Semidefinite Relaxation of Optimal Power Flow for ac-dc Grids

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***Abstract*—The proliferation of technologies operating on dc power has motivated system planners towards integration of dc and ac grids. The optimal power flow (OPF) analysis is widely used to determine the economically efficient operating points of the power grids. The OPF problem in ac-dc grids is a non-convex optimization problem due to the nonlinear power flow equations and the operating constraints imposed by the ac-dc converters. In this paper, we study the OPF problem in ac-dc grids to address the non-convexity of the problem. The objective of the ac-dc OPF problem is to jointly minimize the generation cost and the losses on the lines and converters. The optimization problem is subject to the ac and dc power flow constraints, the limits of the voltages and line flows, and the operating limits of the converters. We use convex relaxation techniques and transform the problem to a semidefinite program. We derive a sufficient condition for zero relaxation gap to obtain the global optimal solution. Simulations are performed on an IEEE 118-bus test system connected to sample dc grids. We show that the zero relaxation gap condition holds for the case study and the global optimal solution can be obtained.**

***Index Terms*—Optimal power flow, ac-dc grid, semidefinite program.**

I. Introduction

The transition from ac grids to integrated ac-dc grids is accelerating due to the advance in power electronics and the increase of the applications for dc power. Many renewable energy resources, such as photovoltaic (PV) systems, generate dc power [1]. High voltage direct current (HVDC) transmission lines facilitate transporting power to or from remote areas [2]. Many electrical loads, such as motor loads, pumps, and lighting consume dc power [3]. The dc microgrid can be a viable alternative as an energy source for data centers [4]. Batteries, supercapacitors, and fuel cells store energy as dc [5]. Another motivation for moving towards ac-dc grids is the increase of energy efficiency by eliminating losses associated with dc-ac-dc conversions [6]. Besides, ac-dc grids can be more reliable than ac grids since the additional conversion steps in ac grids may introduce potential faults [7].

Optimal power flow (OPF) plays an important role in power system operation. In OPF, the output power of the generators is determined by optimizing an objective function such as the total generation cost and the system power losses [8]. The OPF problem in ac grids is subject to physical and operating constraints such as power balance constraints, and voltage magnitude and power flow limits [8]. The OPF in ac grids takes the form of a non-convex optimization problem, and is generally difficult to solve. The non-convexity of the problem arises from the nonlinear power flow equations and quadratic dependency on the set of bus voltages.

It is a challenge to determine the global optimal solution among the multiple local optimal solutions of the OPF problem in ac grids. Recently, semidefinite programming (SDP) and convex relaxation of the OPF problem in ac grids have attracted significant research attention [9]–[16]. These techniques are guaranteed to determine the global optimal solution when the relaxation gap is zero. The work in [9] presents the bus injection and branch flow models of the OPF problem and proves their equivalence. In [10], the convex relaxations of the OPF problem based on SDP, chordal extension, and second-order cone programming (SOCP) are studied. The sufficient conditions that guarantee the exactness of these relaxations are provided. In [11], a nested optimization approach is proposed to decompose the multi-period joint OPF and electric vehicle (EV) charging problem into separable subproblems. The decomposed problem is solved using a nonsmooth separable programming technique. In [12], it is shown that the SDP relaxation for ac grids has zero relaxation gap for practical power grids including IEEE test power systems. The work in [13] presents the geometric properties of the set of all vectors of bus power injections satisfying the operating constraints of a radial power network. The convex relaxation of the OPF problem is obtained and the sufficient condition to determine a global optimal solution of the OPF problem is derived. The work in [14] presents a branch flow model for the analysis of meshed and radial networks. The proposed convex relaxation method is exact for radial networks provided there are no upper bounds on loads or voltage magnitudes. The work in [15] has proposed the SOCP relaxation of the OPF problem for resistive networks. The uniqueness of the optimal solution is characterized for radial and meshed networks. In [16], a model for the power lines capacity is presented. The zero relaxation gap for weakly cyclic networks is studied. The upper bound on the rank of the minimum-rank solution of the SDP relaxation is provided.

In ac-dc grids, the OPF problem includes the constraints imposed by the limits on the voltage and current ratings of the converters in addition to the constraints imposed by the ac and dc grids. The converter losses can add up to a significant

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fraction of the overall system losses. Thus, the converter losses are usually included in the the ac-dc OPF problem [17]. The losses in a converter can be approximated by a quadratic function of its current magnitude [18]. The ac-dc OPF is also a non-convex problem due to the quadratic form of the converter losses and nonlinear power flow equations. Several methods have been proposed to address the ac-dc OPF such as heuristic and interior point methods [19], [20], the Newton-Raphson method [21], SOCP [22], and sequential approaches [23].

Most of the works on OPF in ac-dc power grids (e.g., [20]–[23]) do not guarantee to obtain the global optimal solution. In this paper, our goal is to determine the global optimal solution of the ac-dc OPF problem. The efficiency of SDP in solving the ac OPF problem has motivated us to use this approach to solve the ac-dc OPF problem. In ac-dc systems, the ac and dc buses are connected through three-phase ac-dc converters. To model the ac-dc converters, we consider voltage source converters (VSCs), which are widely used in practice [24]. In this paper, we extend the work we have done in [25] by considering the converter losses and the operating constraints imposed by the VSCs in the ac-dc OPF problem. The challenges include obtaining the SDP form of the power losses and the constraints on the reactive and apparent power flows in the VSCs. Deriving a sufficient condition for zero relaxation gap is another challenge that we address in this paper. The main contributions of this paper are as follows:

- We introduce a general ac-dc OPF problem formulation, which can be used in different scenarios including (a) an ac grid connected to dc microgrids via VSCs, and (b) an ac grid embedded with dc cables such as HVDC lines. We model the losses and the limits on the voltage and current ratings of the VSCs in the ac-dc OPF problem.
- We transform the problem into SDP, solve the problem, and determine the zero relaxation gap condition. We show that the zero relaxation gap condition can hold in practical ac-dc grids including the IEEE test systems connected to some dc grids. We describe how the solution of the original ac-dc OPF problem can be determined from the solution of the SDP form of the ac-dc OPF.
- Simulations are performed on an IEEE 118-bus test system connected to five off-shore wind farms using HVDC lines and one ac-dc microgrid. We show that the SDP form of the OPF problem has zero relaxation gap and it can provide the global optimal solution.

