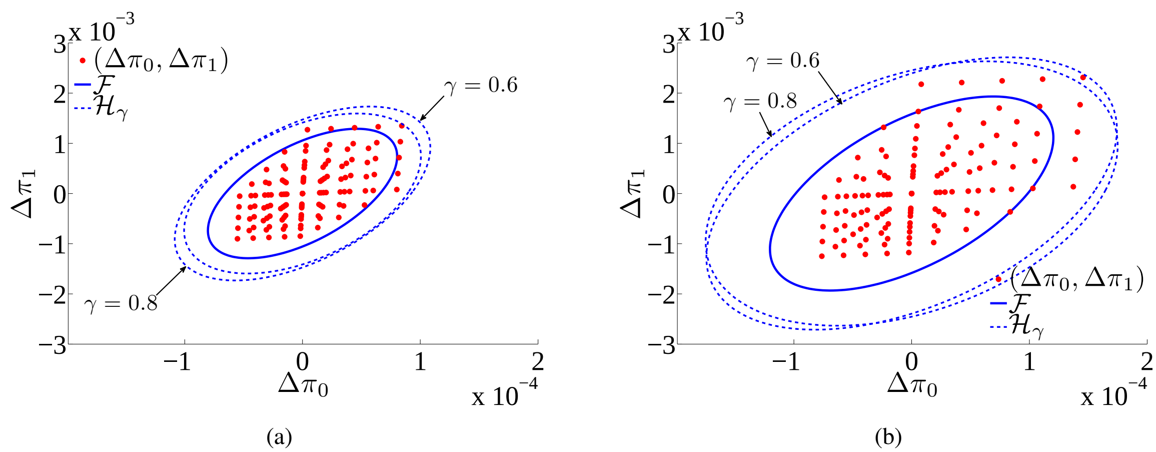


Figure 7. Linear, and second-order ellipsoidal bounds to the stationary distribution assuming (a) 20%, and (b) 30% uncertainty.

second-order ellipsoidal bounds ($\mathcal{F}$, and $\mathcal{H}_\gamma$, respectively) on the uncertainty in the steady-state probabilities. In both cases, we see that a linear approximation is insufficient. The intersection of the $\mathcal{H}_\gamma$ accurately captures all possible variations for 20% uncertainty in the parameters. For 30% uncertainty, there is a single point that lies outside the intersection of the $\mathcal{H}_\gamma$.

Now, suppose that the performance of the system depends on the number of operational components. To model system performance, define a reward model by choosing $\rho = [\rho_0,\ \rho_1,\ \rho_2]^T = [0,\ 1,\ 2]^T$. The long-term reward is then given by $\xi = \pi^T \rho = \pi_1 + 2\pi_2$. Because the parameters are uncertain, we can obtain bounds on ξ by simply projecting the ellipsoidal bounds for the stationary distribution onto the direction defined by ρ. Fig. 8 depicts upper, and lower bounds

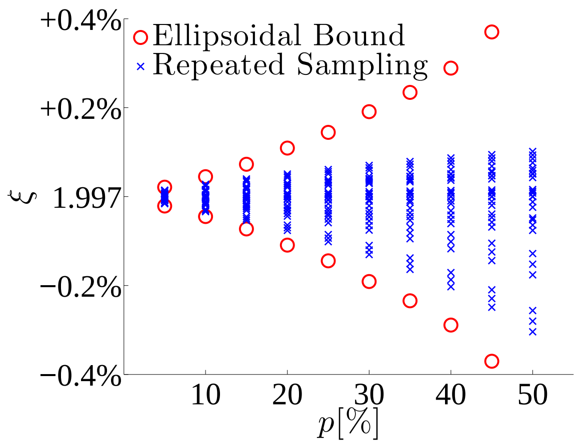


Figure 8. Uncertainty in reward as a function of parametric uncertainty.

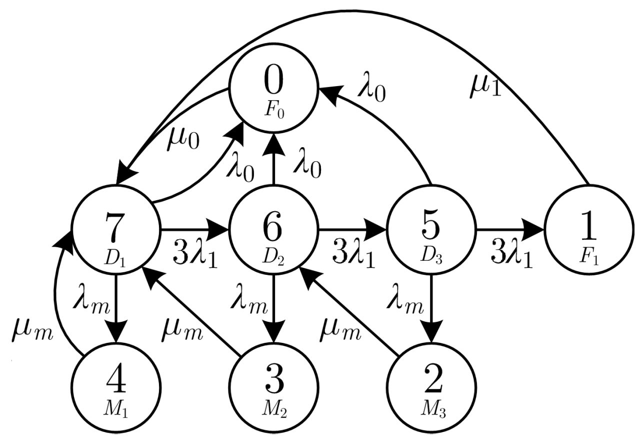


Figure 9. Reliability model for a transformer with deterioration, and preventative maintenance.

to the values that ξ can take as a function of the level of uncertainty p. Results from repeatedly solving (4)-(5) for all possible parameter values are superimposed. Notice that the ellipsoidal bounds accurately capture the impact of uncertainty on the values that ξ can take.

