Correspondence

A Variable Threshold Page Procedure for Detection of Transient Signals

Z. Jane Wang and Peter Willett

***Abstract*—When employed to detect a transient change between known independent identically distributed populations, Page's test is easy to implement and provides reliable performance. However, its application to unknown transient changes is less clear. A Page test can be thought of as a repeated sequential test, and here we propose that each sequential test use a time-varying threshold. The idea is that *short* signals are detected quickly before post-termination data has a chance to refute them; and that evidence for a *long* signal is allowed to build, rather than being summarily discarded too early. Simulations results show that the performance of the proposed scheme comes close to tracing the "envelope" of fixed-style Page tests.**

***Index Terms*—Page test, sequential test, transient detection.**

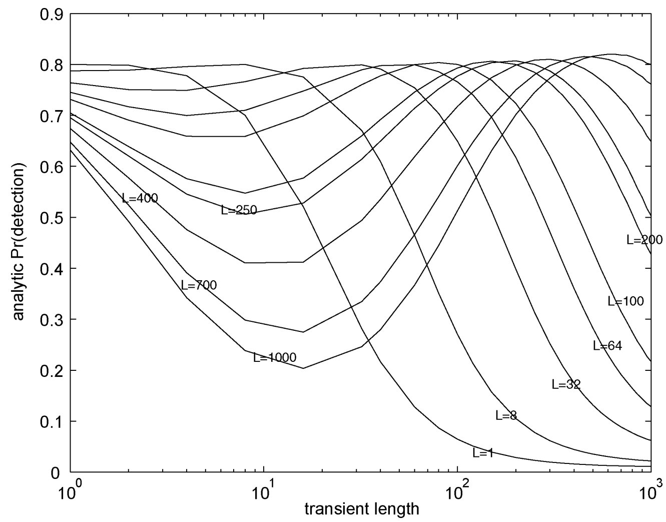

Fig. 1. Detectability of Gaussian shift-in-variance transients using fixed-style Page procedures designed for various but specific transient length L. For each L the transient's aggregate SNR is from Fig. 2, and provides constant detectability. In all cases, the fixed-style Page procedures are designed to provide $P_d = 0.8$, $\bar{T} = 10^6$ and $N = 10^3$. Here, the logarithmic scale is used to demonstrate the category of the transient signal in terms of length (e.g. very short, middle, and long).

I. Introduction

Many signal-processing applications require the detection of a statistical change occurring in a sequence of observed data [3]. If such a change occurs, the Page (or cusum) test [1] is optimal for this in the sense that it minimizes the worst-case average delay to detection given an average wait between false alarms [4]. Now consider further the situation of interest here: the detection of a *transient* signal, which amounts to an alert about a temporary change in distribution. A transient (or burst) can be thought of a two-sided change: the observations switch from obeying probability density function (pdf) f_0 to pdf f_1 at some unknown time n_s, and then switch back to f_0 after an interval n_d. We model this as

$$x_n \text{ has density} \begin{cases} f_0(x_n), & 1 \le n < n_s \text{ and } n_s + n_d \le n \le N \\ f_1(x_n), & n_s \le n < n_s + n_d \end{cases} \quad (1)$$

where the samples in the observation sequence are assumed independent, x_n is the sample at time n, and N is the sample length. Since n_s and n_d, which frame the transient, are not known, a generalized likelihood ratio test (GLRT) is often the choice: this is the Page procedure.

To distinguish it from the *variable threshold* Page (VTP) test proposed later, we refer to the standard one as the *fixed-style* Page test. The fixed-style Page test [1] forms

$$Z_n = max\,\{0, Z_{n-1} + g(x_n)\}\,, \text{ with } Z_0 = 0 \quad (2)$$

and follows a decision rule as

$$\begin{cases} \text{Declare detection,} & \text{if } Z_{n-1} + g(x_n) \ge h \\ \text{Continue test,} & \text{if } 0 < Z_{n-1} + g(x_n) < h \end{cases} \quad (3)$$

where h represents the threshold, and the function $g(x)$ is (asymptotically) optimally the log-likelihood ratio (LLR) $\log(f_1(x)/f_0(x))$ [4]. Suppose both f_0 and f_1 are known, then the Page test (2) using the LLR as $g(x)$ becomes a GLRT with respect to the unknown parameters n_s and n_d [7]. We can see that a Page test (2) is characterized by the update $g(x)$ and the threshold h. For instance, as shown in Section II, $g(x)$ takes the form $(x^2 - b)$ for the case of Gaussian shift-in-variance, thus the Page test is characterized by a threshold h and a bias b. For signal detection, it is desired to calculate the probability of detection (P_d) given a specific false alarm rate. Several techniques were explored to evaluate P_d by Page's test for a transient of a given length and strength in [7]. We are particularly interested in the FFT approach based on Markov-chain analysis since it can be extended to analyze the performance of the proposed adaptive Page test.