The rest of this paper is organized as follows. The models for ac-dc grids and the VSCs are presented in Section II. In Section III, the ac-dc OPF problem is formulated and is transformed to an SDP. The zero relaxation gap condition is also studied. Simulation results are presented in Section IV. The paper is concluded in Section V.

II. System Model and ac-dc OPF Formulation

Consider an ac-dc grid consisting of an ac grid connected to multiple dc grids. We represent an ac-dc grid by a tuple $\mathcal{O}(\mathcal{N},\mathcal{L})$, where $\mathcal{N}$ denotes the set of buses and $\mathcal{L}$ denotes the set of transmission lines. The dc grids are connected to the ac grid using VSCs at some buses. An ac-dc grid with a

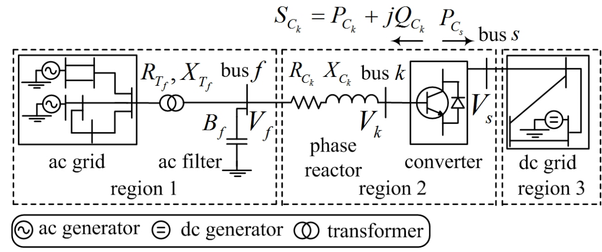

Fig. 1. A VSC station schematic in an ac-dc grid.

VSC station is shown in Fig. 1. The VSC station consists of a transformer, ac filter, phase reactor, and converter. Without loss of generality, the VSC station is assumed to be a two- or three-level converter using the pulse-width modulation (PWM) switching method. Let $\mathcal{N}_{\text{ac}} \subseteq \mathcal{N}$ denote the set of ac buses that are not connected to the converters. Let $\mathcal{N}_{\text{dc}} \subseteq \mathcal{N}$ denote the set of dc buses that are not connected to the converters. Let $\mathcal{G} \subseteq \mathcal{N}$ denote the set of generator buses. Let $\mathcal{N}_{\text{ac}}^{\text{conv}} \subseteq \mathcal{N}$ denote the set of ac side converter buses. Let $\mathcal{N}_{\text{dc}}^{\text{conv}} \subseteq \mathcal{N}$ denote the set of dc side converter buses. Let $\mathcal{N}^{\text{conv}} = \mathcal{N}_{\text{ac}}^{\text{conv}} \cup \mathcal{N}_{\text{dc}}^{\text{conv}}$ denote the set of all converter buses. Let X_{C_k} denote the phase reactor of the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. Let R_{C_k} denote the resistance modeling the losses of the non-ideal phase reactor of the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. Let B_f denote the shunt susceptance for the ac filter connected to the filter bus $f \in \mathcal{N}_{\text{ac}}$. Let X_{T_f} and R_{T_f} denote the reactance and resistance of the transformer connecting the ac grid to the filter bus $f \in \mathcal{N}_{\text{ac}}$, respectively. In Fig. 1, buses k and s are in sets $\mathcal{N}_{\text{ac}}^{\text{conv}}$ and $\mathcal{N}_{\text{dc}}^{\text{conv}}$, respectively. The buses in region 1 are in set $\mathcal{N}_{\text{ac}}$, the buses in region 2 are in set $\mathcal{N}^{\text{conv}}$, and the buses in region 3 are in set $\mathcal{N}_{\text{dc}}$. The following assumptions are made in modeling the VSC station and ac-dc grid.

A1. The VSCs can control the active power (or the dc voltage) and the reactive power (or the ac voltage) magnitude and phase angle of the ac terminal voltages [26].

A2. In a VSC, the resistance R_{C_k} is much smaller than the reactance X_{C_k} for $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. Hence, the conductance of the phase reactor is negligible compared to its susceptance [17].

A3. The difference of the phase angles $\delta_k - \delta_f$ for ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and filter bus $f \in \mathcal{N}_{\text{ac}}$ in a VSC station is small [17].

A4. A dc grid is modeled as an ac grid with purely resistive transmission lines and generators operating at unity power factor.

The converter between ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ converts ac voltage V_k to dc voltage V_s. Let m_a denote the maximum modulation factor, which can be set according to the modulation mode. The voltage magnitude at the converter ac bus is upper bounded by [27]

$$|V_k| \leq m_a |V_s|. \tag{1}$$

Let P_{C_k} and P_{C_s} denote the active power injected into ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$, respectively. Let $P_{\text{loss},k}^{\text{conv}}$ denote the losses of the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. The active power balance equation for the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ is

$$P_{C_k} + P_{C_s} + P_{\text{loss},k}^{\text{conv}} = 0. \tag{2}$$

Let I_k denote the injected current into an arbitrary bus $k \in \mathcal{N}$. For a VSC station, the power losses can be determined from the aggregate losses of the components in the station such as the transformer, ac filter, phase reactor, and converter. The losses of the transformer, ac filter and phase reactor include the losses through the equivalent series resistance, the core losses, and the losses due to the harmonic currents. The losses of the converter include switching losses and conductance losses [28]. The detailed losses model of the VSC station can be found in [29] and [30]. However for the purpose of this paper, the losses in a VSC station with ac converter bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ can be approximated by a quadratic function of the ac current magnitude $|I_k|$ [17]–[19], [30]. Hence,

$$P_{\text{loss},k}^{\text{conv}} = a_k + b_k|I_k| + c_k|I_k|^2, \tag{3}$$

where a_k, b_k, and c_k are positive coefficients representing the constant or no-load losses (e.g., filter losses, transformer core losses), the linear losses (e.g., switching), and the quadratic losses (e.g., transformer, phase reactor copper losses, and conduction losses), respectively. The model in (3) takes into account the losses of every component of a VSC station. The values of the coefficients a_k, b_k, and c_k depend on the components and the power rating of the VSC station station, and can be obtained using various approaches such as online identification or by aggregating the loss patterns of each component. The losses model for a sample HVDC link rated at 600 MW and 300 kV is given in [30, pp. 58-60], wherein it is seen that a quadratic losses model as a function of the phase reactor current offers sufficient accuracy for system-level studies.

