Security-Constrained Unit Commitment for ac-dc Grids with Generation and Load Uncertainty

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***Abstract*—The uncertainties in renewable generators and load demand make it a challenge for system operators to execute the security-constrained unit commitment (SCUC) program in an ac-dc grid. The SCUC is a nonlinear mixed-integer optimization problem due to the power flow equations, constraints imposed by the ac-dc converters, and the binary variables associated with the generators' on/off states. In this paper, we study the SCUC problem in ac-dc grids with generation and load uncertainty. We introduce the concept of conditional value-at risk (CVaR) to limit the risk of deviations in the load demand and renewable generation. We relax the binary variables and introduce a l_1-norm regularization term to the objective function, and then use convex relaxation techniques to transform the problem into a semidefinite program (SDP). We develop an algorithm based on the iterative reweighted l_1-norm approximation that involves solving a sequence of SDPs. Simulations are performed on an IEEE 30-bus test system. Results show that the proposed algorithm returns a solution within 2% gap from the global optimal solution for the underlying test system. When compared with the multi-stage algorithm in the literature, our algorithm has a lower running time and returns a solution with a smaller gap from the global optimal solution.**

***Keywords*: security-constrained unit commitment, conditional value-at risk, l_1-norm regularization, semidefinite program.**

I. Introduction

The advancement in power electronic technologies in ac-dc converters has led to the revival of dc power for high voltage transmission, particularly for connecting off-shore wind farms to the power grid [1]. In an ac-dc grid with renewable energy sources, the *security-constrained unit commitment* (SCUC) program plays an important role to determine the set of operating generators with the minimum cost. In general, the objective of the SCUC problem includes the generators' operation cost and the system's losses [2]. The SCUC problem in an ac-dc grid is typically subject to power balance equations, power flow limits of the transmission lines, bus voltage limits, voltage and current limits of the ac-dc converters, as well as the generators' constraints such as the capacity limits, minimum on/off time requirements, and ramping up/down rate limits.

There are challenges in solving the ac-dc SCUC problem. First, the system operator typically uses the forecasted load demand and the renewable generation. However, the actual system condition may deviate from the presumed condition due to the uncertainties in the load demand and renewable generators. Thus, the system operator requires to take into account some corrective actions such as committing reserve generators. Second, the SCUC is a nonlinear mixed-integer optimization problem with nonconvex constraints imposed by the power flow equations and the ac-dc converters, as well as a large number of binary variables associated with the on/off states of the generators. Therefore, the SCUC is in the class of NP-hard problems [3] and is difficult to be solved optimally.

There have been some efforts in tackling the above challenges. We divide the related literature into two main threads. The first thread is concerned with the solution approaches for the deterministic SCUC problem. One approach is to divide the SCUC problem into a master problem that solves the unit commitment and some sub-problems that evaluate the feasibility of the network constraints. The nonlinear mixed-integer program (MIP) and Lagrange relaxation approach are generally used to formulate the master problem [4], [5]. Different techniques such as the Benders cut method [6], Newton-Raphson method [7], and branch-and-bound method [8] are used to solve the sub-problems. The multi-stage algorithms are not guarantee to converge to a good local optimal, since the unit commitment decisions from the first stage are fixed in the subsequent stages. Furthermore, the MIP and the Lagrange relaxation approach with large number of multipliers for the constraints generally suffers from the curse of dimensionality in large networks. Convex relaxation techniques such as quadratically-constrained quadratic program [9] and semidefinite relaxation [10] are the alternative approaches to solve the SCUC problem. The second thread is concerned with the uncertainties in the load demand and renewable energy generators. The scenario-based stochastic optimization is commonly used to address the uncertainty issues using the approximate probability distribution [11], [12] and the available historical data record [13] for the uncertain variables. Robust optimization methods [14]–[16] are used to address the unavailibility of probability distribution functions of the uncertain variables. Nevertheless, the robust optimization methods consider the worst-case scenario and ignore the severity of other possible scenarios.

In this paper, we focus on a convex relaxation technique incorporated with a risk minimization approach to solve the ac-dc SCUC problem with load and generation uncertainty. We transform the problem into a semidefinite program (SDP) and introduce the conditional value-at-risk (CVAR) [17] to minimize the likelihood of high deviations in the load demand and renewable generation. This paper is an extension of our previous work [18] by taking into account the unit commitment constraints, as well as the load and generation uncertainties.

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The main contributions of this paper are as follows:

- *Addressing the Uncertainty Issues*: To address the uncertainties in the load demand and renewable generator, we introduce a penalty based on CVAR to the objective function. It enables us to limit the risk of high deviations in the net power supply within a confidence level.
- *Novel Solution Approach*: Unlike most of the existing approaches (e.g., [4]–[8], [11]–[16]) that apply either dc or linearized ac power flow models, we consider the full ac power flow model to formulate the SCUC problem as a nonlinear mixed-integer optimization problem. We relax the binary variables and introduce a weighted l_1-norm regularization term to the objective function to enforce the relaxed variables to be either 0 or 1. We use convex relaxation techniques to transform the problem into an SDP, and develop an iterative reweighted l_1-norm approximation algorithm that solves a sequence of SDPs.
- *Performance Evaluation*: Simulations are performed on an IEEE 30-bus system connected to some dc grids. We show that for 1000 different initial conditions, the proposed algorithm returns a near-global optimal solution with 2% gap in all scenarios, and returns the near-global optimal solution with 1% gap in 98% of the scenarios. When compared with the multi-stage deterministic SCUC approaches (e.g., in [6] and [7]) in a number of test systems, our algorithm returns a solution with a smaller gap from the global optimal solution in a lower execution time. When compared with a robust multi-stage SCUC algorithm (e.g., in [14]–[16]), the optimal objective value with our algorithm is smaller, as it takes into account the likelihood of deviations in the net power supply instead of the worst-case scenario.

The rest of this paper is organized as follows. Section II introduces the system model for the ac-dc grid. In Section III, we formulate the SCUC problem and transform it into an SDP. We propose an iterative algorithm to solve the problem. In Section IV, we evaluate the perfomance of the proposed algorithm through simulations. Section V concludes the paper.

II. System Model

Consider an ac-dc grid comprising a set of buses $\mathcal{N}$ and a set of lines $\mathcal{L}$. The components operating on dc power (e.g., photovoltaic (PV) panels, dc loads) are connected to the ac network through three-phase voltage source converters (VSCs). Fig. 1 shows the schematic of a VSC station consisting of an ac-dc converter, phase reactor, ac filter, and transformer. For simplicity in the notations, we partition the set of buses into four distinct subsets: The set $\mathcal{N}_{\rm ac}^{\rm conv} \subseteq \mathcal{N}$ of ac side converter buses, the set $\mathcal{N}_{\rm ac} \subseteq \mathcal{N}$ of ac buses that are not connected to the converters, the set $\mathcal{N}_{\rm dc}^{\rm conv} \subseteq \mathcal{N}$ of dc side converter buses, and the set $\mathcal{N}_{\rm dc} \subseteq \mathcal{N}$ of dc buses that are not connected to the converters. Let $\mathcal{N}^{\rm conv} = \mathcal{N}_{\rm ac}^{\rm conv} \cup \mathcal{N}_{\rm dc}^{\rm conv}$ denote the set of all converter buses. For example, for the VSC station in Fig. 1, we have $s \in \mathcal{N}_{\rm dc}^{\rm conv}$, $k \in \mathcal{N}_{\rm ac}^{\rm conv}$, and $f \in \mathcal{N}_{\rm ac}$. In the following subsections, we introduce the models of the VSC, energy storage system, generator, load, and ac-dc network.

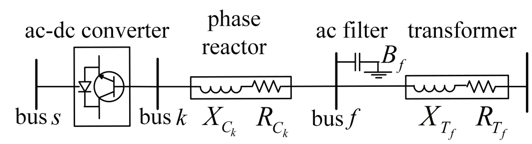

Figure 1. A VSC station schematic with dc converter bus $s \in \mathcal{N}_{\rm dc}^{\rm conv}$, ac converter bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$, and filter bus $f \in \mathcal{N}_{\rm ac}$.

A. VSC Station and Energy Storage System Models

Fig. 1 shows a VSC station with a two- or three-level converter using the pulse-width modulation switching method. In this figure, X_{C_k} denotes the phase reactor of the converter connected to ac bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ and R_{C_k} denotes the resistance modeling the losses of this phase reactor. Let $B_{C_k} = \frac{-X_{C_k}}{R_{C_k}^2 + X_{C_k}^2}$ denote the susceptance of the phase reactor connected to bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$. The ac filter connected to bus $f \in \mathcal{N}_{\rm ac}$ is modeled by the shunt susceptance B_f. The transformer connected to bus $f \in \mathcal{N}_{\rm ac}$ is modeled by its series reactance X_{T_f} and resistance R_{T_f} [19].

We divide the operation cycle into a set $\mathcal{T} = \{1, \ldots, T\}$ of T time slots with equal length. The losses in a VSC station in time slot $t \in \mathcal{T}$ can be approximated by a quadratic function of its ac current magnitude $|I_k(t)|$ injected into ac bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ [20]. Let $P_{{\rm loss},k}^{\rm conv}(t)$ denote the losses of the VSC station with ac bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ in time slot t. We have

$$P_{{\rm loss},k}^{\rm conv}(t) = a_k + b_k|I_k(t)| + c_k|I_k(t)|^2, \tag{1}$$

where a_k, b_k, and c_k are positive coefficients, which depend on the components and the power rating of the VSC station. Let $P_{C_k}(t)$ and $Q_{C_k}(t)$ denote the injected active and reactive power into ac converter bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ in time slot t, respectively. Let $P_{C_s}(t)$ denote the injected active power into dc converter bus $s \in \mathcal{N}_{\rm dc}^{\rm conv}$ in time slot t. The active power balance in time slot $t \in \mathcal{T}$ for the ac-dc converter connected to buses $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ and $s \in \mathcal{N}_{\rm dc}^{\rm conv}$ is

$$P_{C_k}(t) + P_{C_s}(t) + P_{{\rm loss},k}^{\rm conv}(t) = 0. \tag{2}$$

Let $V_k(t)$ denote the voltage at bus $k \in \mathcal{N}$ in time slot t. Let $I_k^{\max}$ and $V_k^{\max}$ denote the maximum current and voltage magnitude at bus $k \in \mathcal{N}$, respectively. Let $S_{C_k}^{\rm nom}$ denote the nominal value of the apparent power of the converter connected to bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$. In time slot $t \in \mathcal{T}$, the active and reactive power flow in a VSC station with converter bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ and filter bus $f \in \mathcal{N}_{\rm ac}$ are bounded by [21], [22]

$$Q_{C_k}(t) \leq |B_{C_k}| V_k^{\max}(V_k^{\max} - |V_f(t)|)\,, \tag{3}$$

$$-\, m_k^{\rm q}\, S_{C_k}^{\rm nom} \leq Q_{C_k}(t), \tag{4}$$

$$(P_{C_k}(t))^2 + (Q_{C_k}(t))^2 \leq (|V_k(t)| I_k^{\max})^2\,, \tag{5}$$

where $m_k^{\rm q}$ is a positive constant and can be determined by the type of the converter connected to bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ [21].

Let $m_{ks}^{\rm v}$ denote the maximum modulation factor of the converter connected to ac bus $k \in \mathcal{N}_{\rm ac}^{\rm conv}$ and dc bus $s \in \mathcal{N}_{\rm dc}^{\rm conv}$. In time slot $t \in \mathcal{T}$, we have [22]

$$|V_k(t)| \leq m_{ks}^{\rm v} |V_s(t)|. \tag{6}$$

Let $P_{B_k}(t)$ denote the active power injected into $(P_{B_k}(t) > 0)$ or absorbed from $(P_{B_k}(t) < 0)$ bus k by the battery in time slot t. The power rating of the battery has limits $P_{B_k}^{\min} < 0$ and $P_{B_k}^{\max} > 0$. That is

$$P_{B_k}^{\min} \le P_{B_k}(t) \le P_{B_k}^{\max}, \qquad k \in \mathcal{N},\, t \in \mathcal{T}. \tag{7}$$

If no battery is connected to bus $k \in \mathcal{N}$, then $P_{B_k}^{\min} = P_{B_k}^{\max} = 0$. Let $B_k^{\text{init}} \ge 0$ denote the initial energy level of the battery in bus $k \in \mathcal{N}$ at the beginning of the operating cycle. The stored energy in the battery until time $T' \le T$ is nonnegative and upper bounded by the limit $B_k^{\max}$. We have

$$0 \le B_k^{\text{init}} - \sum_{t=1}^{T'} P_{B_k}(t) \le B_k^{\max}, \;\; k \in \mathcal{N},\, T' \le T. \tag{8}$$

B. Generator and Load Models

We use binary variable $u_k(t) \in \{0, 1\}$ to indicate whether the generator in bus k is on $(u_k(t) = 1)$ or off $(u_k(t) = 0)$ in time slot t. The generation cost of a *committed* generator in bus k with generation level $P_{G_k}(t) \ge 0$ in time slot t can be approximated by a quadratic function $c_{k2}\,(P_{G_k}(t))^2 + c_{k1} P_{G_k}(t) + c_{k0}$, where c_{k0}, c_{k1}, and c_{k2} are nonnegative coefficients [23]. In addition, a generator in bus k has a fixed startup cost c_k^{su} when it is turned on and a fixed shutdown cost c_k^{sd} when it is turned off. We use a binary variable $s_k(t) \in \{0, 1\}$ to indicate whether the generator in bus k is started up $(s_k(t) = 1)$ or not $(s_k(t) = 0)$ in time slot t. We use a binary variable $d_k(t) \in \{0, 1\}$ to indicate whether the generator in bus k is shut down $(d_k(t) = 1)$ or not $(d_k(t) = 0)$ in time slot t. We have $s_k(t) = d_k(t) = 0$ if generator k is neither started up nor shut down in time slot t. Let vector $\boldsymbol{P}_G(t) = (P_{G_k}(t),\, k \in \mathcal{N})$ denote the generation profile of all generators in time slot t. We define variable vectors $\boldsymbol{u}(t) = (u_k(t),\, k \in \mathcal{N})$, $\boldsymbol{s}(t) = (s_k(t),\, k \in \mathcal{N})$ and $\boldsymbol{d}(t) = (d_k(t),\, k \in \mathcal{N})$ for all generators. The grid-wide operation cost of the generators in time slot t is

$$C_{\text{gen}}(\boldsymbol{P}_G(t), \boldsymbol{u}(t), \boldsymbol{s}(t), \boldsymbol{d}(t)) = \sum_{k \in \mathcal{N}} c_{k2}\,(P_{G_k}(t))^2 + c_{k1} P_{G_k}(t) + c_{k0}\, u_k(t) + c_k^{\text{su}} s_k(t) + c_k^{\text{sd}} d_k(t). \tag{9}$$

We have $c_{k0} = c_{k1} = c_{k2} = c_k^{\text{su}} = c_k^{\text{sd}} = 0$, when either a renewable generator or no generator is connected to bus k.