Naturally if appreciable information about the form of the transient is available, that information should be used. But if little is known about the transient save that some elevation of the power level is to be expected in a number of contiguous samples, it was found in [10] that a Page procedure based on the Gaussian shift-in-variance model is a simple and robust choice for detecting a wide variety of transient signals. It was also found in [10] that the loss, relative to the best "tuned" detector, of the use of such a Page procedure is not at all great. This is encouraging; however, it was noted [10] that the update was tuned commensurate with the transient's strength. Here we are interested in the detection of time-series segments (transients) with unknown location and strength: the segment's controlling pdf f_1 in (1) has the same form as the ambient pdf f_0, yet with a parameter λ that is unknown and whose magnitude controls the detectability of the transient. As shown in Fig. 1, direct application of Page's test is risky when information of the transient strength (and length) is unavailable: as indicated in Fig. 2, a long-and-quiet transient is better served by a small bias and a high

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Z. J. Wang is with the Electrical and Computer Engineering Department, University of British Columbia, Vancouver, BC, V6T 1Z4 Canada (e-mail: zjanew@ece.ubc.ca).

P. Willett is with the Electrical and Computer Engineering Department, University of Connecticut, Storrs, CT 06269-3157 USA (e-mail: willett@engr.uconn.edu).

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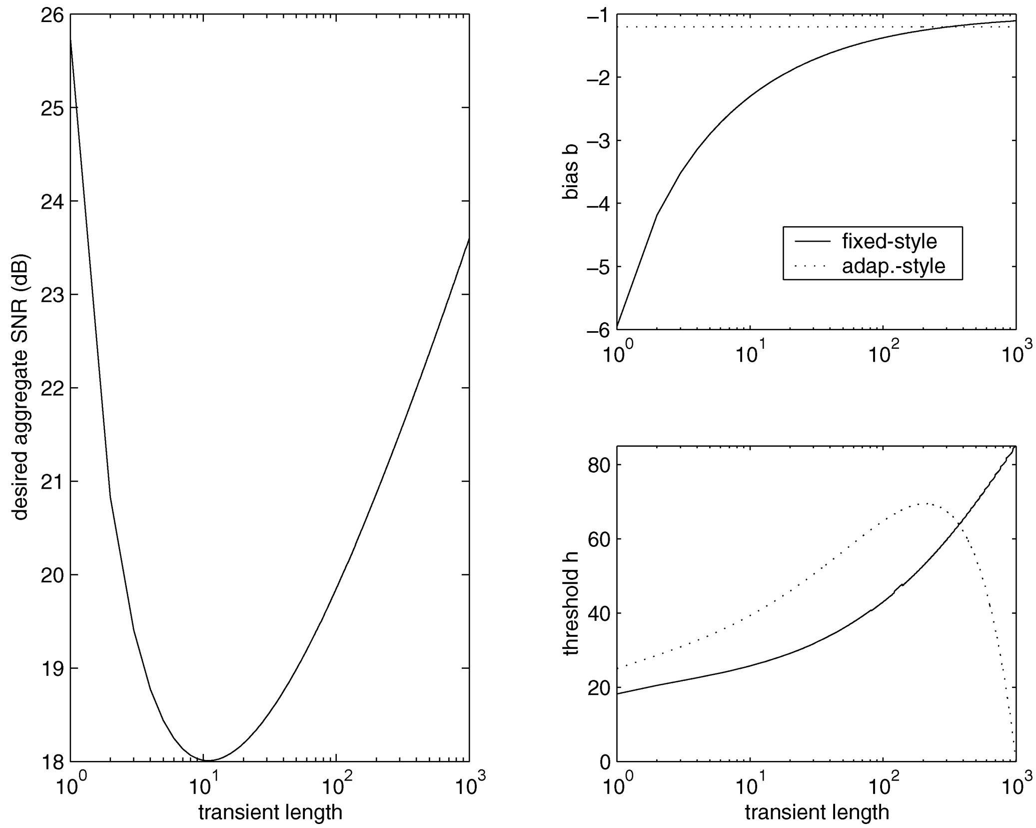

Fig. 2. Gaussian shift-in-variance case: the right plots give comparison of the bias and thresholds used in the new VTP procedure to those of the fixed Page schemes tailored to each specific transient length. In the latter case, note that a Page test designed for a transient length $L = 10$ uses approximate threshold and bias 25.78 and -2.31, respectively, *at all times*. The VTP scheme varies its threshold and bias dynamically as a function of the time since the last reset-to-zero, and thus we wish to stress that although the two pairs of curves are notionally similar, they are functionally completely different. The aggregate SNR S_L necessary to achieve $P_d = 80\%$ and $\bar{T} = 10^6$ in the fixed scheme is plotted in the left figure.

threshold but ill-served by the large bias and low threshold designed for short-and-loud transient signals; and vice versa.