In general, the current magnitude $|I_k|$ of a converter should not exceed an upper limit denoted by $I_k^{\max}$. The upper bound of the current amplitude can be replaced by the maximum apparent power flow as an operation constraint [17]. Let S_{C_k} and Q_{C_k} denote the apparent and reactive power of the converter connected to bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, respectively. We have

$$|S_{C_k}|^2 = (P_{C_k})^2 + (Q_{C_k})^2 \leq (|V_k| I_k^{\max})^2. \tag{4}$$

From assumption A1, a VSC station can control active and reactive powers to adjust the voltage of the ac and dc terminal buses. Therefore, the active and reactive powers P_{C_k} and Q_{C_k} are variables and should satisfy constraint (4).

The operation of the VSC is constrained by the upper and lower limits of the reactive output power of the converter. Let $S_{C_k}^{\text{nom}}$ denote the nominal value of the apparent power of the converter connected to bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. In practical VSCs, the maximum reactive power that the converter can absorb is approximately proportional to the nominal value of its apparent power, $S_{C_k}^{\text{nom}}$ [17]. For the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, we have

$$-m_b S_{C_k}^{\text{nom}} \leq Q_{C_k}, \tag{5}$$

where m_b is a positive constant and can be determined by the type of the converter. Let $V_k^{\max}$ denote the maximum voltage magnitude of bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. Let δ_f and δ_k denote the phase angle of the voltages at filter bus $f \in \mathcal{N}_{\text{ac}}$ and converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ in the VSC station, respectively.

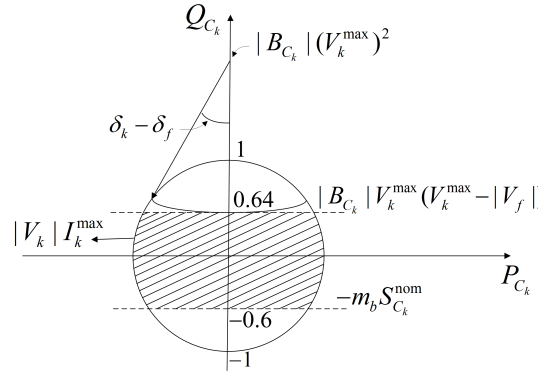

Fig. 2. Q-P characteristic of the converter for $|V_k| = 1\,\text{pu}, V_k^{\max} = 1.05\,\text{pu}, I_k^{\max} = 1\,\text{pu}, |V_f| = 0.95\,\text{pu}, R_{C_k} = 0.001\,\text{pu}, X_{C_k} = 0.1643\,\text{pu}, S_k^{\text{nom}} = 1\,\text{pu}$, and $m_b = 0.6$.

Let $B_{C_k} = \frac{-X_{C_k}}{R_{C_k}^2 + X_{C_k}^2}$ denote the susceptance of the non-ideal phase reactor connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. From assumption A2, the conductance of the phase reactor is negligible compared to its susceptance. For filter bus $f \in \mathcal{N}_{\text{ac}}$ and converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ in the VSC station, the reactive power that the converter can inject is upper bounded by

$$Q_{C_k} \leq |B_{C_k}| V_k^{\max} \left(V_k^{\max} - |V_f| \cos(\delta_k - \delta_f) \right). \tag{6}$$

From assumption A3, we obtain $\cos(\delta_k - \delta_f) \approx 1$. Hence, the upper limit for the injected reactive power can be approximated by the minimum value of the right-hand side of (6). We have

$$Q_{C_k} \leq |B_{C_k}| V_k^{\max} \left(V_k^{\max} - |V_f| \right). \tag{7}$$

The upper bound of the converter apparent power and the upper and lower bounds of the converter reactive power are shown in Fig. 2 for a sample VSC station. Constraint (4) implies that the apparent power is limited by a circle in the Q-P plane. Constraints (5) and (7) indicate the minimum and maximum reactive power capability. The feasible operation region of the VSC is shown by the dashed area.

In the ac-dc OPF problem, we aim to minimize a cost function subject to the constraints imposed by the ac grid, the dc grids, and the VSCs. Let P_{G_k} and Q_{G_k} denote the active and reactive power generation at bus $k \in \mathcal{G}$, respectively. In an ac-dc OPF problem, the variables include the complex voltage V_k for buses $k \in \mathcal{N}$ and P_{G_k}, Q_{G_k} for generator buses $k \in \mathcal{G}$, as well as P_{C_k}, Q_{C_k} and $|I_k|$ for converter buses $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. In the following subsection, we present the objective function and the constraints of the ac-dc OPF problem.

A. Objective Function and Constraints

The objective function includes the total generation cost C_{gen} and the total system losses P_{loss}. The generation cost function for bus $k \in \mathcal{G}$ is denoted by $f_k(P_{G_k})$. It can be approximated by a quadratic function $c_{k2}P_{G_k}^2 + c_{k1}P_{G_k} + c_{k0}$, where c_{k0}, c_{k1}, and c_{k2} are positive coefficients [12]. Thus, the total generation cost is

$$\begin{aligned} C_{\text{gen}} &= \sum_{k \in \mathcal{G}} f_k(P_{G_k}) \\ &= \sum_{k \in \mathcal{G}} \left(c_{k2}P_{G_k}^2 + c_{k1}P_{G_k} + c_{k0} \right). \end{aligned} \tag{8}$$

The total system losses are equal to the total generation minus the total load of the system. It can be expressed as the summation of the injected active power into all buses. Let P_{D_k} denote the active load in bus $k \in \mathcal{N}$. We obtain

$$P_{\text{loss}} = \sum_{k \in \mathcal{N}} \left(P_{G_k} - P_{D_k} \right). \tag{9}$$

In (9), if bus k is not a generator bus, then $P_{G_k} = 0$. In this paper, we only consider the active power losses of the system since the reactive power does not dissipate energy.