B. Preventative Transformer Maintenance

This example examines the preventative maintenance of an electric-power distribution transformer (see [20], [21], and the references therein for a detailed description of this model). Note that similar models have been used to study the impact of preventative maintenance in operational software systems [22]. The state-transition diagram that describes the Markov reliability model

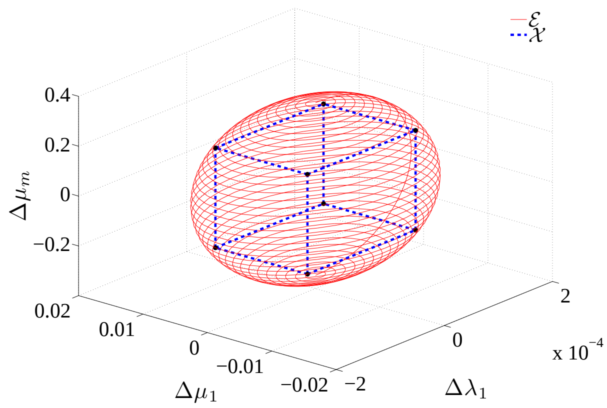


Figure 10. Ellipsoidal upper bound to parameters with 10% uncertainty.

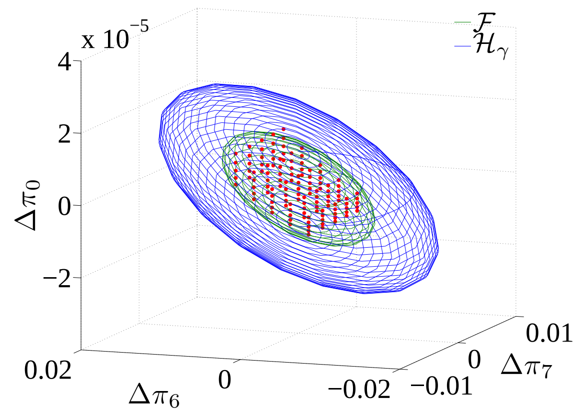


Figure 11. Ellipsoidal bounds to stationary distribution.

is depicted in Fig. 9. The transformer has an ideal operating state denoted by D_1, and two deteriorated states, denoted by D_2 and D_3. Denote the rate at which complete failure due to deterioration is expected by λ_1, which implies that transitions between the deterioration states occur at the rate $3\lambda_1$. Transformer failure due to deterioration is denoted by state F_1. Once in this state, repair at the rate μ_1 restores the transformer to the ideal operating state. Apart from gradual deterioration, a transition to a catastrophic failure state, denoted by F_0, at the rate λ_0 is possible from any of the deteriorated states. From this state, repair restores operation at a rate μ_0. Preventative maintenance can be performed on the transformer when it is in states D_i, $i = 1, 2, 3$. Preventative maintenance in state D_i ($i > 1$) restores operation to state D_{i-1} after passing through the maintenance state M_i. Preventative maintenance is performed at a rate λ_m, and the time required for maintenance is captured by μ_m. The availability of the transformer is given by the sum of the steady-state probabilities in states D_1, D_2, and D_3, i.e., $\xi = \pi_7 + \pi_6 + \pi_5$. Note that this availability can be expressed as $\xi = \pi^T \rho$, where π is the stationary distribution, and the reward vector $\rho = [0, 0, 0, 0, 0, 1, 1, 1]^T$. The problem of interest is to determine the optimal preventative maintenance rate λ_m that maximizes the availability, while taking into account the effect of parametric uncertainty. We show that the proposed method to uncertainty analysis can provide further insight into the problem.

The nominal parameter values are given by $\bar{\lambda}_1 = 1/1000\,\text{days}^{-1}$, $\bar{\mu}_1 = 1/14\,\text{days}^{-1}$, $\bar{\mu}_m =$

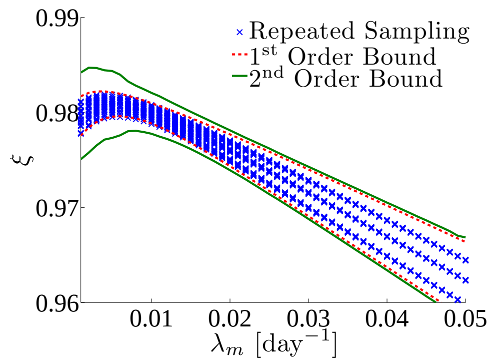


Figure 12. Transformer availability as a function of preventative maintenance rate assuming 10% parametric uncertainty.