We argue here that it is not a practically useful goal simply to be able to detect transients with a particular strength and length at least n_d. Particularly in passive surveillance applications it is desired to detect all transients. A close (and loud) acoustical source, for example, should be detectable even if it appears for a very short time; and while distant (and quiet) sources cannot be discerned quickly, one still does want them if they persist long enough for evidence to mount. The criterion for Page testing that represents false alarm performance is the average-time-between-false-alarms T [2], [7]. We propose that:

> A useful scheme should detect any transient whose overall detectability (P_d), parameterized by λ, exceeds a given value, given the average-time-between-false-alarms be at least $\bar{T}$.

It is reasonable to ask whether the fixed-style Page test can be improved by appropriate modifications.

- One might first estimate the transient power σ^2 and then applies a simple standard Page test designed with the LLR update. However, since the transient is of unknown length and location, accurately estimating σ_2 is nontrivial and probably needs some segmentation procedure. Further, this scheme requires revisit of past data (i.e., more calculation), and its detection performance is uncertain.
- A *bank* of Page tests—designed to cover a range of transient strengths and lengths—is an appealing option. Its performance, as indicated (but not precisely predicted) by the envelope of its constituent tests in Fig. 1, is good. We propose the VTP procedure since the Page-bank approach needs tuning in the number and location of its Page sub-tests, as the VTP test has performance that can be directly and precisely predicted, and because the VTP test is a *single* test and hence more efficient.
- In [13] a VTP-like heuristic was adopted, where a decision was made to absorb signal-to-noise ratio (SNR) behavior into both *threshold* and *bias*: both of these varied with the number of samples since the most recent Page reset, and both were increasing. It turns out that the initially-heavy bias of [13] penalized "long-and-quiet" transient signals too much. Therefore, the constant-bias VTP choice is preferable.

Since a short, loud burst is best served by a low threshold, while a long, quiet transient requires a high one, the VTP varies its threshold as a function of the time since its most recent "reset" in just that way. The VTP does not need the knowledge of transient power, and is lightly affected by transient length/strength trade-offs.

II. The Variable Threshold Page (VTP) Test

A. Motivation

We use the case of Gaussian shift-in-variance problem as the informing example, where we have

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$
$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}. \quad (4)$$

Based on the LLR, we use

$$g(x) = x^2 - \frac{\sigma^2 \log(\sigma^2)}{\sigma^2 - 1} \quad (5)$$

where the second term is the bias. Clearly, "λ" in this situation is σ^2, which is conveniently parameterized by L and S_L as $\sigma^2 = (S_L/L)+1$. Here, L is the length of the transient signal, and the aggregate SNR S_L is the total energy for a transient of length L. The corresponding bias is

$$b_L = \frac{\left(1+\frac{S_L}{L}\right)\log\left(1+\frac{S_L}{L}\right)}{\frac{S_L}{L}}. \tag{6}$$

for this length-L transient.

As noted earlier, the Page test is handicapped in the transient detection problem by the suitability of a heavy bias and low threshold for short-and-loud disturbances, versus that of a light bias and high threshold in the long-and-quiet case. Consequently we seek an *adaptive* test with *time-varying* biases and thresholds. However, not only are there many parameters, but it is also a multi-objective optimization—P_d for each transient length and strength is to be maximized, and we have already seen that the need to optimize these can and does conflict. Thus, we propose a heuristic by which a fixed bias and variable threshold sequence can be set.

The Page test with a constant bias b_c and time-varying thresholds $\{h(k)\}_{k=1}^N$ can be interpreted as composed of iterated generalized sequential probability ratio tests (GSPRTs). For a sequence $\{x_n\}_{n=1}^k$, a GSPRT [5] is given by

$$\tau_l(k) \le \log\left(\frac{f_1(x_1,\ldots,x_k)}{f_0(x_1,\ldots,x_k)}\right) \le \tau_u(k) \tag{7}$$

where $\tau_l(k)$ and $\tau_u(k)$ are the test thresholds that are functions of time k. In the above expression, if the upper (lower) threshold is violated, the test terminates with an acceptance of $H_1(H_0)$. We choose $\tau_l(k)=0$, and each "reset-to-zero" of the adaptive procedure means that the GSPRT is then restarted. If the adaptive Page's test resets to zero at n_s, then we consider the hypotheses at the k^{th} stage to be