Let ω denote a positive scaling coefficient. The objective function f_{obj} of the ac-dc OPF problem is

$$f_{\text{obj}} = C_{\text{gen}} + \omega P_{\text{loss}}. \tag{10}$$

In (10), by increasing the value of ω, the total system losses have a larger weight in the objective function as compared with the total generation cost. The typical value of ω is around the value of coefficients c_{k1}, $k \in \mathcal{G}$ since the total system losses P_{loss} is a linear function of the generators' output power P_{G_k}, $k \in \mathcal{G}$. For an appropriate value of ω, minimizing f_{obj} will enable timely adjustment of control settings to jointly reduce the generation cost, VSC losses and transmission line losses. Thus, it can improve the economic efficiency of the power system operation.

The ac-dc OPF problem is subject to a set of equality and inequality constraints imposed by the ac grid, the dc grids, and the VSCs.

1) Equality constraints: The equality constraints consist of the power balance equations. Let z^* denote the conjugate of an arbitrary complex number z. The active power balance equations for bus $k \in \mathcal{N}$ can be written as

$$P_{G_k} - P_{D_k} = \text{Re}\{V_k I_k^*\}, \qquad \forall\, k \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}} \tag{11a}$$
$$P_{C_k} - P_{D_k} = \text{Re}\{V_k I_k^*\}, \qquad \forall\, k \in \mathcal{N}^{\text{conv}}. \tag{11b}$$

From assumption A4, power balance equations (11a) and (11b) can be used for dc buses by setting $I_k^* = I_k$ and $\text{Re}\{V_k I_k^*\} = V_k I_k$. Let Q_{D_k} denote the active load at bus $k \in \mathcal{N}$. The reactive power balance equations for ac buses are

$$Q_{G_k} - Q_{D_k} = \text{Im}\{V_k I_k^*\}, \qquad \forall\, k \in \mathcal{N}_{\text{ac}} \tag{12a}$$
$$Q_{C_k} - Q_{D_k} = \text{Im}\{V_k I_k^*\}, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}. \tag{12b}$$

In (11a) and (12a), if bus k is not a generator bus, then $P_{G_k} = Q_{G_k} = 0$. Consider the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$. By substituting (11b) into the power balance equation (2), we obtain

$$\text{Re}\{V_k I_k^*\} + \text{Re}\{V_s I_s^*\} + P_{\text{loss},k}^{\text{conv}} + P_{D_k} + P_{D_s} = 0. \tag{13}$$

2) Inequality constraints: The generators output active power P_{G_k}, $k \in \mathcal{G}$, generators output reactive power Q_{G_k}, $k \in \mathcal{G}$, the voltage magnitudes $|V_k|$, $k \in \mathcal{N}$, and the apparent power flowing through the transmission lines S_{lm}, $(l,m) \in \mathcal{L}$ are bounded. We use $P_{G_k}^{\min}$, $P_{G_k}^{\max}$, $Q_{G_k}^{\min}$, $Q_{G_k}^{\max}$, $V_k^{\min}$, and $V_k^{\max}$ to represent the lower and upper bounds on the generator active power, reactive power, and bus voltage at bus k, respectively. If bus k is not a generator bus, then $P_{G_k}^{\min} = P_{G_k}^{\max} = Q_{G_k}^{\min} = Q_{G_k}^{\max} = 0$. $S_{lm}^{\max}$ is the maximum apparent power flow through the line $(l,m) \in \mathcal{L}$. The inequality constraints include

$$P_{G_k}^{\min} \le P_{G_k} \le P_{G_k}^{\max}, \qquad \forall\, k \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}} \tag{14a}$$
$$Q_{G_k}^{\min} \le Q_{G_k} \le Q_{G_k}^{\max}, \qquad \forall\, k \in \mathcal{N}_{\text{ac}} \tag{14b}$$
$$V_k^{\min} \le |V_k| \le V_k^{\max}, \qquad \forall\, k \in \mathcal{N} \tag{14c}$$
$$|S_{lm}| \le S_{lm}^{\max}, \qquad \forall\, (l,m) \in \mathcal{L}. \tag{14d}$$

The inequality constraints imposed by the VSCs are (1), (4), (5) and (7). The ac-dc OPF problem is formulated as follows:

$$\text{minimize} \quad C_{\text{gen}} + \omega P_{\text{loss}} \tag{15a}$$
$$\text{subject to} \;\; (1), (4), (5), (7), \text{ and } (11a)\text{–}(14d). \tag{15b}$$

The minimization is over the complex voltage V_k for all buses $k \in \mathcal{N}$, the active output power P_{G_k} and reactive output power Q_{G_k} for all generator buses $k \in \mathcal{G}$, as well as the active power flow P_{C_k}, injected reactive power Q_{C_k}, and current magnitude $|I_k|$ for converter buses $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$.

III. SDP Form of the ac-dc OPF Problem

In this section, we introduce a semidefinite relaxation of the ac-dc OPF problem (15). Our approach and notations are similar to [12]. However, we need to model the power losses and the constraints imposed by the active and reactive power flow for the VSCs in SDP form. We first introduce the notations. Then, we transform the objective function and the constraints to formulate the SDP form of the ac-dc OPF.