Let $P_{D_k}(t)$ and $Q_{D_k}(t)$ denote the active and reactive load components in bus k in time slot t, respectively. We assume that the reactive power uncertainty can be mitigated using the reactive power compensator. However, the scheduled output power $P_{G_k}(t)$ and the active load $P_{D_k}(t)$ for the next day may not match with the actual output power $\widehat{P}_{G_k}(t)$ and demand $\widehat{P}_{D_k}(t)$ in time slot t, respectively. An alternative is to study the deviation of the actual *net power supply* $\widehat{P}_{G_k}(t) - \widehat{P}_{D_k}(t)$ from its presumed value $P_{G_k}(t) - P_{D_k}(t)$ in bus k in time slot t. The excess net supply can be curtailed or stored in the battery energy storage. We also assume that in those buses with a renewable generator or load demand, there exists a reserve unit that can be turned on to compensate the shortage in the net supply. We model the generation cost for the reserve generator in bus k in time slot t as $c_{\text{res},k}\big[P_{G_k}(t) - P_{D_k}(t) - (\widehat{P}_{G_k}(t) - \widehat{P}_{D_k}(t))\big]^+$, where $[\cdot]^+ = \max\{\cdot, 0\}$ and $c_{\text{res},k}$ is the nonnegative marginal cost in \$/MW of the reserve unit connected to bus k. Let vector $\widehat{\boldsymbol{P}}_G(t) = (\widehat{P}_{G_k}(t),\, k \in \mathcal{N})$ denote the actual generation profile in time slot t. Let vectors $\boldsymbol{P}_D(t) = (P_{D_k}(t),\, k \in \mathcal{N})$ and $\widehat{\boldsymbol{P}}_D(t) = (\widehat{P}_{D_k}(t),\, k \in \mathcal{N})$ denote the presumed and actual load profiles in time slot t, respectively. The total cost of reserve units in time slot t is

$$C_{\text{res}}\big(\boldsymbol{P}_G(t),\, \widehat{\boldsymbol{P}}_G(t),\, \boldsymbol{P}_D(t),\, \widehat{\boldsymbol{P}}_D(t)\big) = \sum_{k \in \mathcal{N}} c_{\text{res},k}\big[P_{G_k}(t) - P_{D_k}(t) - (\widehat{P}_{G_k}(t) - \widehat{P}_{D_k}(t))\big]^+. \tag{10}$$

When a generator in bus k is operating, its output active power $P_{G_k}(t)$ and reactive power $Q_{G_k}(t)$ in time slot t are bounded by the limits $P_{G_k}^{\min}$, $P_{G_k}^{\max}$, $Q_{G_k}^{\min}$, and $Q_{G_k}^{\max}$. That is

$$u_k(t) P_{G_k}^{\min} \le P_{G_k}(t) \le u_k(t) P_{G_k}^{\max}, \qquad k \in \mathcal{N},\, t \in \mathcal{T} \tag{11a}$$
$$u_k(t) Q_{G_k}^{\min} \le Q_{G_k}(t) \le u_k(t) Q_{G_k}^{\max}, \qquad k \in \mathcal{N},\, t \in \mathcal{T}. \tag{11b}$$

Let r_k^{u} and r_k^{d} denote the maximum ramp up and ramp down rates for the generator in bus k, respectively. Let r_k^{su} and r_k^{sd} denote the maximum startup ramp and shutdown ramp rates for the generator in bus k, respectively. For $t \in \mathcal{T}$, we have

$$P_{G_k}(t) - P_{G_k}(t-1) \le u_k(t-1)\, r_k^{\text{u}} + s_k(t)\, r_k^{\text{su}}, \tag{12a}$$
$$P_{G_k}(t-1) - P_{G_k}(t) \le u_k(t-1)\, r_k^{\text{d}} + s_k(t)\, r_k^{\text{sd}}. \tag{12b}$$

Parameters r_k^{u}, r_k^{d}, r_k^{su}, and r_k^{sd} are set to a large number if the generator in bus $k \in \mathcal{N}$ is a renewable energy source.

A generator in bus k has a minimum up time t_k^{u} and down time t_k^{d} to start up and shut down. We have

$$\sum_{t'=t-t_k^{\text{u}}+1}^{t} s_k(t') \le u_k(t), \qquad k \in \mathcal{N},\, t \in \mathcal{T} \tag{13a}$$
$$\sum_{t'=t-t_k^{\text{d}}+1}^{t} d_k(t') \le 1 - u_k(t), \qquad k \in \mathcal{N},\, t \in \mathcal{T}. \tag{13b}$$

The values of $s_k(t)$ and $d_k(t)$, $k \in \mathcal{N}$ at $t \le 0$ are set based on the generators' state before starting the operating horizon. The binary variable $s_k(t)$ is equal to 1 only when generator k is off in time slot $t-1$ and is on in time slot t. The binary variable $d_k(t)$ is equal to 1 only when generator k is on in time slot $t-1$ and is off in time slot t. Thus, we have

$$u_k(t) - u_k(t-1) = s_k(t) - d_k(t), \quad k \in \mathcal{N},\, t \in \mathcal{T}. \tag{14}$$

The binary variables $u_k(t)$, $k \in \mathcal{N}$ at $t = 0$ are set based on the on/off state of the generators before the operating horizon. Constraints (13a), (13b), and (14) enforce variables $s_k(t)$, and $d_k(t)$ to be either 0 or 1 even if we relax variables $s_k(t)$ and $d_k(t)$ to take any value in the interval $[0, 1]$. Thus, we have

$$u_k(t) \in \{0, 1\}, \qquad k \in \mathcal{N},\, t \in \mathcal{T} \tag{15a}$$
$$0 \le s_k(t),\, d_k(t) \le 1, \qquad k \in \mathcal{N},\, t \in \mathcal{T}. \tag{15b}$$

C. ac-dc Network Model

The losses in the ac-dc network are equal to the total generation minus the total load. We obtain

$$P_{\text{loss}}(t) = \sum_{k \in \mathcal{N}} \left(P_{G_k}(t) + P_{B_k}(t) - P_{D_k}(t)\right), \quad t \in \mathcal{T}. \tag{16}$$

Let A^{T} denote the transpose of an arbitrary matrix or vector A. Let Y denote the admittance matrix. For the vector of the bus voltages $\mathbf{v}(t) = (V_1(t), \ldots, V_{|\mathcal{N}|}(t))$ and injected currents $\mathbf{i}(t) = (I_1(t), \ldots, I_{|\mathcal{N}|}(t))$ in time slot t, we have

$$\mathbf{v}(t) = Y\mathbf{i}^{\mathrm{T}}(t). \tag{17}$$

Let z^* denote the conjugate of an arbitrary complex number z. The power balance equations in time slot $t \in \mathcal{T}$ are

$$P_{G_k}(t) + P_{B_k}(t) - P_{D_k}(t) = \mathrm{Re}\{V_k(t)I_k^*(t)\},\ k \in \mathcal{N}\backslash\mathcal{N}^{\mathrm{conv}} \tag{18a}$$

$$P_{C_k}(t) = \mathrm{Re}\{V_k(t)I_k^*(t)\}, \qquad k \in \mathcal{N}^{\mathrm{conv}} \tag{18b}$$

$$Q_{G_k}(t) - Q_{D_k}(t) = \mathrm{Im}\{V_k(t)I_k^*(t)\}, \qquad k \in \mathcal{N}_{\mathrm{ac}} \tag{18c}$$

$$Q_{C_k}(t) = \mathrm{Im}\{V_k(t)I_k^*(t)\}, \qquad k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}. \tag{18d}$$

In (18a) and (18c), if bus k is not a generator bus, then $P_{G_k}(t) = Q_{G_k}(t) = 0$. Let $V_k^{\min}$ denote the lower bound on the bus voltage at bus k. Let $S_{lm}^{\max}$ denote the maximum apparent power flow through the line $(l, m) \in \mathcal{L}$. We have

$$V_k^{\min} \leq |V_k(t)| \leq V_k^{\max}, \qquad k \in \mathcal{N} \tag{19a}$$

$$|S_{lm}(t)| \leq S_{lm}^{\max}, \qquad (l, m) \in \mathcal{L}. \tag{19b}$$

III. Problem Formulation

The system operator aims to jointly minimize the generation cost $C_{\mathrm{gen}}(\cdot)$ in (9), the cost $C_{\mathrm{res}}(\cdot)$ in (10) associated with the net power supply uncertainty, and the total system losses $P_{\mathrm{loss}}(t)$ in (16). The cost $C_{\mathrm{res}}(\cdot)$ in (10) depends on the random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$. We consider the risk measure CVaR [24] to limit the risk of the net supply shortage. For a confidence level $\beta \in (0, 1)$ and vectors $\boldsymbol{P}_G(t)$ and $\boldsymbol{P}_D(t)$ in time slot t, we define

$$\mathrm{CVaR}_\beta\big(\boldsymbol{P}_G(t), \boldsymbol{P}_D(t)\big) = \mathbb{E}\left\{C_{\mathrm{res}}\big(\boldsymbol{P}_G(t), \widehat{\boldsymbol{P}}_G(t), \boldsymbol{P}_D(t), \widehat{\boldsymbol{P}}_D(t)\big) \,\big|\, C_{\mathrm{res}}\big(\boldsymbol{P}_G(t), \widehat{\boldsymbol{P}}_G(t), \boldsymbol{P}_D(t), \widehat{\boldsymbol{P}}_D(t)\big) \geq \alpha_\beta \right\}, \tag{20}$$

where $\mathbb{E}(\cdot)$ is the expectation over the random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$ and $\alpha_\beta = \min\big\{\alpha(t) \,\big|\, \Pr\{C_{\mathrm{res}}(\cdot) \leq \alpha(t)\} \geq \beta\big\}$. In general, the probability distributions of random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$ are not available. It is possible to estimate the CVaR by adopting sample average approximation (SAA) technique [24]. We use the set $\mathcal{J} \triangleq \{1, \ldots, J\}$ of J samples of $\boldsymbol{P}_G^j(t)$ and $\boldsymbol{P}_D^j(t)$ of the random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$ in time slot t from the historical record and obtain $\Pr\big\{\boldsymbol{P}_G^j(t), \boldsymbol{P}_D^j(t)\big\}$, the probability of the scenario with sample $\boldsymbol{P}_G^j(t)$ and $\boldsymbol{P}_D^j(t)$. Then, (20) can be approximated by

$$\mathrm{CVaR}_\beta\big(\boldsymbol{P}_G(t), \boldsymbol{P}_D(t)\big) \approx \min_{\alpha(t) \in \mathbb{R}} \Gamma_\beta\big(\alpha(t), \boldsymbol{P}_G(t), \boldsymbol{P}_D(t)\big), \tag{21}$$

where

$$\Gamma_\beta\big(\alpha(t),\, \boldsymbol{P}_G(t),\, \boldsymbol{P}_D(t)\big) = \alpha(t) + \sum_{j \in \mathcal{J}} \Big(\frac{\Pr\{\boldsymbol{P}_G^j(t),\, \boldsymbol{P}_D^j(t)\}}{1 - \beta} \Big[C_{\mathrm{res}}\big(\boldsymbol{P}_G(t), \boldsymbol{P}_G^j(t), \boldsymbol{P}_D(t), \boldsymbol{P}_D^j(t)\big) - \alpha(t)\Big]^+\Big). \tag{22}$$

Equation (21) implies that to compute the CVaR, it is sufficient to minimize $\Gamma_\beta(\cdot)$ in (22) over the variable $\alpha(t)$ using the historical samples of the random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$.

The objective function f_{obj} of the SCUC problem is

$$f_{\mathrm{obj}} = \sum_{t \in \mathcal{T}} \Big(C_{\mathrm{gen}}\big(\boldsymbol{P}_G(t), \boldsymbol{u}(t), \boldsymbol{s}(t), \boldsymbol{d}(t)\big) + \omega_{\mathrm{loss}} P_{\mathrm{loss}}(t) + \omega_{\mathrm{cvar}}\, \Gamma_\beta\big(\alpha(t), \boldsymbol{P}_G(t), \boldsymbol{P}_D(t)\big)\Big), \tag{23}$$

where ω_{loss} and ω_{cvar} are nonnegative weight coefficients. In (23), by increasing the value of ω_{loss}, the total system losses have a larger weight in the objective function as compared with the total generation cost and CVaR. For small-scale grids, the typical value of ω_{loss} is around the value of coefficients c_{k1}, $k \in \mathcal{N}$, for the generators, since the total system losses $P_{\mathrm{loss}}(t)$, $t \in \mathcal{T}$, is a linear function of the generators' output power. For large-scale test systems, the impact of coefficients c_{k2}, $k \in \mathcal{N}$, on the grid-wide generation cost increases. Hence, one may set ω_{loss} to be a larger value than c_{k1}, $k \in \mathcal{N}$, (e.g., 10 to 100 times larger). By increasing the value of ω_{cvar}, the cost associated with the net power supply shortage has a larger weight in the objective function, and thus the system operator is more risk-averse. The typical value for ω_{cvar} is around the value of $c_{k1}/c_{\mathrm{res},k}$, $k \in \mathcal{N}$, since $\Gamma_\beta\big(\alpha(t),\, \boldsymbol{P}_G(t),\, \boldsymbol{P}_D(t)\big)$ in (22) is a linear function of $c_{\mathrm{res},k} P_{G_k}(t)$.