$$\begin{aligned} H_k &: \{x_{n_s+1}, x_{n_s+2}, \ldots, x_{n_s+k}\} \\ &\quad \text{has a density } \Pi_{i=n_s+1}^{n_s+k} f_1(x_i|\lambda_k) \\ H_0 &: \{x_{n_s+1}, x_{n_s+2}, \ldots, x_{n_s+k}\} \\ &\quad \text{has a density } \Pi_{i=n_s+1}^{n_s+k} f_0(x_i) \end{aligned} \tag{8}$$

where the pdfs $f_1(\cdot)$ and $f_0(\cdot)$ are defined as in (4) and $\lambda_k = \sigma_k^2 = S_k/k + 1$ for the Gaussian shift-in-variance case. Using the bias b_k and the threshold h_k in the fixed-style Page test[1] designed for transients with length k, then presented in a form of an SPRT, the GSPRT of the k^{th} stage amounts to

$$\begin{aligned} 0 &\le \sum_{i=n_s+1}^{n_s+k} \log\left(\frac{f_1(x_i|\lambda_k)}{f_0(x_i)}\right) \\ &= \sum_{i=n_s+1}^{n_s+k} \frac{1}{2}\left(1-\frac{1}{\sigma_k^2}\right)\left(x_i^2 - b_k\right) \\ &\le \tau_u(k) \\ &= \frac{1}{2}\left(1-\frac{1}{\sigma_k^2}\right) h_k \end{aligned} \tag{9}$$

for detecting transients with length k. In our scheme, a constant bias is applied. Taking this into account, we have

$$0 \le \sum_{i=n_s+1}^{n_s+k} \left[x_i^2 - b_c + (b_c - b_k)\right] \le h_k \tag{10}$$

[1]We use h_k to refer to the (fixed) threshold used in a fixed-style Page test designed for a transient of length k. On the other hand, $h(k)$ denotes the threshold used by the VTP for the k^{th} sample since its last reset. We hope by this note to forestall confusion.

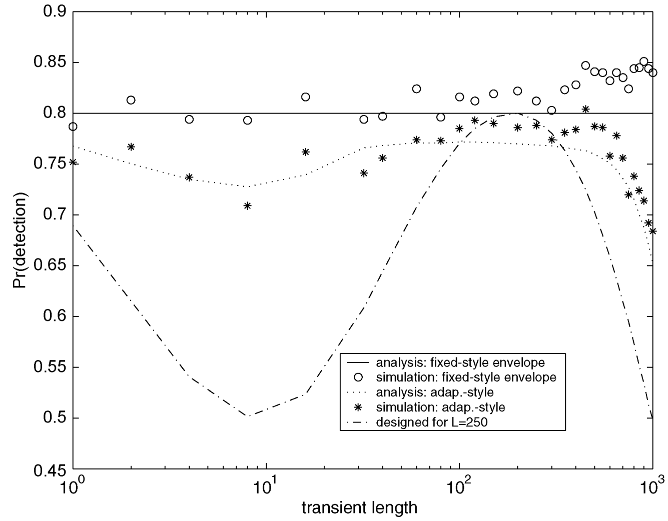

Fig. 3. Performance of the VTP scheme in Gaussian shift-in-variance transient problem. To demonstrate the precision of the FFT-based analytic prediction, the P_d envelope from the normal fixed-style Page procedure—that is, the performance that would be obtained from a Page test clairvoyantly tuned to the correct transient length and strength—and the result of the (not clairvoyant!) VTP test are also simulated based on 10^4 runs. The performance of the Page test optimized for transient length $L = 250$ is repeated from Fig. 1 for comparison only.

where b_c is the constant bias. Forcing the lower threshold to be zero, and subtracting $k(b_c - b_k)$ from the above expression to maintain the easy Page implementation, we have the test

$$0 \le \sum_{i=n_s+1}^{n_s+k} \left(x_i^2 - b_c\right) \triangleq L(n_s,k) \le h_k + k(b_k - b_c) \triangleq h(k) \tag{11}$$

meaning

$$\begin{cases} \text{Reset test and decide } H_0, & \text{if } L(n_s,k)\le 0 \\ \text{Stop test and decide } H_k, & \text{if } L(n_s,k)\ge h(k) \\ \text{Continue test to the } (k+1)\text{th stage}, & \text{otherwise.} \end{cases}$$

Therefore, for the VTP procedure in the case of the Gaussian shift-in-variance, we have

$$\begin{aligned} b &= b_c \\ h(k) &= h_k + k\left(\frac{\left(1+\frac{S_k}{k}\right)\log\left(1+\frac{S_k}{k}\right)}{\frac{S_k}{k}} - b_c\right) \\ &\text{for } k = 1,2,\ldots N. \end{aligned} \tag{12}$$

From Fig. 3, it is clear that this new VTP scheme works very well at detecting transients with unknown length and strength. The above development is applicable to more general models, such as the Gaussian shift-in-mean and exponential shift-in-scale cases.