Let matrix Y denote the admittance matrix. For $k \in \mathcal{N}$ and $(l,m) \in \mathcal{L}$, e_k is the k^{th} basis vector in $\mathbb{R}^{|\mathcal{N}|}$, e_k^T is its transposed vector, and $Y_k = e_k e_k^T Y$. The row k of matrix Y_k is equal to the row k of Y. The other entries of Y_k are zero. We use the Π model of the transmission lines (l,m) [31]. Let y_{lm} and $\bar{y}_{lm}$ denote the value of the series and shunt elements at bus l connected to bus m, respectively. We define $Y_{lm} = (\bar{y}_{lm} + y_{lm}) e_l e_l^T - (y_{lm}) e_l e_m^T$, where the entries (l,l) and (l,m) of Y_{lm} are equal to $\bar{y}_{lm} + y_{lm}$ and $-y_{lm}$, respectively. The other entries of Y_{lm} are zero. We define matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$, $\mathbf{Y}_{lm}$, $\bar{\mathbf{Y}}_{lm}$, $\mathbf{M}_k$ and $\mathbf{M}_{lm}$ as follows. These matrices will be used to write the SDP form of the ac-dc OPF problem:

$$\mathbf{Y}_k = \frac{1}{2} \begin{bmatrix} \text{Re}\{Y_k + Y_k^T\} & \text{Im}\{Y_k^T - Y_k\} \\ \text{Im}\{Y_k - Y_k^T\} & \text{Re}\{Y_k + Y_k^T\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2} \begin{bmatrix} \text{Im}\{Y_k + Y_k^T\} & \text{Re}\{Y_k - Y_k^T\} \\ \text{Re}\{Y_k^T - Y_k\} & \text{Im}\{Y_k + Y_k^T\} \end{bmatrix},$$

$$\mathbf{Y}_{lm} = \frac{1}{2} \begin{bmatrix} \text{Re}\{Y_{lm} + Y_{lm}^T\} & \text{Im}\{Y_{lm}^T - Y_{lm}\} \\ \text{Im}\{Y_{lm} - Y_{lm}^T\} & \text{Re}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_{lm} = -\frac{1}{2} \begin{bmatrix} \text{Im}\{Y_{lm} + Y_{lm}^T\} & \text{Re}\{Y_{lm} - Y_{lm}^T\} \\ \text{Re}\{Y_{lm}^T - Y_{lm}\} & \text{Im}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix},$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix},$$

$$\mathbf{M}_{lm} = \begin{bmatrix} (e_l - e_m)(e_l - e_m)^T & 0 \\ 0 & (e_l - e_m)(e_l - e_m)^T \end{bmatrix}.$$

We define the following matrix for converter bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$

$$\mathbf{S}_k = \begin{bmatrix} a_k & \frac{b_k}{2} \\ \frac{b_k}{2} & c_k \end{bmatrix}.$$

We define the variable column vector $\mathbf{x}$ as the real and imaginary values of the vector of the complex bus voltages $\mathbf{v} = (V_1, \ldots, V_{|\mathcal{N}|})$.

$$\mathbf{x} = \begin{bmatrix} \text{Re}\{\mathbf{v}\}^T & \text{Im}\{\mathbf{v}\}^T \end{bmatrix}^T.$$

We define variable matrix $\mathbf{W} = \mathbf{x}\mathbf{x}^T$. We define the variable column vector $\mathbf{i}_k$ as follows

$$\mathbf{i}_k = \begin{bmatrix} 1 & |I_k| \end{bmatrix}^T, \quad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}.$$

We also define variable matrix $\mathbf{I}_k = \mathbf{i}_k \mathbf{i}_k^T$ for converter bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. We use the notation $\text{Tr}\{A\}$ to represent the trace of an arbitrary square matrix A. In [12], it is shown that

$$\text{Re}\{V_k I_k^*\} = \text{Tr}\{\mathbf{Y}_k \mathbf{W}\}, \quad \forall\, k \in \mathcal{N} \tag{16a}$$
$$\text{Im}\{V_k I_k^*\} = \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}, \quad \forall\, k \in \mathcal{N} \tag{16b}$$
$$|V_k|^2 = \text{Tr}\{\mathbf{M}_k \mathbf{W}\}, \quad \forall\, k \in \mathcal{N} \tag{16c}$$
$$|V_l - V_m|^2 = \text{Tr}\{\mathbf{M}_{lm} \mathbf{W}\}, \quad \forall\, (l,m) \in \mathcal{L} \tag{16d}$$
$$|S_{lm}|^2 = \text{Tr}\{\mathbf{Y}_{lm} \mathbf{W}\}^2 + \text{Tr}\{\bar{\mathbf{Y}}_{lm} \mathbf{W}\}^2, \ \forall\, (l,m) \in \mathcal{L} \tag{16e}$$
$$P_{\text{loss},k}^{\text{conv}} = \text{Tr}\{\mathbf{S}_k \mathbf{I}_k\}, \quad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}. \tag{16f}$$

We will use (16a)−(16f) to rewrite the objective function and the constraints in the ac-dc OPF problem in terms of variable matrices $\mathbf{W}$ and $\mathbf{I}_k$, $k \in \mathcal{N}^{\text{conv}}$.

A. Transforming the Objective Function

Substituting (16a) into (11a) for $k \in \mathcal{G}$, we have $P_{G_k} = \text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k}$. The generation cost function (8) becomes

$$C_{\text{gen}} = \sum_{k \in \mathcal{G}} \Big(c_{k2} \left(\text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k} \right)^2 + c_{k1} \left(\text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k} \right) + c_{k0} \Big). \tag{17}$$

In (17), the generation cost is expressed as a quadratic function of matrix $\mathbf{W}$. However, in the SDP form, the objective function must be linear. We can replace C_{gen} with $\sum_{k \in \mathcal{G}} \beta_k$, which is a linear function of auxiliary variables β_k, $k \in \mathcal{G}$. Then, we can include inequalities $f_k(P_{G_k}) \leq \beta_k$ into the constraints set for all generator buses. Let $\tau_k = c_{k1} P_{D_k} + c_{k0}$, then the matrix form of $f_k(P_{G_k}) \leq \beta_k$ for bus $k \in \mathcal{G}$ is

$$\begin{bmatrix} \beta_k - c_{k1}\text{Tr}\{\mathbf{Y}_{\text{k}} \mathbf{W}\} - \tau_k & \sqrt{c_{k2}}(\text{Tr}\{\mathbf{Y}_{\text{k}} \mathbf{W}\} + P_{D_k}) \\ \sqrt{c_{k2}}(\text{Tr}\{\mathbf{Y}_{\text{k}} \mathbf{W}\} + P_{D_k}) & 1 \end{bmatrix} \succeq 0. \tag{18}$$