Let $\boldsymbol{\psi} = (V_k(t),\, I_k(t),\, u_k(t),\, s_k(t),\, d_k(t), P_{G_k}(t), Q_{G_k}, P_{D_k}(t), P_{B_k}(t),\, k \in \mathcal{N}, S_{lm}(t),\, (l, m) \in \mathcal{L},\, \alpha(t),\, P_{C_k}(t), Q_{C_k}(t),\, P_{C_s}(t),\, k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}, s \in \mathcal{N}_{\mathrm{dc}}^{\mathrm{conv}}, t \in \mathcal{T})$ denote decision variables vector. The SCUC problem is formulated as

$$\begin{aligned} &\underset{\boldsymbol{\psi}}{\text{minimize}} \quad f_{\mathrm{obj}} \\ &\text{subject to } (1)-(8) \text{ and } (11a)-(19b). \end{aligned} \tag{24}$$

Problem (24) is a nonlinear mixed-integer optimization problem, which is difficult to be solved. We use the l_1-norm and convex relaxation techniques to relax the binary variables $u_k(t)$, $k \in \mathcal{N}$, $t \in \mathcal{T}$ and transform problem (24) into an SDP. Some of our notations are similar to [18] and [23]. The approaches given in [18] and [23] are not directly applicable to solve problem (24), since we have binary decision variables.

We define the variable column vector $\mathbf{x}(t) = \big[\mathrm{Re}\{\mathbf{v}(t)\}^{\mathrm{T}}\ \ \mathrm{Im}\{\mathbf{v}(t)\}^{\mathrm{T}}\big]^{\mathrm{T}}$ as the real and imaginary values of the vector of the bus voltages $\mathbf{v}(t)$ in time slot t. We define variable matrix $\mathbf{W}(t) = \mathbf{x}(t)(\mathbf{x}(t))^{\mathrm{T}}$. The VSC losses are the function of ac converter current. We define the variable column vector $\mathbf{i}_k(t) = \big[\frac{I_k^{\max} + |I_k(t)|}{2} \quad I_k^{\max} - |I_k(t)|\big]^{\mathrm{T}}$, $k \in \mathcal{N}_{\mathrm{ac}}$, $t \in \mathcal{T}$. We define variable matrix $\mathbf{I}_k(t)$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, $t \in \mathcal{T}$ as $\mathbf{I}_k(t) = \mathbf{i}_k(t)\mathbf{i}_k(t)^{\mathrm{T}}$. In Appendix A, we present the SDP form of problem (24). First, we express constraints (1)$-$(8) and (11a)$-$(19b) in terms of matrix variables $\mathbf{W}(t)$ and $\mathbf{I}_k(t)$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, $t \in \mathcal{T}$. Second, we relax the variable $u_k(t)$ to take any value in the interval $[0, 1]$ and include the following l_0-norm constraint into the constraint set.

$$\|u_k(t)\|_0 + \|1 - u_k(t)\|_0 = 1, \qquad k \in \mathcal{N},\ t \in \mathcal{T}, \tag{25}$$

where $\|\mathbf{z}\|_0$ is the l_0-norm of an arbitrary vector $\mathbf{z}$. When $\mathbf{z}$ is a nonnegative scalar, we have $\|z\|_0 = 0$ for $z = 0$, otherwise $\|z\|_0 = 1$. Third, we obtain $f_{\mathrm{obj}}^{\mathrm{SDP}}$, the SDP form of the objective function in (23). We introduce the auxiliary variables $\vartheta_k(t)$, $k \in \mathcal{N}$, $t \in \mathcal{T}$ and replace $C_{\mathrm{gen}}(\cdot)$ in (9) with $\sum_{k \in \mathcal{N}} \vartheta_k(t) + c_{k0} u_k(t) + c_k^{\mathrm{su}} s_k(t) + c_k^{\mathrm{sd}} d_k(t)$. We write the system losses in (16) in terms of $\mathbf{W}(t)$. We introduce the auxiliary variable $\mu^j(t)$ for sample j in time slot t to upper bound each term $[C_{\text{res}}(\cdot) - \alpha(t)]^+$ in (22). We introduce the auxiliary variable $\eta_k^j(t)$ for bus k and sample j in time slot t to upper bound the term $[P_{G_k}(t) - P_{D_k}(t) - (\widehat{P}_{G_k}(t) - \widehat{P}_{D_k}(t))]^+$ in function $C_{\text{res}}(\cdot)$ in (21). We replace function $\Gamma_\beta(\cdot)$ in (23) with $\alpha(t) + \frac{1}{1-\beta}\sum_{j\in\mathcal{J}} \Pr\{\boldsymbol{P}_G^j(t),\, \boldsymbol{P}_D^j(t)\}\mu^j(t)$. In Appendix A, we obtain the feasible set Φ^{SDP} of decision variables $\boldsymbol{\phi} = (\vartheta_k(t),\, u_k(t),\, s_k(t),\, d_k(t),\, P_{B_k}(t),\, \mu^j(t),\, \eta_k^j(t),\, j \in \mathcal{J},\, k \in \mathcal{N},\, \alpha(t), \mathbf{W}(t),\, \mathbf{I}_k(t),\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T})$ excluding the l_0-norm constraint (25). We obtain the SDP form of problem (24) as follows:

$$
\begin{aligned}
&\underset{\boldsymbol{\phi}\in\Phi^{\text{SDP}}}{\text{minimize}} \;\; f_{\text{obj}}^{\text{SDP}} && && \text{(26a)}\\
&\text{subject to constraint (25),} && && \text{(26b)}\\
&\qquad \operatorname{rank}(\mathbf{I}_k(t)) = 1, && k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}, && \text{(26c)}\\
&\qquad \operatorname{rank}(\mathbf{W}(t)) = 1, && t \in \mathcal{T}, && \text{(26d)}\\
&\qquad \mathbf{I}_k(t) \succeq 0, && k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}, && \text{(26e)}\\
&\qquad \mathbf{W}(t) \succeq 0, && t \in \mathcal{T}. && \text{(26f)}
\end{aligned}
$$

Problem (26) is equivalent to the SCUC problem (24), and is a nonconvex optimization problem due to the l_0-norm constraint (26b) and the rank constraints (26c) and (26d). We tackle the nonconvexity of problem (26) in the following subsections.

A. l_1-norm Relaxation of the SCUC Problem

We remove the nonconvex constraint (26b) from the constraint set of problem (26) and introduce the penalty $\sum_{k\in\mathcal{N}}\sum_{t\in\mathcal{T}} \|u_k(t)\|_0 + \|1-u_k(t)\|_0$ with a weight coefficient ς to the objective function. We have the following l_0-regularized SCUC problem:

$$
\underset{\boldsymbol{\phi}\in\Phi^{\text{SDP}}}{\text{minimize}} \;\; f_{\text{obj}}^{\text{SDP}} + \varsigma \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}} \left(\|u_k(t)\|_0 + \|1-u_k(t)\|_0\right) \quad (27)
$$

subject to constraints (26c)−(26f).

The value of $\|u_k(t)\|_0 + \|1-u_k(t)\|_0$ is 1, if $u_k(t)$ is binary. Otherwise, we have $\|u_k(t)\|_0 + \|1-u_k(t)\|_0 = 2$. Thus, in problem (27), with a sufficiently large ς, the solution to variables $u_k(t),\, k \in \mathcal{N},\, t \in \mathcal{T}$ are all binary. The term $\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}} \|u_k(t)\|_0 + \|1-u_k(t)\|_0$ is equal to $T|\mathcal{N}|$ for binary variables $u_k(t),\, k \in \mathcal{N},\, t \in \mathcal{T}$. Hence, for a sufficiently large ς, problems (26) and (27) are equivalent.

Next, we replace the terms $\|u_k(t)\|_0 + \|1-u_k(t)\|_0$ with the l_1-regularization term $\theta_{k1}(t)\,\|u_k(t)\|_1 + \theta_{k2}(t)\,\|1-u_k(t)\|_1$ for $k \in \mathcal{N}$ in the objective function of problem (27) [25], where $\theta_{k1}(t)$ and $\theta_{k2}(t)$ are weight coefficients and $\|\mathbf{z}\|_1$ is the l_1-norm of an arbitrary vector $\mathbf{z}$. When $\mathbf{z}$ is a nonnegative scalar, $\|z\|_1 = z$. We define vectors $\boldsymbol{\theta}_1(t) = (\theta_{k1}(t),\, k \in \mathcal{N})$ and $\boldsymbol{\theta}_2(t) = (\theta_{k2}(t),\, k \in \mathcal{N})$ in time slot t. We also define vectors $\boldsymbol{\theta}_1 = (\boldsymbol{\theta}_1(t),\, t \in \mathcal{T})$ and $\boldsymbol{\theta}_2 = (\boldsymbol{\theta}_2(t),\, t \in \mathcal{T})$. Under the given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, the objective function of the SCUC problem with an l_1-regularization term becomes

$$
\begin{aligned}
f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG}} = f_{\text{obj}}^{\text{SDP}} + \varsigma \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}} \big(&\theta_{k1}(t)\|u_k(t)\|_1 \\
&+ \theta_{k2}(t)\,\|1-u_k(t)\|_1\big). \quad (28)
\end{aligned}
$$

The l_1-regularized SCUC problem is as follows:

$$
\underset{\boldsymbol{\phi}\in\Phi^{\text{SDP}}}{\text{minimize}} \;\; f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG}} \quad (29)
$$

subject to constraints (26c)−(26f).

In problem (29), if $\theta_{k1}(t)$ is sufficiently larger than $\theta_{k2}(t)$, then $u_k(t)$ will be equal to zero. If $\theta_{k2}(t)$ is sufficiently larger than $\theta_{k1}(t)$, then $u_k(t)$ will be equal to one. Hence, there always exist vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, for which the optimal solution to problem (29) is equal to the optimal solution to problem (27). Problem (29) is still a nonconvex optimization problem due to the rank constraints (26c) and (26d). In the next subsection, we propose the SDP relaxation form of problem (29) and discuss the zero relaxation gap conditions. Finally, we propose an SCUC algorithm to determine the appropriate vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ to obtain a near-global optimal solution to the SCUC problem (27) (or the original SCUC problem (24)).

B. SDP Relaxation of the l_1-regularized SCUC Problem

We relax the rank constraints (26c) and (26d) in problem (29) and only keep constraints (26e) and (26f). The SDP relaxation of the SCUC problem is obtained as follows:

$$
\begin{aligned}
&\underset{\boldsymbol{\phi}\in\Phi^{\text{SDP}}}{\text{minimize}} \;\; f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG}} && \text{(30a)}\\
&\text{subject to constraints (26e) and (26f).} && \text{(30b)}
\end{aligned}
$$

Under the given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, problem (30) is an SDP and can be solved efficiently. Let $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$ and $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\; k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}$ denote the solution matrices to problem (30) under the given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. There are some difficulties in determining the correct solution to the original SCUC problem (26) by solving problem (30). First, we need to determine vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ leading to binary values for $u_k(t),\, k \in \mathcal{N},\, t \in \mathcal{T}$. Second, we need to obtain the conditions for zero relaxation gap between problems (29) and (30). In Theorem 1, we show that problem (30) always returns rank two solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\; k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}$. Hence, the rank constraint in (26c) is not satisfied and the relaxation gap between problems (29) and (30) is not zero.

Theorem 1 *Under the given $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, the solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t), k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, t \in \mathcal{T}$ to problem (30) are all rank two.*

The proof can be found in Appendix B. We use the *trace norm* regularization technique to obtain rank one matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\; k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}$. Minimizing the trace norm of a matrix induces sparsity to its vector of eigenvalues, which can lead to reducing the rank of the matrix. We use a penalty coefficient ε and introduce the penalty functions $\varepsilon\,\text{Tr}\{\mathbf{I}_k(t)\},\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\, t \in \mathcal{T}$ to the objective function of problem (30). We obtain the following trace norm-regularized SCUC problem:

$$
\begin{aligned}
&\underset{\boldsymbol{\phi}\in\Phi^{\text{SDP}}}{\text{minimize}} \;\; f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG}} + \varepsilon \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \text{Tr}\{\mathbf{I}_k(t)\} && \text{(31a)}\\
&\text{subject to constraints (26e) and (26f).} && \text{(31b)}
\end{aligned}
$$

Let $b^{\max}$ denote the maximum value for b_k and $I^{\min}$ denote the minimum value for $I_k^{\max}$ among all VSC stations. Let $c_1^{\max}$, $c_2^{\max}$, and $P_G^{\max}$ denote the maximum value for c_{k1}, c_{k2}, and

$P_{G_k}^{\max}$ among all generators, respectively. In Theorem 2, we provide an approximation for the penalty coefficient ε to obtain rank one solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\ t \in \mathcal{T}$.