B. VTP Design and Operation Summary

1) Select a performance level that is acceptable. That is, select $\bar{T}$, the average number of samples between false alarms; and select the probability of detection P_d beneath which an "alert" would be of unacceptably low fidelity. Example values here might be $\bar{T} = 10^6$ and $P_d = 80\%$.
2) It is assumed that the pdfs f_0 and f_1 differ only in a single parameter λ. Determine this parameter λ_k and a corresponding

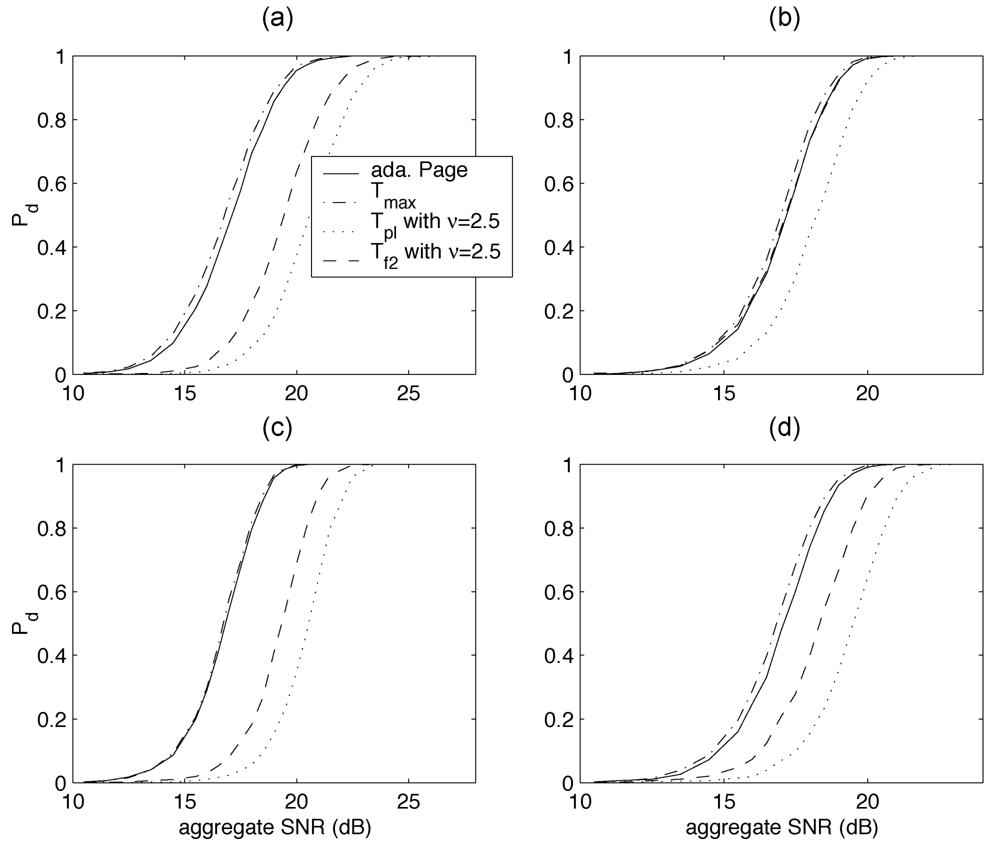

Fig. 4. Detection performances of the VTP scheme. The transient duration is $M = 30$ samples; different panels refer to different transient signals, with (a) transient signal s_1, (b) transient signal s_2, (c) transient signal s_3, and (d) transient signal s_4. Note: the "maximum" detector $T_{\max}$ is tuned to the true transient length $M = 30$.

threshold such that a fixed optimal Page test would detect a transient signal (of length k) in which f_1 has parameter λ_k with performance[2] $\bar{T}$ and $P_d, \forall k \in \{1, N\}$.

3) The VTP **implementation** is as a series of sequential tests according to

$$\begin{aligned} Z_n &= Z_{n-1} + g_0(x_n) - b \\ k &= k+1 \\ (Z_n \leq 0) &\longrightarrow \text{set}(k=0) \text{ and } (Z_n = 0) \\ (Z_n \geq h(k)) &\longrightarrow \text{Declare detection.} \end{aligned} \tag{13}$$

Therefore, the update is $g(x) = g_0(x) - b$, where $g_0(\cdot)$ is a fixed memoryless operation without the bias term. Note that n represents (absolute) sample number, while k refers to the number of samples since the last reset.