To represent the total system losses, we can substitute (16a) into (9). Thus, we obtain

$$P_{\text{loss}} = \sum_{k \in \mathcal{N}} \text{Tr}\{\mathbf{Y}_k \mathbf{W}\}. \tag{19}$$

The SDP form of the objective function in (10) can be expressed as

$$f_{\text{obj}}^{\text{SDP}} = \sum_{k \in \mathcal{G}} \beta_k + \omega \sum_{k \in \mathcal{N}} \text{Tr}\{\mathbf{Y}_k \mathbf{W}\}. \tag{20}$$

B. Transforming the Constraints

The active power balance equation in (11a) can be combined with constraint (14a). Substituting (16a) into (14a), for $k \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}}$, we obtain

$$P_{G_k}^{\min} - P_{D_k} \leq \text{Tr}\{\mathbf{Y}_k \mathbf{W}\} \leq P_{G_k}^{\max} - P_{D_k}. \tag{21}$$

Similarly, for ac buses $k \in \mathcal{N}_{\text{ac}}$, we have

$$Q_{G_k}^{\min} - Q_{D_k} \leq \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} \leq Q_{G_k}^{\max} - Q_{D_k}. \tag{22}$$

Substituting (16c) into (14c), for $k \in \mathcal{N}$, we obtain

$$(V_k^{\min})^2 \leq \text{Tr}\{\mathbf{M}_k \mathbf{W}\} \leq (V_k^{\max})^2. \tag{23}$$

Substituting (16e) into (14d), for $(l,m) \in \mathcal{L}$, we have

$$\text{Tr}\{\mathbf{Y}_{lm} \mathbf{W}\}^2 + \text{Tr}\{\bar{\mathbf{Y}}_{lm} \mathbf{W}\}^2 \leq (S_{lm}^{\max})^2. \tag{24}$$

The matrix form of inequality (24) is

$$\begin{bmatrix} (S_{lm}^{\max})^2 & \text{Tr}\{\mathbf{Y}_{lm}\mathbf{W}\} & \text{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}\} \\ \text{Tr}\{\mathbf{Y}_{lm}\mathbf{W}\} & 1 & 0 \\ \text{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}\} & 0 & 1 \end{bmatrix} \succeq 0. \tag{25}$$

For the converter connected to ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$, the SDP form of constraint (1) is

$$\text{Tr}\{\mathbf{M}_k \mathbf{W}\} \leq m_a^2 \text{Tr}\{\mathbf{M}_s \mathbf{W}\}. \tag{26}$$

The SDP form of constraint (5) is

$$-m_b S_{C_k}^{\text{nom}} \leq \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}. \tag{27}$$

Let $\rho_k = -|B_{C_k}|\,(V_k^{\max})^2 + Q_{D_k}$ and $\xi_k = (B_{C_k} V_k^{\max})^2$. Let $\mathbf{C}_k = (2\rho_k + 1)\bar{\mathbf{Y}}_k - \xi_k \mathbf{M}_f$ for ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ connected to filter bus $f \in \mathcal{N}_{\text{ac}}$. In Appendix A, we prove that constraint (7) is equivalent to the following matrix inequality:

$$\begin{bmatrix} \rho_k^2 + \text{Tr}\{\mathbf{C}_k \mathbf{W}\} & \frac{1}{\sqrt{2}}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & \frac{1}{\sqrt{2}}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} \\ \frac{1}{\sqrt{2}}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & 0 & \sqrt{2}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} \\ \frac{1}{\sqrt{2}}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & 0 & \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & 0 \\ \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & \sqrt{2}\text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} & 0 & 1 \end{bmatrix} \succeq 0. \tag{28}$$

The matrix form of inequality (4) is

$$\begin{bmatrix} (I_k^{\max})^2 \text{Tr}\{\mathbf{M}_k \mathbf{W}\} & \text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k} & \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + Q_{D_k} \\ \text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k} & 1 & 0 \\ \text{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + Q_{D_k} & 0 & 1 \end{bmatrix} \succeq 0. \tag{29}$$

Substituting (16a) and (16f) into (13), we obtain

$$\text{Tr}\{\mathbf{Y}_k \mathbf{W}\} + \text{Tr}\{\mathbf{Y}_s \mathbf{W}\} + \text{Tr}\{\mathbf{S}_k \mathbf{I}_k\} + P_{D_k} + P_{D_s} = 0. \tag{30}$$

Let $\mathbf{I}_k^{22}$ denote the entry in the second row and the second column of matrix $\mathbf{I}_k$. We can obtain $\mathbf{I}_k^{22} = |I_k|^2 = (R_{C_k}^2 + X_{C_k}^2)^{-1} |V_k - V_f|^2$ for ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and filter bus $f \in \mathcal{N}_{\text{ac}}$ of in a same VSC station. From (16d), we obtain

$$\mathbf{I}_k^{22} = (R_{C_k}^2 + X_{C_k}^2)^{-1} \text{Tr}\{\mathbf{M}_{kf} \mathbf{W}\}. \tag{31}$$

Let $\mathbf{I}_k^{12}$ denote the entry in the first row and the second column of matrix $\mathbf{I}_k$. For $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, we have

$$\mathbf{I}_k^{12} \geq 0. \tag{32}$$

We can write the equivalent SDP form of the ac-dc OPF problem (15) as follows:

$$\underset{}{\text{minimize}} \sum_{k\in\mathcal{G}} \beta_k + \omega \sum_{k\in\mathcal{N}} \text{Tr}\{\mathbf{Y}_k\mathbf{W}\} \tag{33a}$$
$$\text{subject to (18), and (21)}-\text{(32)}, \tag{33b}$$
$$\text{rank}(\mathbf{I}_k) = 1, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \tag{33c}$$
$$\text{rank}(\mathbf{W}) = 1. \tag{33d}$$