Theorem 2 *To obtain rank one solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\ t \in \mathcal{T}$ to problem (31), the penalty coefficient ε can be approximated by*

$$\varepsilon \approx b^{\max}(2c_2^{\max}P_G^{\max} + c_1^{\max} + \omega_{\text{loss}})\frac{(I^{\min}+1)}{3\,(I^{\min})^2}. \tag{32}$$

The proof can be found in Appendix C. We can show that if the solution matrices $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\ t \in \mathcal{T}$ to problem (31) are at most rank two, then we can construct a rank one solution matrix to problem (31) as well. Thus, the relaxation gap between problems (29) and (31) is zero. In [18, Theorem 4], it is shown that the relaxation gap is zero for the ac-dc optimal power flow (OPF) problem in practical ac-dc networks, including the IEEE test systems connected to some dc grids. We adopt the result in [18, Theorem 4] to show that problem (31) returns matrices $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\ t \in \mathcal{T}$ with rank of at most two for practical ac-dc networks.

Theorem 3 *The solution matrices $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t),\ t \in \mathcal{T}$ to SCUC problem (31) are at most rank two, if for all set of operating generators, the SDP relaxation gap for the OPF problem in the underlying ac-dc grid is zero.*

The proof can be found in Appendix D. If the condition in Theorem 3 holds, then under the given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, the solution to problem (31) becomes a feasible solution to problem (29). The optimal value of problem (31) may not be the same as the optimal value for the objective function of problem (29) due to the trace norm regularization term in the objective function of problem (31). Let $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}$ and $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}$ denote the optimal values for the objective functions of problems (29) and (31), respectively. In The following theorem provides an upper bound for the difference between the optimal values $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}$ and $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}$.

Theorem 4 *The difference between the optimal values of problems (29) and (31) is bounded by*

$$0 \le f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31} - f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29} \le 0.45\,T\,\varepsilon \textstyle\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} (I_k^{\max})^2. \tag{33}$$

The proof can be found in Appendix E. In Section IV, we show that the value for ε in (32) is small, and we can approximate the solution to problem (29) by the solution to problem (31).

In Fig. 2, we summarize the steps to solve the original SCUC problem (24), which is equivalent to the l_0-regularized SCUC problem (27). We formulate problem (29) to relax the l_0-norm regularization term. Next, we obtain the SDP relaxation form of SCUC problem (30). To guarantee the zero relaxation gap, we formulate the trace norm-regularized SCUC problem (31). In the final step, we determine a (local) optimal solution of problem (27) by solving problem (31).

C. SCUC Algorithm Design

We propose Algorithm 1 based on the iterative reweighted l_1-algorithm in [25] to determine the solution to the SCUC problem. Let i denote the iteration index. In iteration i, we solve problem (31) under the given vectors $\boldsymbol{\theta}_1^i$ and $\boldsymbol{\theta}_2^i$ and use

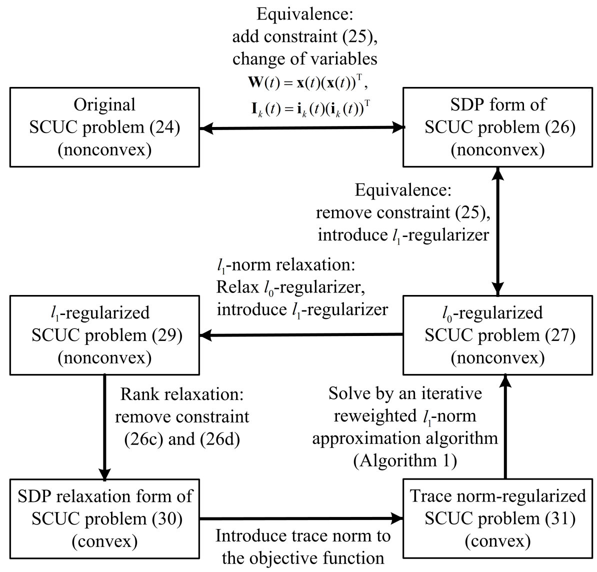

Figure 2. The procedure of solving the original SCUC problem (24).

the solution variables $u_{k,\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}^{\text{opt},i}(t),\ k \in \mathcal{N},\ t \in \mathcal{T}$ to determine the updated coefficients $\boldsymbol{\theta}_1^{i+1}$ and $\boldsymbol{\theta}_2^{i+1}$ as follows:

$$\theta_{k1}^{i+1}(t) = \frac{1}{u_{k,\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}^{\text{opt},i}(t)+\sigma}, \tag{34a}$$

$$\theta_{k2}^{i+1}(t) = \frac{1}{1-u_{k,\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}^{\text{opt},i}(t)+\sigma}, \tag{34b}$$

where σ is a positive constant. If we obtain binary values for $u_{k,\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}^{\text{opt},i-1}(t),\ k \in \mathcal{N},\ t \in \mathcal{T}$ in Line 6, then the algorithm returns matrices $\mathbf{W}^{\text{opt}}(t) := \mathbf{W}_{\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}^{\text{opt},i-1}(t)$ and $\mathbf{I}_k^{\text{opt}}(t) := \mathbf{I}_{k,\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}^{\text{opt},i-1}(t),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\ t \in \mathcal{T}$ in Line 7. According to Theorem 3, for the coefficient ε in (32), matrices $\mathbf{I}_k^{\text{opt}}(t),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\ t \in \mathcal{T}$ are all rank one. If the solution matrices $\mathbf{W}^{\text{opt}}(t),\ t \in \mathcal{T}$ are at most rank two, then the SDP relaxation gap is zero. In Line 9, we recover the solution vectors $\mathbf{x}^{\text{opt}}(t)$ and $\mathbf{i}_k^{\text{opt}}(t),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}},\ t \in \mathcal{T}$ as follows. If matrix $\mathbf{W}^{\text{opt}}(t)$ is rank two for some $t \in \mathcal{T}$, then we calculate two nonzero eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ with corresponding eigenvectors $\boldsymbol{\nu}_1(t)$ and $\boldsymbol{\nu}_2(t)$ of each rank two matrix $\mathbf{W}^{\text{opt}}(t)$. It can be shown that the rank one matrix $\mathbf{W}_1^{\text{opt}}(t) := (\lambda_1(t)+\lambda_2(t))\boldsymbol{\nu}_1(t)\boldsymbol{\nu}_2^{\text{T}}(t)$ is also the solution of problem (31). We replace each rank two matrix $\mathbf{W}^{\text{opt}}(t)$ with the rank one solution matrix $\mathbf{W}_1^{\text{opt}}(t)$, and calculate the eigenvalue $\lambda(t)$ with corresponding eigenvector $\boldsymbol{\nu}(t)$ of the resulting rank one matrices for $t \in \mathcal{T}$. We determine the solutions as $\mathbf{x}^{\text{opt}}(t) = \sqrt{\lambda(t)}\boldsymbol{\nu}(t)$ and $\mathbf{I}_k^{\text{opt}}(t) = \mathbf{i}_k^{\text{opt}}(t)\mathbf{i}_k^{\text{opt}}(t)^{\text{T}}$. If the rank of $\mathbf{W}^{\text{opt}}(t)$ is greater than two for some $t \in \mathcal{T}$, then the SDP relaxation gap may not be zero. One may use a heuristic method to enforce the low-rank solution of problem (31) to become rank one or rank two [26].

Theorem 5 *If the solution matrices $\mathbf{W}_{\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}^{\text{opt},i}(t),\ t \in \mathcal{T}$ to problem (31) are at most rank two, then for sufficiently large parameter ς and small parameter σ, Algorithm 1 converges to a local optimal solution of the original SCUC problem (24).*

Algorithm 1 ac-dc SCUC algorithm.

Set $i := 1$. Initialize σ and ς. Randomly initialize $\boldsymbol{\theta}_1^1$ and $\boldsymbol{\theta}_2^1$. Obtain J samples of the random variables $\widehat{\boldsymbol{P}}_G(t)$ and $\widehat{\boldsymbol{P}}_D(t)$.
Repeat
  Solve problem (31) under given vectors $\boldsymbol{\theta}_1^i$ and $\boldsymbol{\theta}_2^i$.
  Determine vectors $\boldsymbol{\theta}_1^{i+1}$ and $\boldsymbol{\theta}_2^{i+1}$ according to (34a) and (34b).
  $i := i + 1$
Until variables $u^{\text{opt},i-1}_{k,\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}(t) \in \{0, 1\}$ for $k \in \mathcal{N}$, $t \in \mathcal{T}$.
Determine solution matrices $\mathbf{W}^{\text{opt}}(t) := \mathbf{W}^{\text{opt},i-1}_{\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}(t)$ and $\mathbf{I}_k^{\text{opt}}(t) := \mathbf{I}^{\text{opt},i-1}_{k,\boldsymbol{\theta}_1^{i-1},\boldsymbol{\theta}_2^{i-1}}(t)$ for $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, $t \in \mathcal{T}$.
If solution matrices $\mathbf{W}^{\text{opt}}(t)$, $t \in \mathcal{T}$ are at most rank two
  Calculate solution vectors $\mathbf{x}^{\text{opt}}(t)$ and $\mathbf{i}_k^{\text{opt}}(t)$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, t \in \mathcal{T}$.

The proof can be found in Appendix F. The obtained solution from Algorithm 1 depends on the initial point of the algorithm, as well as the values of σ and ς [25], [27, Ch. 5], and [28]. By simulations, we show that an appropriate initialization to determine the near-optimal solution to problem (24) is $\theta_{k1}^1(t) = \theta_{k2}^1(t)$, $k \in \mathcal{N}$, $t \in \mathcal{T}$, which corresponds to the convex relaxations of the binary variables in the original SCUC problem (24). The value of $\sigma = 10^{-3}$ is sufficiently small to be used in Algorithm 1. Regarding the parameter ς, the weight of l_1-regularization term $\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}(\theta_{k1}(t)\|u_k(t)\|_1 + \theta_{k2}(t)\,\|1-u_k(t)\|_1)$ increases in the objective function (28) when ς increases. Hence, Algorithm 1 may not converge to a local optimal with small objective value $f_{\text{obj}}^{\text{SDP}}$. A proper value for ς can be chosen such that the value of $\varsigma\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}(\theta_{k1}(t)\|u_k(t)\|_1 + \theta_{k2}(t)\,\|1-u_k(t)\|_1)$ is around the value of $f_{\text{obj}}^{\text{SDP}}$. The value of $\varsigma\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}\big(\theta_{k1}(t)\|u_k(t)\|_1 + \theta_{k2}(t)\,\|1-u_k(t)\|_1\big)$ is approximately equal to $\varsigma\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}(\|u_k(t)\|_0 + \|1-u_k(t)\|_0)$. The value of $\|u_k(t)\|_0 + \|1-u_k(t)\|_0$ is equal to 1 if $u_k(t)$ is binary. Otherwise, we have $\|u_k(t)\|_0 + \|1-u_k(t)\|_0 = 2$. Hence, the value of $\varsigma\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}(\|u_k(t)\|_0 + \|1-u_k(t)\|_0)$ is at most $2\varsigma T|\mathcal{N}|$. We can approximate the value of $f_{\text{obj}}^{\text{SDP}}$ from the solution of the first iteration of Algorithm 1. Thus, we can choose ς such that $2\varsigma T|\mathcal{N}|$ is greater than or equal to the value of $f_{\text{obj}}^{\text{SDP}}$ in the first iteration of Algorithm 1.

IV. Performance Evaluation

In this section, we evaluate the performance of Algorithm 1 in solving the SCUC problem. The test system is shown in Fig. 3, which is an IEEE 30-bus test system connected to three wind farms in buses 14 and 30, two PV panels in buses 3 and 7, and one dc microgrid in bus 28. The data for the IEEE 30-bus test system is from [29].

The base power of the system is 100 MVA. The generators' specifications are given in Table I. The coefficients c_{k0}, c_{k1}, and $c_{k2}, k \in \mathcal{N}$ for the generation cost function in (9) of the conventional generators can be found in [29]. The marginal cost of the reserve units is set to $c_{\text{res},k} = 200$ \$/pu. The resistance of the high voltage direct current (HVDC) lines is 0.06 pu. The resistance of the dc cables is 0.001 pu. The maximum apparent power flow through the HVDC lines and other transmission lines is 1.1 pu. The data for the VSC stations are given in Table II. The base power of the VSC station is 100 MVA. To obtain the samples for the load demands and the output power of the wind farms and PV panels, we use the historical available data from Ontario, Canada power grid database [30] from June 1 to August 1, 2016. For each bus, we scale the available historical data such that the mean value is equal to the load demand given in [29] for that bus. We consider the fixed power factor for the loads. Fig. 4 (a) shows the average overall load demand of all buses over 24 hours. For each renewable generator, we scale down the available historical data such that the maximum output power of each wind farm and PV panel over the historical data is equal to 20 MW and 15 MW, respectively. Figs. 4 (b) and (c) show the average output power of the PV panels and wind farms over 24 hours, respectively. Each renewable generator is equipped with a battery energy storage system with capacity of 4 MW, maximum charging/discharging rate of 1 MW, and initial energy level of 2 MW. For the underlying test system, the base power is 100 MVA and the value of c_{k1}, $k \in \mathcal{N}$ is 20 \$/MW [29]. Hence, the weight coefficients ω_{loss} is set to 2×10^3 \$/pu. Furthermore, we have $c_{\text{res},k} = 200$ \$/pu, and thus ω_{cvar} is set to $100\times(c_{k1}/c_{\text{res},k}) = 10$. Unless stated otherwise, β in (22) is set to 0.9. We assume that the conventional generators are turned on in all time slots before the planning horizon. For the benchmark scenario, we consider the test system without renewable generators and batteries and with the average load profiles in each bus. We perform simulations using MATLAB/CVX with MOSEK solver in a PC with processor Intel(R) Core(TM) i7-3770K CPU@3.5 GHz.