4) In (13), the $b = b_c$ is constant, and the variable threshold $\{h(k)\}$ is

$$h(k) = h_k + k(b_k - b_c) \tag{14}$$

for $k = 1, \ldots, N$, where b_k and h_k are the constant biases and thresholds designed for transients with length exactly k. The setting of the bias b_c is described in Section II-C.

5) To meet the $\bar{T}$ constraint,[3] the following recursive approach is used.

a) Set $\alpha = 1$, and $T_0 = T_{\rm des}/\alpha$, where $T_{\rm des}$ is the desired average time between false alarm $\bar{T}$.

b) Using the fixed-style Page scheme, with "minimum-detectable" λ_k (and thus b_k), based on the FFT approach introduced in [7], we design $\{h_k\}$, $k = 1, \ldots, N$, to satisfy $\bar{T} = T_0$ $(\forall k)$.

c) Then for the VTP scheme, the bias b and thresholds $h(k)$, $k = 1, \ldots, N$, are obtained as in (14). We calculate $\bar{T}$ via the FFT approach from the Appendix, and record it as T_1.

d) If $T_1 \approx T_{\rm des}$ (meaning the mismatch is small enough), stop. Otherwise, update $\alpha = T_1/T_0$, and $T_0 = T_{\rm des}/\alpha$, and go to step (b).

Normally, the above steps need only a few iterations.

The VTP idea is similar to [2] in which a good compromise bias was chosen for a fixed-style Page test. Here, however, we allow the threshold to vary.

C. About the Bias

From the experience in [13] it is tempting to choose a light bias, and to tune for performance via the threshold. However, unlike the fixed-style Page test the initial condition on the Page statistic at the beginning of a transient becomes important. Consider that the most

[2]Admittedly, this step requires considerable programming and iteration, using the FFT-based analysis procedure. For an user who does not wish to invest the time in that, one can select an approximate S_k from fixed-length test considerations, or even from central limit theorem arguments. However, this step is only required once and implemented in the off-line design stage.

[3]The user should be aware that the average number of samples between false alarms $(\bar{T})$ for the VTP scheme with the above bias and thresholds is not the same as for the "fixed" Page procedures designed for particular transient length L.

recent reset of the Page statistic was several samples before the transient's onset. On one hand, this means that the "initial" value of the Page statistic is nonzero, which would seem to imply that a detection will be hastened. On the other hand—and more importantly—it means that the time-varying threshold has already begun its ascent before the transient begins. The probability of a premature reset of the Page statistic climbs, particularly for a low-power transient, and naturally detectability suffers.

We note that a too-heavy bias hinders the detection of a long and quiet transient signal, while a too light bias hurts the detection of a short and loud transient. On one hand, we wish the bias b_c be light enough to satisfy the condition $\mathcal{E}(g(x)|H_1) > 0$ for all transients, where $\mathcal{E}$ means the expectation operation. On the other hand, we wish to make the bias b_c as heavy as possible, such that the VTP resets often in the absence of a transient, and consequently such that the most recent reset happens as near to the actual transient onset as possible. To counteract a heavy bias, a lowered threshold must be used; we naturally, however, demand that this threshold must always be positive. Therefore, assuming a maximum transient length N, our solution is to use

$$b_c = b_N + \frac{h_N}{N+1} \tag{15}$$

in which b_N and h_N are, respectively, the bias and threshold tuned for a fixed Page test designed for a transient of length N, where N is the assumed maximum transient length. Now, both this choice and its assumption of a maximum transient signal length N are open to challenge. However, it has been observed that there is little harm from choosing N quite large, for example $N = 1000$. Furthermore, although there may be a better strategy, the difference between that and (15) must be small, since as will be seen, the performance of VTP Page test almost traces that of the envelope of optimum tests "tuned" to the transient strength/length pairs.

III. Results for the Variance-Based VTP Test

In the following sections we discuss the performance of VTP with white Gaussian transients, and then task it to detect more types of signals.

A. VTP Test for the Case of Gaussian Shift-in-Variance

In the case of Gaussian shift-in-variance problem described earlier, the update is $g(x) = x^2 - b$, and the biases and thresholds are plotted in Fig. 2, where $N = 1000$, $P_d = 0.8$ and $\bar{T} = 10^6$. The detection performance of Page procedures with constant bias and threshold are shown in Fig. 1. It is noted that no fixed Page test provides constant detectability of equally-detectable transients. There is, however, a tendency for a Page test designed for an intermediate transient length (e.g., $L = 250$) to provide a good tradeoff for a wide range of signal strength and length; however, it is shown that the VTP scheme yields a performance improvement even over this. From Fig. 3, a clear improvement over a wide range of signal strengths is observed. The new VTP scheme achieves nearly the performance of the "envelope" (see Fig. 1) of all fixed Page tests designed for specific transient signal lengths. From the good match between analysis and simulation results, it appears that the Markov-chain analysis of [7] (synopsized in the Appendix) provides a good estimate of P_d when the number of quantization levels is 2^{13}.