The minimization is over variables β_k, $k \in \mathcal{G}$, $\mathbf{I}_k$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, and $\mathbf{W}$. The rank constraints (33c) and (33d) in problem (33) are not convex. We propose a SDP relaxation of the ac-dc OPF problem. This optimization problem is obtained from the problem (33) by relaxing the rank constraints (33c) and (33d) and replacing them with constraints $\mathbf{I}_k \succeq 0,\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $\mathbf{W} \succeq 0$, respectively. Hence, the SDP relaxation of the ac-dc OPF problem is obtained as follows:

$$\text{minimize} \sum_{k\in\mathcal{G}} \beta_k + \omega \sum_{k\in\mathcal{N}} \text{Tr}\{\mathbf{Y}_k\mathbf{W}\} \tag{34a}$$
$$\text{subject to (18), and (21)}-\text{(32)}, \tag{34b}$$
$$\mathbf{I}_k \succeq 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \tag{34c}$$
$$\mathbf{W} \succeq 0. \tag{34d}$$

In the following theorem, we state that the solution matrices $\mathbf{I}_k^{\text{opt}},\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ to problem (34) return zero values for the current magnitude of the converters.

Theorem 1 *The solution matrices* $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ *to problem (34) are all symmetric with* $\mathbf{I}_k^{12,\text{opt}} = 0$.

The proof can be found in Appendix B. From the definition of the variable matrix $\mathbf{I}_k$, we have $\mathbf{I}_k^{12} = |I_k|$. Theorem 1 states that the solutions $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ return zero values for the current magnitude $|I_k|$, $k \in \mathcal{N}_{\text{ac}}$. Hence, they are rank two. If we enforce problem (34) to return symmetric rank one matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ with $\mathbf{I}_k^{12,\text{opt}} \geq 0$, then we can determine a correct solution for $\mathbf{W}^{\text{opt}}$ as well. Motivated by the proof of Theorem 1, we introduce a penalty function in problems (33) and (34) to obtain a modified ac-dc OPF problem and its modified SDP relaxation form. The modified ac-dc OPF problem is obtained as follows:

$$\text{minimize} \sum_{k\in\mathcal{G}} \beta_k + \omega \sum_{k\in\mathcal{N}} \text{Tr}\{\mathbf{Y}_k\mathbf{W}\} - \varepsilon \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12} \tag{35a}$$
$$\text{subject to (18), and (21)}-\text{(32)}, \tag{35b}$$
$$\text{rank}(\mathbf{I}_k) = 1, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \tag{35c}$$
$$\text{rank}(\mathbf{W}) = 1. \tag{35d}$$

By relaxing the rank constraints (35c) and (35d), the modified SDP relaxation form of the ac-dc OPF is obtained as

$$\text{minimize} \sum_{k\in\mathcal{G}} \beta_k + \omega \sum_{k\in\mathcal{N}} \text{Tr}\{\mathbf{Y}_k\mathbf{W}\} - \varepsilon \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12} \tag{36a}$$
$$\text{subject to (18), and (21)}-\text{(32)}, \tag{36b}$$
$$\mathbf{I}_k \succeq 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \tag{36c}$$
$$\mathbf{W} \succeq 0, \tag{36d}$$

where ε is a positive penalty coefficient. In problems (35) and (36), we have used a penalty function to obtain rank one solution matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ with $\mathbf{I}_k^{12,\text{opt}} \geq 0$. The feasible set in problems (33) and (35) are the same. However, the solution to problem (35) is not the same as the solution to problem (33) since their objective function are not the same. Let $f_{\text{obj}}^{\text{SDP},33}$ and $f_{\text{obj}}^{\text{SDP},35}$ denote the optimal values for the objective functions of problems (33) and (35), respectively. In Appendix C, we show that the difference between the optimal values $f_{\text{obj}}^{\text{SDP},33}$ and $f_{\text{obj}}^{\text{SDP},35}$ is bounded by

$$0 \leq f_{\text{obj}}^{\text{SDP},35} - f_{\text{obj}}^{\text{SDP},33} \leq \varepsilon M, \tag{37}$$

where

$$M = \min\Bigg\{ \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} I_k^{\max}, \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \Bigg(\mathbf{I}_k^{12,35} - \sqrt{\mathbf{I}_k^{22,34}} + I_k^{\max}\left(\sqrt{1+\frac{b_k}{c_k I_k^{\max}}} - 1\right)\Bigg)\Bigg\}, \tag{38}$$

and $\mathbf{I}_k^{22,34}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $\mathbf{I}_k^{12,35}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are the solutions to problems (34) and (35), respectively. Inequality (37) implies that the optimal value $f_{\text{obj}}^{\text{SDP},35}$ obtained from problem (35) may not be equal to the optimal value $f_{\text{obj}}^{\text{SDP},33}$ obtained from problem (33). However, if εM approaches zero, then the difference between the optimal values $f_{\text{obj}}^{\text{SDP},35} - f_{\text{obj}}^{\text{SDP},33}$ will also approach zero. The value of M in (38) is small when the number of converters in the system is small. In fact, the value of $\mathbf{I}_k^{12,35} - \sqrt{\mathbf{I}_k^{22,34}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ is generally small. The value of $I_k^{\max}\big(\sqrt{1+\frac{b_k}{c_k I_k^{\max}}} - 1\big)$ is also small in practice. For example, for a VSC with typical values of $I_k^{\max} = 1.0526$ pu, $c_k = 0.0036$ pu, and $b_k = 0.0037$ pu [32], we have $I_k^{\max}\big(\sqrt{1+\frac{b_k}{c_k I_k^{\max}}} - 1\big) = 0.427$ pu. In Theorem 2, we show that the value of ε is small in practice. Besides, problems (33) and (35) have the same feasible sets. Thus, the optimal solution to problems (33) and (35) are approximately equal. Let $b^{\max}$ denote the maximum value for coefficient b_k among all VSC stations in the system. Let $c_1^{\max}$, $c_2^{\max}$, and $P_G^{\max}$ denote the maximum value for c_{k1}, c_{k2}, and P_{G_k} among all generators $k \in \mathcal{G}$, respectively. In the following theorem, we give an approximation for ε to obtain rank one solution matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ with $\mathbf{I}_k^{12,\text{opt}} \geq 0$.