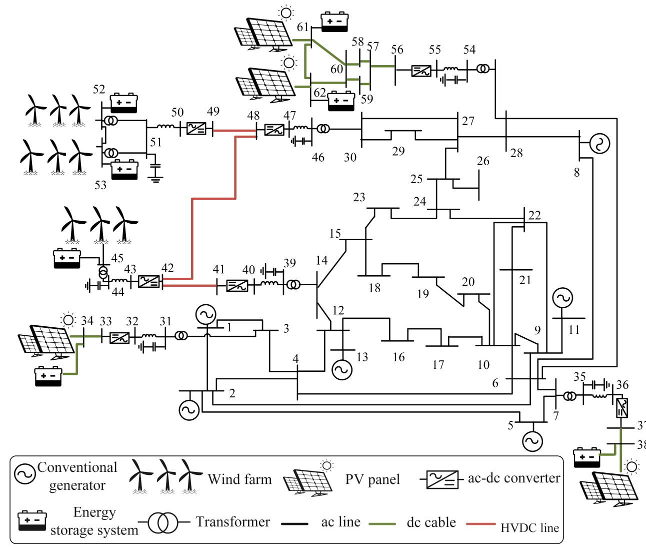

Figure 3. An IEEE 30-bus test system connected to three wind farms in buses 14 and 30, two PV panels in buses 3 and 7, and one dc microgrid in bus 28.

Table I
Generators' Specifications

k	c_k^{su} (\$)	c_k^{sd} (\$)	r_k^{u} (pu)	r_k^{d} (pu)	r_k^{su} (pu)	r_k^{sd} (pu)	t_k^{u} (pu)	r_k^{d} (pu)
1	1000	100	0.5	0.5	0.7	0.7	3	3
2	1500	200	0.7	0.7	0.7	0.7	1	1
5	1000	100	0.5	0.5	0.7	0.7	3	3
8	1500	200	0.7	0.7	0.7	0.7	1	1
11	1000	100	0.5	0.5	0.7	0.7	1	1
13	1500	200	0.5	0.5	0.7	0.7	1	1

Table II
VSC Station Parameters with Converter ac Bus k, Converter dc Bus s, and Filter Bus f

VSC parameters (pu)			
$R_{T_k} = 0.0005$	$X_{T_k} = 0.0125$	$B_f = 0.2$	$S_{C_k}^{\text{nom}} = 1$
$R_{C_k} = 0.00025$	$X_{C_k} = 0.04$	$V_k^{\max} = 1.06$	$I_k^{\max} = 1.0526$
$m_k^{\text{q}} = 0.5$		$m_{ks}^{\text{v}} = 1.1$	
VSC losses data (pu)			
$a_k = 0.0053$	$b_k = 0.0037$	$c_k = 0.0018$	

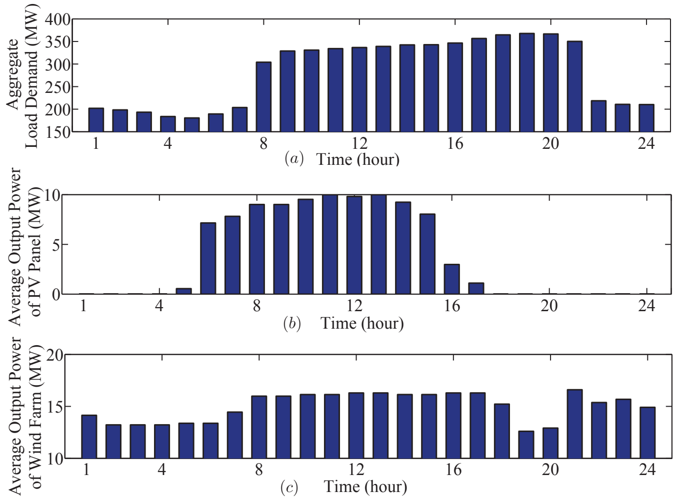

Figure 4. (a) Aggregate load demand, (b) average output power of the PV panels, (c) average output power of the wind farms over 24 hours.

To evaluate the proposed SDP relaxation technique, we study the results of Theorems 1 to 4. We first solve problem (30) for different coefficients $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. All solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, $t \in \mathcal{T}$, becomes rank two, which confirms the result of Theorem 1. Then, we use the result of Theorem 2 to compute the appropriate value for the penalty factor ε. For our case study, we have $\omega_{\text{loss}} = 2000$, $b^{\max} = 0.0037$ pu, $c_1^{\max} = 40$ \$/MW pu, $c_2^{\max} = 0.25$ \$/MW^2, $I^{\min} = 1.0526$ pu and $P_G^{\max} = 180$ MW. Hence from (32), we have $\varepsilon = 4.86$. We solve problem (31) with $\varepsilon = 4.86$ for several coefficients $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. We obtain rank one solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, $t \in \mathcal{T}$ and rank two solution matrices $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $t \in \mathcal{T}$. Thus, the relaxation gap between problems (29) and (31) is zero. This confirms the result of Theorem 3, as the SDP relaxation gap is zero for ac-dc OPF in the modified IEEE test systems [18]. The optimal value of problem (31) is \$601,508.1. According to the upper bound in (33) in Theorem 4, $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31} - f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}$ is at most \$407.085, which is about 0.068% of the optimal value of problem (31). Thus, we can approximate the solution to problem (29) by the solution to problem (31), and the approximation is tight.

We use Algorithm 1 (with nonlinear ac power flow model) to evaluate the solutions to the SCUC problem for the benchmark ac grid without renewable generator and the ac-dc grid with renewable generators. Algorithm 1 guarantees to return a local optimal solution of the SCUC problem (31). Our goal is to demonstrate that Algorithm 1 most often converges to a near-global optimal solution. As it is mentioned in Section III-C, parameter $\sigma = 10^{-3}$ is sufficiently small to be used in Algorithm 1. There are five conventional generators, T is equal to 24, and the value of $f_{\text{obj}}^{\text{SDP}}$ in the first iteration of Algorithm 1 is \$213,182. Hence, parameter ς in the update rules (34a) and (34b) is set to 10^3, which is greater than $f_{\text{obj}}^{\text{SDP}}/240$. We run Algorithm 1 in both cases for 1000 randomly chosen initial weight coefficients θ_{k1}^1 and θ_{k2}^1, $k \in \mathcal{N}$, $t \in \mathcal{T}$ from the interval $[0, 50]$. For both case studies, the smallest obtained objective value corresponds to the global optimal solution. For both cases and for all initial conditions, Algorithm 1 returns a near-global optimal solution within 2% gap from the global optimal solution. Moreover, for both case studies, Algorithm 1 returns the near-global optimal solution within 1% gap in 98% of the initial conditions. We emphasize that such a result for the gap from the global optimal solution is only valid for the underlying test cases, and in general, we cannot guarantee a specific gap. Nevertheless, Theorem 5 guarantees the convergence of Algorithm 1 to a local optimal solution. Algorithm 1 returns the *global* optimal solution for both case studies when θ_{k1} and θ_{k2}, $k \in \mathcal{N}$, $t \in \mathcal{T}$ are (approximately) equal. The term in (28) is a constant when $\theta_{k1} = \theta_{k2}$, $k \in \mathcal{N}$, $t \in \mathcal{T}$. Thus, in the first iteration, the binary variables $u_k(t)$, $s_k(t)$, and $d_k(t)$, $k \in \mathcal{N}$, $t \in \mathcal{T}$ are relaxed to take any value in the interval $[0, 1]$. Hence, the optimal value of problem (31) in iteration 1 becomes the lower bound for the global optimal solution of the SCUC problem. In our case studies, when Algorithm 1 starts from the lower bound for the global optimal solution, it converges to the global optimal solution. The number of iterations is between 3 to 12 and the average running time is 35 seconds.

We consider the global optimal solutions to the benchmark ac grid without renewable generator and the ac-dc grid with renewable generators. The output power of all conventional generators are reduced in most of the time slots in Fig. 5. The generation cost and system losses are given in Table III. The generation cost is lower by 23.4% in a system with renewable generators. However, the risk of using renewable generators is \$10,430, i.e., 6.2% of the generation cost. The system losses in the grid without renewable generators are due to the losses on the transmission lines and are equal to 191.994 MW. Using the renewable generators, the total system losses become 188.577 MW, which include 73.177 MW of losses on the transmission lines and 115.4 MW of losses on the VSC stations. Hence, using the renewable generators can reduce the losses on the transmission lines by about 60%. However, the VSC losses can add up to a significant fraction of total losses (65% of the total losses) and have to be included in the SCUC problem.

Next, we apply Algorithm 1 to evaluate the solutions to the SCUC problem for the benchmark ac grid without renewable generator in the following case studies by using (i) the dc power flow equations, (ii) the linearized ac power flow equations, and (iii) the nonlinear full ac power flow equations (as in problems (29) and (31)). The dc and linearized ac power flow equations are commonly used in the literature to solve the SCUC problem [4]–[8], [11]–[16]. For these case studies, we apply the proposed iterative reweighted l_1-norm approximation

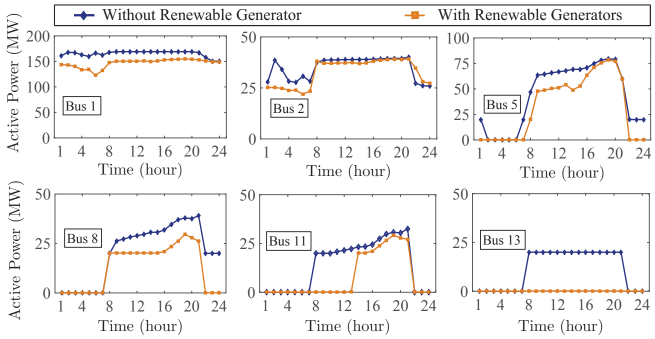

Figure 5. The output active power of the generators for the grid without renewable energy generator and the grid with renewable energy generators.

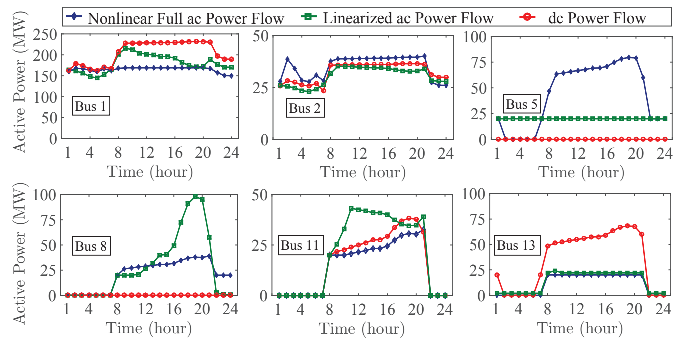

Figure 6. The output active power of the generators for the scenarios with nonlinear ac power flow, linearized ac power flow, and dc power flow.

Table III
THE GENERATION COST, SYSTEM LOSSES, AND CVAR OBTAINED FOR THE TEST SYSTEMS WITH AND WITHOUT RENEWABLE GENERATORS.

Case study	Generation cost ($)	Total losses (MW)	Converter losses (MW)	CVaR ($)
Without renewable generator	217,620.1	191.944	0	0
With renewable generators	166,570.8	188.577	115.4	10,430

Table IV
THE OPTIMAL VALUE AND AVERAGE CPU TIME FOR THE DETERMINISTIC MULTI-STAGE ALGORITHM AND OUR PROPOSED ALGORITHM.

	Our proposed algorithm		Multi-stage algorithm	
Test system	$f_{\text{obj}}^{\text{SDP,opt}}$ (\$)	CPU time (s)	$f_{\text{obj}}^{\text{SDP,opt}}$ (\$)	CPU time (s)
IEEE 14-bus	207,550.1	9	216,826.2	22
IEEE 30-bus	221,458.9	35	230,317.3	81
IEEE 118-bus	3,372,717.8	225	3,627,233.8	395
IEEE 300-bus	21,510,112.1	520	22,277,870.5	905
Polish 2383wp	45,933,712.9	4,450	49,552,733.3	6,000
Polish 3012wp	63,991,443.8	6,700	72,188,966.1	8,950

algorithm to obtain the binary solution of the SCUC problem.

Fig. 6 shows the output active power of the generators. By using the dc power flow model, the generators in buses 5 and 8 are turned off during the operation cycle due to their high generation cost. With the obtained output active power profiles for the generators, no feasible solution for the ac power flow problem can be obtained. By using the linearized ac power flow model, the generators in buses 5 and 8 are turned on during time slots 8 to 21. When we solve the full ac power flow problem with the obtained output power profiles for the generators, the objective value is \$222,124.5. By using the nonlinear full ac power flow model, the output power of the generator in bus 5 increases during peak load period. The objective value is \$221,458.9, which is smaller than the objective value with linearized ac power flow model. The comparison of these scenarios shows that the generators' schedule with the dc power flow model can be infeasible. The generators' schedule with the linearized ac power flow models can deviate from the actual optimal generators' schedule with the full ac power flow model. These results justify the use of nonlinear full ac model in the SCUC problem formulation.

For the purpose of comparison, we use the multi-stage optimization technique (e.g., in [4], [6], [7]) to solve the *deterministic* SCUC problem in an ac grid without generation and load uncertainty. We perform simulations on six test systems shown in Table IV. The data for the test systems can be found in [29] and [31]. We randomly assign the specifications in Table I to the generators. We use the average load profiles over 24 hours in each bus. For IEEE 14-bus and 30-bus test systems, we set $\omega_{\text{loss}} = 2 \times 10^3$ \$/pu. For Other four test systems, we set $\omega_{\text{loss}} = 2 \times 10^5$ \$/pu. The multi-stage optimization technique involves iterative procedure between the master problem and the sub-problems. The linearized ac power flow model is used [4]. The objective of the master problem is to minimize the grid-wide generation cost. The constraints include the system power balance and operation constraints of the generators. The master problem can be formulated as an MIP. We use CPLEX 12.6 as the MIP solver to solve the master problem [6], [7]. We apply the Benders cut method and formulate a linear program for checking the network constraints [6]. In Table IV, we compare the average CPU time and the lowest objective value among 10 runs with different initial conditions. MIP has a low convergence speed in large networks. Furthermore, in the multi-stage algorithm, the unit commitment decisions from the first stage are fixed in the second stage. Thus, it is not guaranteed to converge to a good local optimal. Whereas, Algorithm 1 is based on the convex relaxation method and the global optimality of the solution in each iteration of Algorithm 1 leads to converging to a near-global optimal solution.