B. More General Transient Types

The purpose of this section is to study the performance of the new VTP test for several types of transient signals. The signal model is

$$\begin{aligned} &H_0 : x(n) = w(n) \\ &H_1 : \begin{cases} x(n) = w(n), & n < n_s \text{ and } n \geq n_s + M \\ x(n) = s(n) + w(n), & n_s \leq n \leq n_s + M \end{cases} \end{aligned} \tag{16}$$

in which $w(n)$ is white Gaussian noise with zero mean and unit variance and $s(n)$ is the transient signal of interest. The transients are of short duration M compared with the observation length N. In our simulations, we always use $N = 128$, $M = 30$, $f_s = 16$, and $\lambda = 0.5$, unless stated otherwise. The transients are as follows.

1) S_1—White burst: The transient signal s_1 is white and Gaussian with zero mean.
2) S_2—Single exponentially-decaying sinusoid: $s_2(i) = e^{-(\lambda \cdot i/f_s)} cos(2\pi f \cdot i/f_s + \phi)$ with the phase, ϕ, randomly chosen from $[0, 2\pi]$, and the frequency, f, from $[1, f_s/4]$.
3) S_3—Exponentially-enveloped white burst: $s_3(i) = e^{-(\lambda \cdot i/f_s)} s_1(i)$.
4) S_4—Narrowband burst: s_4 is white Gaussian noise passed through a 0.3π bandwidth filter.

Certainly, this is not an exhaustive menu of transients, but we hope some typical examples are covered.

We apply the VTP detector to the above transients' detection. The b and $\{h(k)\}$ derived according to (14) are used, where $P_d = .8$, $\bar{T} = 10^6$ and $N = 128$. The assumptions on which the VTP procedure is built are those of S_1; the detector is weakly suited to S_3, and would seem to be ill-suited for either S_2 or S_4. To illustrate the performance of the VTP detector, we compare it to the basic and improved "power-law" statistics [9], [12], and the "maximum" detector [8]. In [10], it was found that the "maximum" detector $T_{\max}$ provides the best performance over a wide range of transients. In [6], it is shown (although not explicitly stated) that $T_{\max}$ is min-max optimal in the sense of Baygun and Hero for detection of a white burst of length at least M. The "power-law" statistic T_{pl} [9] works particularly well for narrowband transient signals. In fact, several new power-law detectors were developed in [12]; for instance, we consider T_{f2} here by combining two adjacent FFT bins. It was found that for most practical transient signals T_{f2} is preferable to T_{pl}.

We plot P_d versus the aggregate SNR in Fig. 4, in which $P_{fa} = 10^{-4}$. It is noted that the VTP procedure provides very close performance to that of the "maximum" detector in all four situations, which is best in all cases as in [10]; we recall that M is tuned in the "maximum" detector (and the detector is sensitive to its choice) while the VTP test requires no such prior information. It is additionally noted that the VTP procedure provides performance superior to even the improved power-law detector T_{f2} in most cases, with the exception of S_2 (essentially a tie) in which the transient is highly narrowband.

IV. Summary

What tends to unite transient signals of practical interest is that they are an organized agglomeration of energy into contiguous (or nearby) time samples. Now, assuming a unit-variance ambient (as would be available after normalization), a transient detector that assumes nothing but this local scale-change—and one that is reasonably insensitive to other characteristics such as spectrum—is that based on the Page structure for a Gaussian shift-in-variance. This detector was previously shown to be robust against transient type; its disadvantage was the tuning that it needed in terms of the strength (power) of the transient. The VTP test developed here has removed even the need for that knowledge; the VTP procedure uses a constant bias, but has a threshold that changes with the number of samples since the most recent reset. Here the application focus is on a shift in variance between Gaussians and the VTP performs well, in many cases nearly tracing the "envelope" of performances achievable with the best Page processors tuned to each transient length; in [11] even greater improvement accrues in the mean-shift problems for Gaussian and exponential models. The VTP proposal is *ad hoc*, but apparently we could do little better.