$1/0.5\,\text{days}^{-1}$, $\bar{\lambda}_0 = 1/500\,\text{days}^{-1}$, and $\bar{\mu}_0 = 1/7\,\text{days}^{-1}$. Suppose the parameters λ_1, μ_1, and μ_m are unknown but bounded around their nominal values. Assuming 10% uncertainty around the nominal values, Fig. 10 depicts the set $\mathcal{X}$ (and corresponding upper-bounding minimum-volume ellipsoid $\mathcal{E}$) that contains all values that the parameters can take. Following the methods outlined in Section III-C, we determine bounds on the stationary distribution. Fig. 11 depicts linear, and second-order ellipsoidal bounds ($\mathcal{F}$, and $\mathcal{H}_\gamma$, respectively) on the values that the steady-state probabilities can take (without loss of generality, we just depict variability in π_7, π_6, and π_0). The exact solution, determined by repeated simulation, is superimposed on the ellipsoidal bounds. We see that a linear approximation is insufficient, and the second-order approximation provides a better (albeit conservative) bound. For 10% uncertainty around the nominal parameter values, Fig. 12 plots the availability of the transformer as a function of λ_m. From this figure, we see that a maintenance rate in the order of 0.005 days^{-1} maximizes the transformer availability. Notice that the second-order bound is more conservative, and in general lower bounds are more accurate than upper bounds.

Now, consider the state-transition diagram depicted in Fig. 9, except with an arbitrary number of deterioration states. We compare the execution time of the proposed method with the execution time involved in obtaining the solution by running repeated simulations as the model order is increased (i.e., as the number of deterioration states is increased). We utilize a first-order Taylor series expansion for this experiment because the programs in (18)-(19) are not

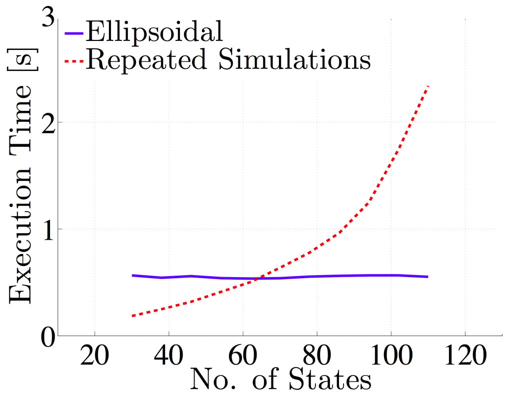


Figure 13. Execution time for the ellipsoidal method compared to exhaustive simulations as a function of model order.

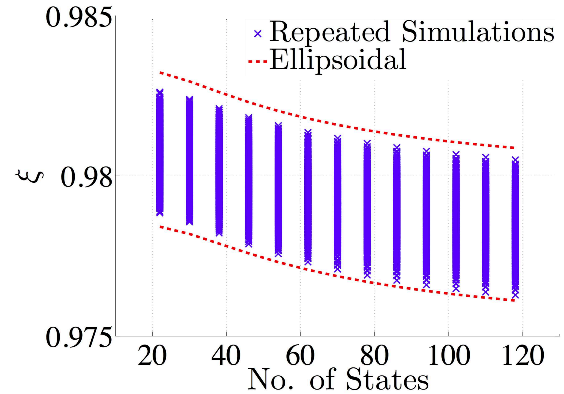


Figure 14. Ellipsoidal bounds and results from exhaustive simulations as a function of model order.

optimized for execution time (this is grounds for future work). Consider that all parameters λ_m, μ_m, λ_1, μ_1, λ_0, μ_0 are uncertain up to 5% around their nominal values. To perform the repeated simulations, for each parameter we sample the nominal value, and two extreme values. The experiment is performed on a PC with a 2.66 GHz Intel® Core™2 Quad CPU processor with 4 GB memory in the MATLAB® environment. The execution time as a function of the number of deterioration states for the two methods is plotted in Fig. 13. Fig. 14 superimposes the bounds obtained with the ellipsoidal method to the results of the repeated simulations. The results indicate that, for large models (as the number of states is increased beyond (approximately) 60 in this case), the proposed method has lower execution time compared to exhaustive simulation of all possible parameter values.

V. Concluding Remarks, and Directions for Future Work

A set-theoretic method based on the Taylor series expansion of the Markov chain stationary distribution is proposed to propagate parametric uncertainty to reliability and performability indices in Markov reliability models. The proposed method allows estimating bounds on reliability and performability indices of interest given bounds on the uncertain model parameters. The main advantage of the proposed framework is that only the generator matrix is required as the input. Additionally, the method eliminates the need for repeated simulations as the Taylor-series coefficients need to be evaluated just once at the nominal parameter values.