We now study the impact of confidence level β on the net power supply variations. We consider the probability distribution $\Pr\{C_{\text{res}}(\boldsymbol{P}_G^{\text{opt}}(t), \widehat{\boldsymbol{P}}_G(t), \boldsymbol{P}_D^{\text{opt}}(t), \widehat{\boldsymbol{P}}_D(t)) = \hat{C}_{\text{res}}\}$ for all $\hat{C}_{\text{res}} \geq 0$ under the given solution vectors $\boldsymbol{P}_G^{\text{opt}}(t)$ and $\boldsymbol{P}_D^{\text{opt}}(t)$ to problem (31). Such a probability distribution can be approximated by computing the value of $C_{\text{res}}(\cdot)$ from the available historical data for the load demand and output power of the renewable generators in [30]. Fig. 7 shows the probability distribution function of $C_{\text{res}}(\cdot)$ for $\beta = \{0.3, 0.6, 0.9\}$. When β increases, the lower values of $C_{\text{res}}(\cdot)$ will have higher probabilities. The system operator uses the conventional

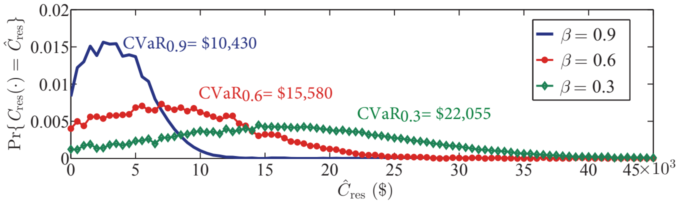

Figure 7. The probability distribution function of $C_{\rm res}(\cdot)$ and the value of CVaR for $\beta = 0.3$, 0.6, and 0.9.

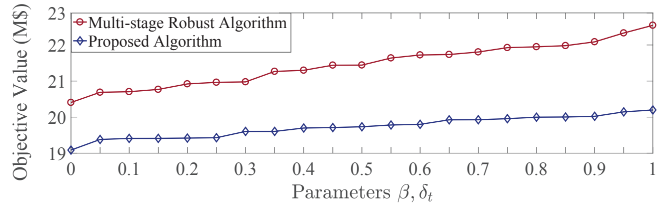

Figure 8. The objective value for different values of parameter β with Algorithm 1 and δ_t, $t \in \mathcal{T}$ with the multi-stage robust algorithm in [16].

generators instead of the renewable generators, and sets higher values to the load demand to limit the risk of power supply shortage. Thus, the value of CVaR decreases when β increases.

Finally, we compare the performance of Algorithm 1 with the multi-stage robust optimization technique (e.g., in [14]–[16])) in solving the *stochastic* SCUC problem (24) in an IEEE 300-bus test system connected to five PV panels, five off-shore wind farms, and five dc microgrids in different buses. We apply the proposed two-level algorithm in [14]. The outer level employs the Benders decomposition algorithm to obtain optimal commitment decision using the results from the inner level, which approximately solves the bilinear optimization problem using an outer approximation algorithm. To make our comparison fair, we consider the penalty for the shortage in the net power supply and assume that the uncertainty set in time slot $t \in \mathcal{T}$ is a polyhedral with parameter Δ_t that takes values between 0 and the number of buses with load or generation uncertainty. In our case study, 208 buses have uncertainty. Thus, we can set $\Delta_t = 208\,\delta_t$ for $t \in \mathcal{T}$, where δ_t can take any value in the interval $[0, 1]$. $\delta_t = 0$ corresponds to the least conservative case study, in which the uncertainty set only includes the maximum possible net power supply in the historical data record. The value $\delta_t = 1$ corresponds to the most conservative case study that takes into account all possible deviations in the load demand and generation. Hence, $\delta_t = 0$ and $\delta_t = 1$ correspond to the scenarios with $\beta = 0$ and $\beta = 1$ in Algorithm 1, respectively. We set $c_{{\rm res},k} = 2000$ \$/pu, $k \in \mathcal{N}$ and $\omega_{\rm cvar} = 1$ in Algorithm 1 and compare the smallest objective value among 10 runs with different initial conditions. Fig. 8 shows that the optimal value for different values of parameter β with Algorithm 1 is smaller than the optimal value for different δ_t, $t \in \mathcal{T}$ with the multi-stage robust algorithm. Two reasons can be given. First, Algorithm 1 generally returns an optimal solution with a smaller gap from the global optimal solution. Whereas, the proposed multi-stage robust algorithm in [14] uses Benders decomposition and may not return a near-global optimal solution. Second, the CVaR takes into account the probability distribution of the scenarios. Whereas, in the robust optimization technique, the worst-case scenario is considered, which can have a small probability. Thus, the multi-stage robust algorithm returns a conservative solution with a larger objective value.

V. Conclusion

In this paper, we studied the SCUC problem for ac-dc grids. The uncertainty in the load demand and renewable generation were addressed by introducing a penalty based on CVaR in the objective function to limit the risk of deviations in the load demand and renewable generation. The SCUC problem was a nonlinear mixed-integer optimization problem. We used l_0-norm to model the constraints with binary variables, and then applied l_1-norm relaxation to obtain a problem with continuous variables. Finally, we used convex relaxation techniques to obtain the SDP form of the problem. An algorithm based on the iterative reweighted l_1-norm approximation was proposed to determine the local optimal solution to the original problem. Simulation results on a modified IEEE 30-bus test system showed that the proposed algorithm with different initial conditions returns the solution with at most 2% gap from the global optimal solution for the underlying test system. When compared with the multi-stage algorithm in the literature, our proposed algorithm returned a solution with lower gap from the global optimal solution in a lower CPU time. For future work, we plan to extend our proposed algorithm to unbalanced three-phase microgrids.

Appendix

A. Transforming Problem (24) into an SDP

For $k \in \mathcal{N}$, let e_k denote the $k^{\rm th}$ basis vector in $\mathbb{R}^{|\mathcal{N}|}$ and $Y_k = e_k e_k^{\rm T} Y$. The row k of matrix Y_k is equal to the row k of the admittance matrix Y. The other entries of Y_k are zero. We use the Π model of the transmission lines. Let y_{lm} and $\bar{y}_{lm}$ denote the value of the series and shunt elements at bus l connected to bus m, respectively. We define $Y_{lm} = (\bar{y}_{lm}+y_{lm})e_l e_l^{\rm T} - (y_{lm})e_l e_m^{\rm T}$, where the entries (l,l) and (l,m) of Y_{lm} are equal to $\bar{y}_{lm} + y_{lm}$ and $-y_{lm}$, respectively. The other entries of Y_{lm} are zero. We define matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$, $\mathbf{Y}_{lm}$, $\bar{\mathbf{Y}}_{lm}$, $\mathbf{M}_k$ and $\mathbf{M}_{lm}$ as follows.

$$\mathbf{Y}_k = \frac{1}{2}\begin{bmatrix} {\rm Re}\{Y_k + Y_k^{\rm T}\} & {\rm Im}\{Y_k^{\rm T} - Y_k\} \\ {\rm Im}\{Y_k - Y_k^{\rm T}\} & {\rm Re}\{Y_k + Y_k^{\rm T}\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2}\begin{bmatrix} {\rm Im}\{Y_k + Y_k^{\rm T}\} & {\rm Re}\{Y_k - Y_k^{\rm T}\} \\ {\rm Re}\{Y_k^{\rm T} - Y_k\} & {\rm Im}\{Y_k + Y_k^{\rm T}\} \end{bmatrix},$$

$$\mathbf{Y}_{lm} = \frac{1}{2}\begin{bmatrix} {\rm Re}\{Y_{lm} + Y_{lm}^{\rm T}\} & {\rm Im}\{Y_{lm}^{\rm T} - Y_{lm}\} \\ {\rm Im}\{Y_{lm} - Y_{lm}^{\rm T}\} & {\rm Re}\{Y_{lm} + Y_{lm}^{\rm T}\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_{lm} = -\frac{1}{2}\begin{bmatrix} {\rm Im}\{Y_{lm} + Y_{lm}^{\rm T}\} & {\rm Re}\{Y_{lm} - Y_{lm}^{\rm T}\} \\ {\rm Re}\{Y_{lm}^{\rm T} - Y_{lm}\} & {\rm Im}\{Y_{lm} + Y_{lm}^{\rm T}\} \end{bmatrix},$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^{\rm T} & 0 \\ 0 & e_k e_k^{\rm T} \end{bmatrix},$$

$$\mathbf{M}_{lm} = \begin{bmatrix} (e_l - e_m)(e_l - e_m)^{\rm T} & 0 \\ 0 & (e_l - e_m)(e_l - e_m)^{\rm T} \end{bmatrix}.$$

We use the notation $\mathrm{Tr}\{A\}$ to represent the trace of an arbitrary square matrix A. It can be shown that

$$\mathrm{Re}\{V_k(t)I_k^*(t)\} = \mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}, \qquad k\in\mathcal{N} \tag{35a}$$
$$\mathrm{Im}\{V_k(t)I_k^*(t)\} = \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}, \qquad k\in\mathcal{N} \tag{35b}$$
$$|V_k(t)|^2 = \mathrm{Tr}\{\mathbf{M}_k\mathbf{W}(t)\}, \qquad k\in\mathcal{N} \tag{35c}$$
$$|V_l(t)-V_m(t)|^2 = \mathrm{Tr}\{\mathbf{M}_{lm}\mathbf{W}(t)\}, \qquad (l,m)\in\mathcal{L} \tag{35d}$$
$$|S_{lm}(t)|^2 = \mathrm{Tr}\{\mathbf{Y}_{lm}\mathbf{W}(t)\}^2+\mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}(t)\}^2, \ (l,m)\in\mathcal{L}. \tag{35e}$$

Since $\mathbf{I}_k(t) = \mathbf{i}_k(t)\mathbf{i}_k(t)^{\mathrm{T}}$, for a VSC station with ac converter bus $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, we have

$$\mathbf{I}_k(t) = \begin{bmatrix} \dfrac{(I_k^{\max}+|I_k(t)|)^2}{4} & \dfrac{(I_k^{\max})^2-|I_k(t)|^2}{2} \\ \dfrac{(I_k^{\max})^2-|I_k(t)|^2}{2} & (I_k^{\max}-|I_k(t)|)^2 \end{bmatrix}. \tag{36}$$

We define symmetric matrix $\mathbf{S}_k$ for $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ as

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{S}_k^{11} & \mathbf{S}_k^{12} \\ \mathbf{S}_k^{21} & \mathbf{S}_k^{22} \end{bmatrix}, \tag{37}$$

where $\mathbf{S}_k^{11} = \frac{a_k+b_kI_k^{\max}+c_k}{(I_k^{\max})^2}$, $\mathbf{S}_k^{22} = \frac{a_k-b_k+c_k}{4\left(I_k^{\max}\right)^2}$, and $\mathbf{S}_k^{12} = \mathbf{S}_k^{21} = \frac{2a_k+b_k(I_k^{\max}-1)+c_k\left(2-4(I_k^{\max})^2\right)}{4\left(I_k^{\max}\right)^2}$. The VSC losses in (1) can be written as

$$P_{\mathrm{loss},k}^{\mathrm{conv}}(t) = \mathrm{Tr}\{\mathbf{S}_k\mathbf{I}_k(t)\}, \qquad k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}},\, t\in\mathcal{T}. \tag{38}$$

In the following parts, we transform the constraints and objective function of the SCUC problem (24) into an SDP.