APPENDIX
EVALUATING THE PERFORMANCE OF THE VARIABLE THRESHOLD PAGE TEST

We first briefly review the "FFT" approach introduced in [7] to obtain the detection performance P_d for a fixed-style Page test. Consider Page's test as an iterated sequential test with lower and upper thresholds 0 and h. Due to the update rule $Z_n = Z_{n-1} + g(x_n)$, the pdf of Z_n is expressed as

$$\psi_n(z) = f_{n-1}(z) * f_g(z) \tag{17}$$

where $f_g(.)$ is the pdf of the update $g(x_n)$, f_{n-1} denotes the pdf of Z_{n-1} given that the test has continued to time n, and $*$ denotes convolution—the convolution can be made both accurate and quick via a fast Fourier transform (FFT). Then, we compute

$$f_n(z) = \frac{\psi_n(z)}{\int_0^h \psi_n(z)dz}, \quad 0 < z < h \tag{18}$$

as a direct normalization. Based on such computed $\psi_n(z)$ and $f_n(z)$, we can easily calculate such quantities as $p_{on}(n) \equiv Pr$(ST will continue to time-step $n+1$), $p_0(n) \equiv Pr$(ST ends at n and decides H_0), and $p_1(n) \equiv Pr$(ST ends at n and decides H_1). Naturally, we have $f_0(z) = \delta(z)$ and $p_{on}(0) = 1$. As in [7], under the H_0 hypothesis, one can express $\bar{T}$ as

$$\bar{T} = \frac{E_0(N)}{\sum_{n=1}^{\infty} p_1(n)} + E_1(N) \tag{19}$$

where $E_i(N)$ is the expected number of samples to a decision for hypothesis H_i. Under the H_1 hypothesis, assuming the standard situation, we have

$$\begin{aligned} &P_d(n_d) \\ &\quad = \sum_{k=0}^{n_d-1} Pr(\text{detect, k resets}), \text{ with} \end{aligned} \tag{20}$$

$$\begin{aligned} &Pr(\text{detect, k resets}) \\ &\quad = \sum_{m=1}^{n_d} \sum_{n=1}^{n_d-m} p_1(n) p^{(k)}(m) \\ &p^{(k)}(m) \\ &\quad = \begin{cases} p_0(m) * p^{(k-1)}(m), & 1 \le m \le (n_d - 1), \\ 0, & \text{else} \end{cases} \end{aligned} \tag{21}$$

where n_d is the transient length. Using the above procedure we can calculate h_L for transient duration L ($L = 1, \ldots, N$) given $\bar{T}$.

The procedure to calculate the probability of detection for the VTP scheme is complicated by the fact that the Page statistic can (and often will) be nonzero at the onset of a change. Since the threshold index i plays an important role in the VTP scheme, a nonzero initialization could result in a different detection decision. It is thus necessary to calculate detection probabilities for different threshold indices i preceding the start point n_s. Overall, under the H_1 hypothesis, we have

$$P_d(n_d) = \sum_{i=1}^{\infty} p(i) P_d(n_d|i) \tag{22}$$

where i is the threshold index corresponding to the start point n_s. According to the definition, the probability mass function (pmf) $p(i)$ is decided by the characteristics of the test under H_0

$$p(i) = \frac{p_{on}(i-1|H_0)}{\sum_{n=0}^{\infty} p_{on}(n|H_0)} \tag{23}$$

where $p_{on}(n|H_0) \equiv Pr$(ST will continue to time-step $n+1|H_0$). Under the H_0 hypothesis, assuming $f_0(z) = \delta(z)$, with the update $g(\cdot)$ we can calculate the pdf $f_n(z|H_0)$ according to (18), and thus calculate $p_{on}(n|H_0)$ correspondingly.

For each index i, to calculate the corresponding $P_d(n_d|i)$ in (22), we also need to study stopping probabilities for the case $f_0(z) = f_{i-1}(z|H_0)$. We consider a decision rule in which Z_n is sequentially compared to the threshold $h(n+i-1)$, and based on this decision rule we compute the quantities

$$\begin{aligned} p_0^i(n) = Pr\ (&\text{ST ends at } n \text{ and decides} \\ &H_0|f_0(z) = f_{i-1}(z|H_0)) \\ p_1^i(n) = Pr\ (&\text{ST ends at } n \text{ and decides} \\ &H_1|f_0(z) = f_{i-1}(z|H_0)). \end{aligned}$$

For the case that $f_0(z) = \delta(z)$, we consider a decision rule in which Z_n is sequentially compared to the threshold $h(n-1)$, and compute the corresponding quantities $p_0^0(n)$ and $p_1^0(n)$. Now, using $p_0^i(n)$'s, $p_1^i(n)$'s, $p_0^0(n)$, and $p_1^0(n)$, we can calculate $P_d(n_d|i)$ as in (20), and finally calculate the overall $P_d(n_d)$ in (22). Theoretically, we should use ∞ as the upper bound of i in (22); however, we use a finite number in practical calculations.

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