Future work could include tailoring the method to the case where more general sets are used to describe parametric uncertainty. Optimization of the methods proposed to handle the second-order terms for computational speed could also be explored further. In addition to parametric uncertainty analysis, the sensitivities and the ellipsoidal set propagation methods could be employed in other application areas. For example, similar to the methods in [2], we could explore identifying model aspects that are most prone to errors, as well as bottleneck analysis and optimal system design.

Appendix

Derivation of result in (12)

Consider that the ergodic CTMC is associated with a discrete time Markov chain (DTMC) whose distribution is governed by

$$p_{k+1} = p_k P, \tag{31}$$

where $P = I + \delta\Lambda$ is a row-stochastic, irreducible, primitive matrix (with an appropriate choice of δ). Define the matrix

$$A = I - P = -\delta\Lambda, \tag{32}$$

and denote the group inverse of A by $A^{\#}$. The stationary distribution of the DTMC satisfies $pA = 0$. Differentiating the expression $pA = 0$ with respect to the parameter θ_i, $i = 1, \ldots, m$ yields

$$\frac{\partial p}{\partial \theta_i} A + p \frac{\partial A}{\partial \theta_i} = 0. \tag{33}$$

Now differentiate (33) with respect to θ_j, $j \neq i$ to obtain

$$\frac{\partial^2 p}{\partial \theta_j \partial \theta_i} A + \frac{\partial p}{\partial \theta_i} \frac{\partial A}{\partial \theta_j} + \frac{\partial p}{\partial \theta_j} \frac{\partial A}{\partial \theta_i} + p \frac{\partial^2 A}{\partial \theta_j \partial \theta_i} = 0. \tag{34}$$

From the construction in (32), it is clear that the entries of A can be expressed as a linear combination of the model parameters $\theta_i, i = 1, \ldots, m$. Consequently, $\partial^2 A / \partial \theta_i \theta_j = 0, \forall i, j$, which allows us to simplify (34) as

$$\frac{\partial^2 p}{\partial\theta_j\partial\theta_i}A = p\frac{\partial A}{\partial\theta_i}A^{\#}\frac{\partial A}{\partial\theta_j} + p\frac{\partial A}{\partial\theta_j}A^{\#}\frac{\partial A}{\partial\theta_i}, \tag{35}$$

where we have used $\partial p/\partial\theta_i = -p(\partial A/\partial\theta_i)A^{\#}$ (this expression for the first-order sensitivity of the stationary distribution follows from [15]). Following along the lines of Theorem 3.2 in [15], it follows that

$$\frac{\partial^2 p}{\partial\theta_j\partial\theta_i} = p\frac{\partial A}{\partial\theta_i}A^{\#}\frac{\partial A}{\partial\theta_j}A^{\#} + p\frac{\partial A}{\partial\theta_j}A^{\#}\frac{\partial A}{\partial\theta_i}A^{\#}, \tag{36}$$

where $A^{\#}$ is the group inverse of A. Because the stationary distributions of the CTMC and DTMC match, we get

$$\frac{\partial\pi^2}{\partial\theta_j\partial\theta_i} = \pi^T\left(\frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#}\frac{\partial\Lambda}{\partial\theta_i}\Lambda^{\#} + \frac{\partial\Lambda}{\partial\theta_i}\Lambda^{\#}\frac{\partial\Lambda}{\partial\theta_j}\Lambda^{\#}\right). \tag{37}$$

Computing the stationary distribution, and group inverse

The stationary distribution π, and group inverse $\Lambda^{\#}$, are obtained by a QR factorization of the generator matrix Λ [15]. Factor Λ as $\Lambda = QR$, where, Q, $R \in \mathbb{R}^{n+1\times n+1}$. We have

$$R = \begin{bmatrix} U & -Ue \\ 0 & 0 \end{bmatrix}, \tag{38}$$

where $U \in \mathbb{R}^{n\times n}$ is a nonsingular upper-triangular matrix, and $e \in \mathbb{R}^n$ is a column vector with all elements equal to one. The stationary distribution is obtained by normalizing the last column of $Q = [q_1, q_2, \ldots, q_{n+1}]$, i.e.,

$$\pi = \frac{q_{n+1}}{\sum\limits_{i=1}^{n+1} q_{i,n+1}}. \tag{39}$$

The group inverse is related to Q and R as

$$\Lambda^{\#} = (I - e\pi^T)\begin{bmatrix} U^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^T(I - e\pi^T). \tag{40}$$

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