1) Transforming the Constraints: Substituting (35a) into (18b), we have $P_{C_k}(t) = \mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}$, $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ and $P_{C_s}(t) = \mathrm{Tr}\{\mathbf{Y}_s\mathbf{W}(t)\}$, $s\in\mathcal{N}_{\mathrm{dc}}^{\mathrm{conv}}$. We also substitute $P_{\mathrm{loss},k}^{\mathrm{conv}}(t) = \mathrm{Tr}\{\mathbf{S}_k\mathbf{I}_k(t)\}$ into (2). For $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, $s\in\mathcal{N}_{\mathrm{dc}}^{\mathrm{conv}}$, and $t\in\mathcal{T}$, we obtain

$$\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+\mathrm{Tr}\{\mathbf{Y}_s\mathbf{W}(t)\}+\mathrm{Tr}\{\mathbf{S}_k\mathbf{I}_k\} = 0. \tag{39}$$

We define $\rho_k(t) = -|B_{C_k}|(V_k^{\max})^2$, $\xi_k = (B_{C_k}V_k^{\max})^2$, and matrix $\mathbf{C}_k(t) = (2\rho(t)_k+1)\bar{\mathbf{Y}}_k-\xi_k\mathbf{M}_f$ for ac bus $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ connected to filter bus $f\in\mathcal{N}_{\mathrm{ac}}$. In [18], it is shown that constraint (3) is equivalent to the following matrix inequality:

$$\begin{bmatrix} \rho_k^2(t)+\mathrm{Tr}\{\mathbf{C}_k(t)\mathbf{W}(t)\} & \frac{\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}}{\sqrt{2}} & \frac{\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}}{\sqrt{2}} & \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} \\ \frac{\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}}{\sqrt{2}} & \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} & 0 & \sqrt{2}\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} \\ \frac{\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}}{\sqrt{2}} & 0 & \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} & \sqrt{2}\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} & 0 & 1 \end{bmatrix} \succeq 0. \tag{40}$$

The SDP form of constraint (4) is

$$-m_k^{\mathrm{q}}S_{C_k}^{\mathrm{nom}} \le \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}. \tag{41}$$

The matrix form of inequality (5) is

$$\begin{bmatrix} (I_k^{\max})^2\mathrm{Tr}\{\mathbf{M}_k\mathbf{W}(t)\} & \mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} \\ \mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\} & 1 & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\} & 0 & 1 \end{bmatrix} \succeq 0. \tag{42}$$

The SDP form of constraint (6) is

$$\mathrm{Tr}\{\mathbf{M}_k\mathbf{W}(t)\} \le (m_{ks}^{\mathrm{v}})^2\mathrm{Tr}\{\mathbf{M}_s\mathbf{W}(t)\}. \tag{43}$$

The active power balance equation in (18a) can be combined with constraint (11a). Substituting (35a) into (11a), for $k\in\mathcal{N}\setminus\mathcal{N}^{\mathrm{conv}}$ and $t\in\mathcal{T}$, we obtain

$$u_k(t)P_{G_k}^{\min}+P_{B_k}(t)-P_{D_k}(t)\le\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}, \tag{44a}$$
$$\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}\le u_k(t)P_{G_k}^{\max}+P_{B_k}(t)-P_{D_k}(t). \tag{44b}$$

For ac buses $k\in\mathcal{N}_{\mathrm{ac}}$ and $t\in\mathcal{T}$, constraint (11b) becomes

$$u_k(t)Q_{G_k}^{\min}-Q_{D_k}(t)\le\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}, \tag{45a}$$
$$\mathrm{Tr}\{\bar{\mathbf{Y}}_k\mathbf{W}(t)\}\le u_k(t)Q_{G_k}^{\max}-Q_{D_k}(t). \tag{45b}$$

The active power balance equation in (18a) for time slots $t-1$ and t can be combined with constraints (12a) and (12b). By using (35a), for $k\in\mathcal{N}$ and $t\in\mathcal{T}$, we obtain

$$\begin{aligned}&\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}-\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t-1)\}-P_{B_k}(t)+P_{B_k}(t-1)\\&\le u_k(t-1)r_k^{\mathrm{u}}+s_k(t)r_k^{\mathrm{su}}+P_{D_k}(t-1)-P_{D_k}(t),\end{aligned} \tag{46a}$$

$$\begin{aligned}&\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t-1)\}-\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}-P_{B_k}(t-1)+P_{B_k}(t)\\&\le u_k(t-1)r_k^{\mathrm{d}}+s_k(t)\,r_k^{\mathrm{sd}}+P_{D_k}(t)-P_{D_k}(t-1).\end{aligned} \tag{46b}$$

By introducing constraint (25), the variable $u_k(t)$ can take any value in the interval [0, 1]. We have

$$0\le u_k(t)\le 1, \qquad k\in\mathcal{N},\, t\in\mathcal{T}. \tag{47}$$

Substituting (35c) into (19a), for $k\in\mathcal{N}$, we obtain

$$(V_k^{\min})^2\le\mathrm{Tr}\{\mathbf{M}_k\mathbf{W}\}\le(V_k^{\max})^2. \tag{48}$$

Substituting (35e) into (19b) and constructing its matrix form, for $(l,m)\in\mathcal{L}$, we have

$$\begin{bmatrix} (S_{lm}^{\max})^2 & \mathrm{Tr}\{\mathbf{Y}_{lm}\mathbf{W}(t)\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}(t)\} \\ \mathrm{Tr}\{\mathbf{Y}_{lm}\mathbf{W}(t)\} & 1 & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}(t)\} & 0 & 1 \end{bmatrix} \succeq 0. \tag{49}$$

Let $\mathbf{I}_k^{12}(t)$ denote the entry in the first row and the second column of matrix $\mathbf{I}_k(t)$ in (36). We have $I_k^{\max}-2\mathbf{I}_k^{12}(t) = |I_k(t)|^2 = (R_{C_k}^2+X_{C_k}^2)^{-1}|V_k(t)-V_f(t)|^2$ in time slot t for ac bus $k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ and filter bus $f\in\mathcal{N}_{\mathrm{ac}}$ in a VSC station. From (35d), we obtain

$$\mathbf{I}_k^{12}(t) = \frac{\mathrm{Tr}\{\mathbf{M}_{kf}\mathbf{W}(t)\}}{2\left(I_k^{\max}-(R_{C_k}^2+X_{C_k}^2)\right)}. \tag{50}$$

Let $\mathbf{I}_k^{11}(t)$ and $\mathbf{I}_k^{22}(t)$ denote the diagonal entries of matrix $\mathbf{I}_k(t)$ in (36). From (36), we have

$$\mathbf{I}_k^{11}(t)+\frac{\mathbf{I}_k^{22}(t)}{4} = (I_k^{\max})^2-\mathbf{I}_k^{12}(t), \ \ k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}},\, t\in\mathcal{T} \tag{51a}$$
$$\mathbf{I}_k^{11}(t)\ge\frac{(I_k^{\max})^2}{4}, \qquad k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}},\, t\in\mathcal{T}. \tag{51b}$$

We can show that constraints (51a) and (51b) are sufficient to obtain matrices $\mathbf{I}_k(t), k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}, t\in\mathcal{T}$ with the form in (36).

2) Transforming the Objective Function: Substituting (35a) into (18a) for $k\in\mathcal{N}$, we have $P_{G_k}(t) = \mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)$. Let vector $\boldsymbol{P}_B(t) = (P_{B_k}(t),\, k\in\mathcal{N})$ denote the profile of injected active power from the energy storage systems in time slot t. We have

$$C_{\text{gen}}\big(\mathbf{W}(t),\,\boldsymbol{P}_D(t),\,\boldsymbol{P}_B(t),\,\boldsymbol{u}(t),\,\boldsymbol{s}(t),\,\boldsymbol{d}(t)\big)=\sum_{k\in\mathcal{N}}\Big(c_{k2}\big(\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)\big)^2+c_{k1}\big(\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)\big)+c_{k0}u_k(t)+c_k^{\text{su}}s_k(t)+c_k^{\text{sd}}d_k(t)\Big).\quad(52)$$

We introduce the auxiliary variables $\vartheta_k(t)$, $k\in\mathcal{N}$, $t\in\mathcal{T}$ and replace $C_{\text{gen}}(\cdot)$ with $\sum_{k\in\mathcal{N}}\vartheta_k(t)+c_{k0}u_k(t)+c_k^{\text{su}}s_k(t)+c_k^{\text{sd}}d_k(t)$. Then, we include the matrix form of inequality $c_{k2}\left(\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)\right)^2+c_{k1}\big(\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)\big)\le\vartheta_k(t)$ into the constraints set for all generators. For $k\in\mathcal{N}$ and $t\in\mathcal{T}$, we have

$$\begin{bmatrix}\vartheta_k(t)-c_{k1}\omega_k(t) & \sqrt{c_{k2}}\,\omega_k(t)\\ \sqrt{c_{k2}}\,\omega_k(t) & 1\end{bmatrix}\succeq 0,\quad(53)$$

where $\omega_k(t)=\text{Tr}\{\mathbf{Y}_\text{k}\mathbf{W}(t)\}+P_{D_k}(t)-P_{B_k}(t)$ in time slot t.

By introducing the auxiliary variables $\mu^j(t)$ for each sample j in time slot t and $\eta_k^j(t)$ for each bus k and sample j in time slot t, we can replace function $\Gamma_\beta(\cdot)$ in (23) with $\alpha(t)+\frac{1}{1-\beta}\sum_{j\in\mathcal{J}}\text{Pr}\big\{\boldsymbol{P}_G^j(t),\,\boldsymbol{P}_D^j(t)\big\}\mu^j(t)$. Then, we include the following inequalities for bus $k\in\mathcal{N}$, sample $j\in\mathcal{J}$, and time slot $t\in\mathcal{T}$ into the constraints set:

$$\textstyle\sum_{k\in\mathcal{N}}c_{\text{res},k}\eta_k^j(t)\le\mu^j(t)+\alpha(t),\quad(54a)$$
$$P_{G_k}(t)-P_{D_k}(t)-P_{G_k}^j(t)+P_{D_k}^j(t)\le\eta_k^j(t).\quad(54b)$$

To represent the total system losses, we can substitute (35a) into (16). Thus, we obtain

$$P_{\text{loss}}(t)=\textstyle\sum_{k\in\mathcal{N}}\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}.\quad(55)$$

The SDP form of (23) can be expressed as

$$f_{\text{obj}}^{\text{SDP}}=\sum_{t\in\mathcal{T}}\Bigg(\sum_{k\in\mathcal{N}}\big(\vartheta_k(t)+c_{k0}u_k(t)+c_k^{\text{su}}s_k(t)+c_k^{\text{sd}}d_k(t)\big)+\omega_{\text{loss}}\sum_{k\in\mathcal{N}}\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\}+\omega_{\text{cvar}}\Big(\alpha(t)+\frac{1}{1-\beta}\sum_{j\in\mathcal{J}}\text{Pr}\big\{\boldsymbol{P}_G^j(t),\boldsymbol{P}_D^j(t)\big\}\mu^j(t)\Big)\Bigg).\quad(56)$$

Constraints (7), (8), (13a), (13b), (14), (15b), (39)−(51b), and (53)−(54b) define the feasible set Φ^{SDP} for the decision variables $\boldsymbol{\phi}=(\vartheta_k(t),\,u_k(t),\,s_k(t),\,d_k(t),\,P_{B_k}(t),\,\mu^j(t),\,\eta_k^j(t),j\in\mathcal{J},\,k\in\mathcal{N},\,\alpha(t),\mathbf{W}(t),\,\mathbf{I}_k(t),\,k\in\mathcal{N}_{\text{ac}}^{\text{conv}},\,t\in\mathcal{T})$.

B. Proof of Theorem 1

We show that the solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ to problem (30) are rank two to minimize the losses in the system. The losses on the VSC station with converter bus $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ in time slot $t\in\mathcal{T}$ are

$$P_{\text{loss},k}^{\text{conv}}(t)=\text{Tr}\{\mathbf{S}_k\mathbf{I}_k(t)\}=\mathbf{I}_k^{11}(t)\,\mathbf{S}_k^{11}+2\,\mathbf{I}_k^{12}(t)\,\mathbf{S}_k^{12}+\mathbf{I}_k^{22}(t)\,\mathbf{S}_k^{22}.\quad(57)$$

To minimize the converter losses in (57), problem (30) returns solution matrix $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ with minimum value for $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)\,\mathbf{S}_k^{11}+\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22}(t)\,\mathbf{S}_k^{22}$ subject to constraints (51a) and (51b). From (51a), we have

$$\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22}(t)=4\left(\left(I_k^{\max}\right)^2-\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{12}(t)-\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)\right).\quad(58)$$

Substituting (58) into $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)\,\mathbf{S}_k^{11}+\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22}(t)\,\mathbf{S}_k^{22}$ and performing some algebraic manipulations, we obtain

$$\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)\,\mathbf{S}_k^{11}+\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22}(t)\,\mathbf{S}_k^{22}=\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)\,(\mathbf{S}_k^{11}-4\,\mathbf{S}_k^{22})+4\left(\left(I_k^{\max}\right)^2-\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{12}(t)\right)\mathbf{S}_k^{22}.\quad(59)$$

We have $\mathbf{S}_k^{11}=\frac{a_k+b_kI_k^{\max}+c_k}{(I_k^{\max})^2}$ and $\mathbf{S}_k^{22}=\frac{a_k-b_k+c_k}{4\left(I_k^{\max}\right)^2}$. Thus, we obtain $\mathbf{S}_k^{11}\ge 4\mathbf{S}_k^{22}$. To minimize (59), the entry $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)$ should be minimized. From constraint (51b), the minimum value of $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11}(t)$ is $\frac{(I_k^{\max})^2}{4}$. Therefore, the solution matrix $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ does not have the form in (36), and it is not rank one. This completes the proof. ■

C. Proof of Theorem 2

From the proof of Theorem 1, the solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ to problem (30) are all rank two because we have $\mathbf{S}_k^{11}\ge 4\mathbf{S}_k^{22}$. We also have $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)=\frac{(I_k^{\max})^2}{4}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$. We introduce the trace norm $\varepsilon\,\text{Tr}\{\mathbf{I}_k(t)\}$, for all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ to the objective function of problem (31) to make the solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ all rank one.

Consider a coefficient ε, for which the solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ to problem (31) are all rank one. Let $\Delta\,\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)$ and $\Delta\,\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)$ denote the difference in the optimal values of entries $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)$ and $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)$ of solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ to problems (30) and (31), respectively. When the rank two solution matrices $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ in problem (30) becomes rank one for problem (31), the system losses increase by approximately $\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Delta\,\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\,\mathbf{S}_k^{11}+\Delta\,\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)\,\mathbf{S}_k^{22}$. When the losses increase, the generation level in the system will increase as well. Let $P_{G_k}^{30}(t)$ and $P_{G_k}^{31}(t)$ denote the output power of the generator in bus k in time slot t by solving problems (30) and (31), respectively. Let $\Delta P_{G_k}(t)=P_{G_k}^{31}(t)-P_{G_k}^{30}(t)$ denote the change in the output power of the generator in bus k in time slot t. In the worst case, the increase in the system losses is compensated by the conventional generators. Hence, the CVaR term in (23) can be assumed to be unchanged in the worst-case scenario. Let ΔC_{gen} denote the change in the total generation cost in (9). Using the first-order approximation near $P_{G_k}^{30}(t)$, $k\in\mathcal{N}$, $t\in\mathcal{T}$, we have

$$\Delta C_{\text{gen}}\approx\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}\big(2c_{k2}P_{G_k}^{30}(t)+c_{k1}\big)\,\Delta P_{G_k}(t).\quad(60)$$

Substituting $c_1^{\max}$, $c_2^{\max}$, and $P_G^{\max}$ into (60), we obtain

$$\Delta C_{\text{gen}}\le K\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}\Delta P_{G_k}(t),\quad(61)$$

where $K=2c_2^{\max}P_G^{\max}+c_1^{\max}$. The increase in the generation levels is equal to the increase in the losses. Hence, we have

$$\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}}\Delta P_{G_k}(t)=$$

$$\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\mathbf{S}_k^{11}+\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)\mathbf{S}_k^{22}. \tag{62}$$

Thus, we obtain

$$\Delta C_{\text{gen}}\le K\Bigg(\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\mathbf{S}_k^{11}+\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)\mathbf{S}_k^{22}\Bigg). \tag{63}$$

Furthermore, the change in other terms in the objective function of problem (31) is equal to

$$\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Big(\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\left(\omega_{\text{loss}}\mathbf{S}_k^{11}+\varepsilon\right) + \Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)\left(\omega_{\text{loss}}\mathbf{S}_k^{22}+\varepsilon\right)\Big). \tag{64}$$

For an appropriate value of ε, the change in the objective function of problem (31) is negative. Therefore, we have

$$\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{11}+\varepsilon\right) + \Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t)\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{22}+\varepsilon\right)\le 0. \tag{65}$$

Moreover, equation (59) implies that $\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{22,\text{opt}}(t) = -4\,\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)$. Hence, inequality (65) is equivalent to

$$\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Delta\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{11,\text{opt}}(t)\Big(\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{11}+\varepsilon\right) - 4\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{22}+\varepsilon\right)\Big)\le 0. \tag{66}$$

If ε is chosen such that $\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{11}+\varepsilon\right) - 4\left(\left(K+\omega_{\text{loss}}\right)\mathbf{S}_k^{22}+\varepsilon\right)\le 0$, then inequality (66) holds. Thus, it is sufficient to have

$$\varepsilon\ge\frac{(K+\omega_{\text{loss}})}{3}\,(\mathbf{S}_k^{11}-4\,\mathbf{S}_k^{22}). \tag{67}$$

From $\mathbf{S}_k^{11}=\frac{a_k+b_kI_k^{\max}+c_k}{(I_k^{\max})^2}$ and $\mathbf{S}_k^{22}=\frac{a_k-b_k+c_k}{4\left(I_k^{\max}\right)^2}$, we obtain $\mathbf{S}_k^{11}-4\,\mathbf{S}_k^{22}=\dfrac{(I_k^{\max}+1)b_k}{(I_k^{\max})^2}$. Thus, (67) is equivalent to

$$\varepsilon\ge b_k\left(K+\omega_{\text{loss}}\right)\frac{(I_k^{\max}+1)}{3(I_k^{\max})^2}. \tag{68}$$

Inequality (68) must hold for all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$. Thus, b_k, c_{k1} and c_{k2}, and $P_{G_k}^{\max}$ are replaced with their maximum values, and $I_k^{\max}$ is relplaced with its minimum value. Thus, ε can be approximated by (32). The proof is completed. ■

D. Proof of Theorem 3

Let $\boldsymbol{\phi}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}$ be an optimal solution to problem (31) under the given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. Consider time slot $t\in\mathcal{T}$. We show that matrix $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$ is at most rank two if for all set of operating generators, the SDP relaxation gap for the ac-dc OPF problem in the underlying ac-dc grid is zero. We construct an ac-dc OPF problem from the SCUC problem (31) by replacing all variables with their optimal values in problem (31) except for matrices $\mathbf{W}(t)$ and $\mathbf{I}_k(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, and variables $\vartheta_k(t)$, $k\in\mathcal{N}$. Let $f_{\text{obj}}^{\text{OPF}}$ denote the objective function of the obtained ac-dc OPF problem. We have

$$f_{\text{obj}}^{\text{OPF}}=\sum_{k\in\mathcal{N}}\vartheta_k(t)+\omega_{\text{loss}}\sum_{k\in\mathcal{N}}\text{Tr}\{\mathbf{Y}_k\mathbf{W}(t)\} + \varepsilon\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\text{Tr}\{\mathbf{I}_k(t)\}. \tag{69}$$

Let $\widetilde{\boldsymbol{\phi}}=(\mathbf{W}(t),\,\mathbf{I}_k(t),\,k\in\mathcal{N}_{\text{ac}}^{\text{conv}},\,\vartheta_k(t),\,k\in\mathcal{N})$ denote the decision variables in the obtained ac-dc OPF problem for time slot t. Let Φ^{OPF} denote the feasible set for decision variables $\widetilde{\boldsymbol{\phi}}$. It can be obtained from the feasible set Φ^{SDP} for the SCUC problem (31) when all variables except decision variables $\widetilde{\boldsymbol{\phi}}$ are replaced with their optimal values in problem (31). Hence, the ac-dc OPF problem in time slot t obtained from problem (31) can be written as

$$\underset{\widetilde{\boldsymbol{\phi}}\in\Phi^{\text{OPF}}}{\text{minimize}}\quad f_{\text{obj}}^{\text{OPF}} \tag{70a}$$

$$\text{subject to}\quad \mathbf{I}_k(t)\succeq 0,\qquad k\in\mathcal{N}_{\text{ac}}^{\text{conv}}, \tag{70b}$$

$$\mathbf{W}(t)\succeq 0. \tag{70c}$$

The feasible set of problem (70) is a subset of the feasible set of problem (31). The decision variables for problem (31) other than those in $\widetilde{\boldsymbol{\phi}}$ are set to their optimal values in problem (70). Hence, the solution matrix $\mathbf{W}^{\text{opt}}(t)$ to problem (70) is equal to the solution matrix $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$ to problem (31). If the SDP relaxation gap for the OPF problem (70) is zero, then problem (70) has a solution $\mathbf{W}^{\text{opt}}(t)$ with rank of at most two [18]. Thus, the solution matrix $\mathbf{W}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt}}(t)$ to problem (31) is at most rank two in time slot t. Using the same approach for all time slots $t\in\mathcal{T}$ completes the proof. ■

E. Proof of Theorem 4

$f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}$ is the optimal value of problem (29). Hence, the value of $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG}}$ is greater than or equal to $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}$ for any feasible solutions of problem (29). The optimal solution of problem (31) that satisfies the rank constraints (26c) and (26d) is a feasible solution of problem (29). Thus, we have

$$f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}\le f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}, \tag{71}$$

which proves the left-hand side of (33).

Let $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},29}(t)$ and $\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},31}(t)$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, $t\in\mathcal{T}$ denote the solution matrices to problems (29) and (31), respectively. Since $f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}+\varepsilon\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},31}(t)\}$ is the optimal objective value of problem (31), we have

$$f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}+\varepsilon\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},31}(t)\} \le f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29}+\varepsilon\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},29}(t)\}. \tag{72}$$

After rearranging the terms, inequality (72) becomes

$$f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},31}-f_{\text{obj},\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{REG},29} \le\varepsilon\sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Big(\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},29}(t)\}-\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},31}(t)\}\Big). \tag{73}$$

We determine the upper bound for $\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},29}(t)\}-\text{Tr}\{\mathbf{I}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}^{\text{opt},31}(t)\}$. According to Theorem 1, rank one matrices

$\mathbf{I}^{\text{opt},29}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)$ and $\mathbf{I}^{\text{opt},31}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)$ have the form in (36). Therefore, there exist nonnegative numbers $|I_k^{29}(t)|$ and $|I_k^{31}(t)|$ such that

$$\text{Tr}\{\mathbf{I}^{\text{opt},29}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\} = \frac{\left(I_k^{\max} + |I_k^{29}(t)|\right)^2}{4} + (I_k^{\max} - |I_k^{29}(t)|)^2, \tag{74a}$$

$$\text{Tr}\{\mathbf{I}^{\text{opt},31}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\} = \frac{\left(I_k^{\max} + |I_k^{31}(t)|\right)^2}{4} + (I_k^{\max} - |I_k^{31}(t)|)^2. \tag{74b}$$

We can show that $I_k^{29}(t) = 0$ maximizes $\text{Tr}\{\mathbf{I}^{\text{opt},29}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\}$ in (74a). We can also show that $I_k^{31}(t) = \frac{3}{5} I_k^{\max}$ minimizes $\text{Tr}\{\mathbf{I}^{\text{opt},31}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\}$ in (74b). Hence, we have

$$\begin{aligned}\text{Tr}\{\mathbf{I}^{\text{opt},29}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\} - \text{Tr}\{\mathbf{I}^{\text{opt},31}_{k,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(t)\} &\leq \frac{5}{4}(I_k^{\max})^2 - \frac{4}{5}(I_k^{\max})^2 \\ &= 0.45\,(I_k^{\max})^2. \end{aligned} \tag{75}$$

By substituting the right-hand side of (75) into (73), we obtain the upper bound in (33). The proof is completed. ■

F. Proof of Theorem 5

Consider problem (27). We approximate $\|u_k(t)\|_0$ and $\|1-u_k(t)\|_0$ by $\frac{\log\left(1+\frac{\|u_k(t)\|_1}{\sigma}\right)}{\log\left(1+\frac{1}{\sigma}\right)}$ and $\frac{\log\left(1+\frac{\|1-u_k(t)\|_1}{\sigma}\right)}{\log\left(1+\frac{1}{\sigma}\right)}$, respectively. For sufficiently small σ, the approximation is tight. We use the first order Taylor approximation of functions $\log\left(1+\frac{\|u_k(t)\|_1}{\sigma}\right)$ and $\log\left(1+\frac{\|1-u_k(t)\|_1}{\sigma}\right)$ near an arbitrary point $\widehat{u}_k(t)$ as follows:

$$\log\left(1+\frac{\|u_k(t)\|_1}{\sigma}\right) \approx \log\left(1+\frac{\|\widehat{u}_k(t)\|_1}{\sigma}\right) + \frac{\|u_k(t)\|_1 - \|\widehat{u}_k(t)\|_1}{\|\widehat{u}_k(t)\|_1 + \sigma}, \tag{85a}$$

$$\log\left(1+\frac{\|1-u_k(t)\|_1}{\sigma}\right) \approx \log\left(1+\frac{\|1-\widehat{u}_k(t)\|_1}{\sigma}\right) + \frac{\|1-u_k(t)\|_1 - \|1-\widehat{u}_k(t)\|_1}{\|1-\widehat{u}_k(t)\|_1 + \sigma}. \tag{85b}$$

We substitute (85a) and (85b) into the objective function of problem (27) and remove the constant terms. We have the following optimization problem.

$$\underset{\phi\in\Phi^{\text{SDP}}}{\text{minimize}}\; f_{\text{obj}}^{\text{SDP}} + \sum_{t\in\mathcal{T}}\sum_{k\in\mathcal{N}} \frac{\varsigma\,\|u_k(t)\|_1}{\|\widehat{u}_k(t)\|_1+\sigma} + \frac{\varsigma\,\|1-u_k(t)\|_1}{\|1-\widehat{u}_k(t)\|_1+\sigma} \tag{86}$$

subject to constraints (26c)−(26f).

The coefficients $\frac{\varsigma}{\|\widehat{u}_k(t)\|_1+\sigma}$ and $\frac{\varsigma}{\|1-\widehat{u}_k(t)\|_1+\sigma}$ in the objective function of problem (86) correspond to the coefficients $\theta_{k1}(t)$ and $\theta_{k2}(t)$ in the objective function of problem (29), respectively. The linear approximation in the right-hand side of (85a) and (85b) are tangent majorant of the logarithmic functions in the left-hand side of (85a) and (85b), respectively. The right-hand sides majorize the left-hand sides with equality at $\widehat{u}_k(t)$. Hence, we can apply the majorization-minimization (MM) algorithm [28] to obtain vector $\widehat{\boldsymbol{u}}$ in an iterative fashion. Lines 1 to 6 of Algorithm 1 correspond to the iterations in the MM algorithm. We start with an arbitrary initial vector $\widehat{\boldsymbol{u}}^1$, which corresponds to the initial value for coefficients $\boldsymbol{\theta}_1^1$ and $\boldsymbol{\theta}_2^1$. In iteration i, we solve the convex optimization problem (31) under the given vector $\widehat{\boldsymbol{u}}_k^i$ (which corresponds to coefficients $\boldsymbol{\theta}_1^i$ and $\boldsymbol{\theta}_2^i$) and obtain the updated vector $\widehat{\boldsymbol{u}}^{i+1}$ as $\widehat{\boldsymbol{u}}^{i+1} = (u^{\text{opt},i}_{k,\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}(t),\, k\in\mathcal{N},\, t\in\mathcal{T})$. This procedure corresponds to the loop within Lines 2 to 6 in Algorithm 1. The objective function in each iteration of Algorithm 1 is convex and continuously differentiable and the feasible set is closed and convex. Hence, if the optimal solution to problem (31) can be obtained in each iteration (i.e., matrices $\mathbf{W}^{\text{opt},i}_{\boldsymbol{\theta}_1^i,\boldsymbol{\theta}_2^i}(t),\, t\in\mathcal{T}$ are at most rank two), then for sufficiently small σ and large ς, the proposed algorithm converges to local optimal solution [28]. The proof is completed. ■

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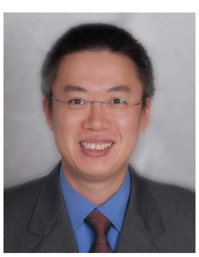

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