Theorem 2 *To obtain rank one solution matrices* $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ *to problem (36) with* $\mathbf{I}_k^{12,\text{opt}} \geq 0$, *the penalty coefficient* ε *can be approximated by*

$$\varepsilon \approx b^{\max}\left(2c_2^{\max} P_G^{\max} + c_1^{\max} + \omega\right). \tag{39}$$

The proof can be found in Appendix D. The value given in (39) is generally not a tight approximation for the penalty coefficient ε. In Section IV, we show that penalty coefficient ε in (39) is a small number. Therefore, the difference between the optimal values given in (37) is negligible. Problems (15) and (33) are equivalent. Besides, problems (33) and (35) are approximately equivalent. However, the relaxation gap between problems (35) and (36) may not always be zero. In the following theorem, we give a sufficient condition for zero relaxation gap.

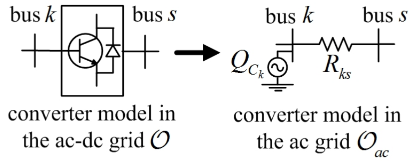

Fig. 3. The converter model in $\mathcal{O}$ and its approximated model in $\mathcal{O}_{\text{ac}}$.

Theorem 3 *Let $\mathbf{W}^{opt}$ and $\mathbf{I}_k^{opt}$, $k \in \mathcal{N}_{ac}^{conv}$ be the solution to problem (36). If the rank of $\mathbf{W}^{\text{opt}}$ is less than or equal to two, and the rank of $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are all one, then the SDP relaxation gap will be zero.*

The proof can be found in Appendix E. The sufficient condition in Theorem 3 is the generalized form of the sufficient condition proposed in [12] for the ac OPF problem. In [12], it is shown that the sufficient condition holds for practical ac grids including the IEEE test systems. We approximate the ac-dc grid by an ac grid. Then, we use the results in [12] to show that the sufficient condition in Theorem 3 holds in practical ac-dc grids as well. We approximate the ac-dc grid $\mathcal{O}$ with an ac grid $\mathcal{O}_{\text{ac}}$. As shown in Fig. 3, the converter between buses $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ is replaced by a small resistor R_{ks} (e.g., 10^{-5} pu). This resistor is used to maintain the connectivity of the resistive part of the grid $\mathcal{O}_{\text{ac}}$. It also implies that buses $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ have almost the same voltage in $\mathcal{O}_{\text{ac}}$. We also connect a generator with only reactive output power to the ac converter buses to model the reactive compensation capability of the converters. Besides, a dc grid in $\mathcal{O}$ is considered as an equivalent ac grid in $\mathcal{O}_{\text{ac}}$ with resistive lines and generators operating at unity power factor.

To formulate the OPF problem for the ac grid $\mathcal{O}_{\text{ac}}$, the converter losses and the operating constraints imposed by the converters will be removed. In Theorem 3, we relate the zero relaxation gap in $\mathcal{O}_{\text{ac}}$ to the zero relaxation gap in $\mathcal{O}$.

Theorem 4 *If $\mathbf{W}^{\text{opt}}$ is at most rank two for the ac OPF in $\mathcal{O}_{\text{ac}}$, then it is at most rank two for the ac-dc OPF in $\mathcal{O}$.*

The proof can be found in Appendix F. Theorem 4 implies that $\mathbf{W}^{\text{opt}}$ is at most rank two for practical ac-dc grids since an ac-dc grid $\mathcal{O}$ can be approximated by an ac grid $\mathcal{O}_{\text{ac}}$, and $\mathbf{W}^{\text{opt}}$ is rank two for practical ac grids [12]. By solving problem (36), we can obtain solution matrices $\mathbf{W}^{\text{opt}}$ and $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, where $\mathbf{W}^{\text{opt}}$ is at most rank two and matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are all symmetric rank one with $\mathbf{I}_k^{12,\text{opt}} \geq 0$. In Algorithm 1, it is explained how to determine the vector of bus voltages $\mathbf{x}^{\text{opt}}$ and the vectors of injected currents $\mathbf{i}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$. The steps in Algorithm 1 are derived from the proof of Theorem 3 in Appendix E. In Line 1, problem (36) is solved. In Line 2, if the solution matrices $\mathbf{W}^{\text{opt}}$ and $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ to problem (36) are all rank one, then in Line 3, we calculate the nonzero eigenvalue φ with eigenvector $\boldsymbol{\psi}$ of $\mathbf{W}^{\text{opt}}$. Going to Line 9, we calculate the solutions as $\mathbf{x}^{\text{opt}} = \sqrt{\varphi}\boldsymbol{\psi}$ and $\mathbf{I}_k^{\text{opt}} = \mathbf{i}_k^{\text{opt}}(\mathbf{i}_k^{\text{opt}})^T$. In Line 4, if matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are all rank one, but matrix $\mathbf{W}^{\text{opt}}$ is rank two, then in Line 5, we calculate two nonzero eigenvalues ϕ_1 and ϕ_2 of $\mathbf{W}^{\text{opt}}$ with the corresponding eigenvectors $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. It can be shown that the rank one matrix $\mathbf{W}_1^{\text{opt}} = (\phi_1 + \phi_2)\boldsymbol{\nu}_1\boldsymbol{\nu}_1^T$ is also the solution of problem (36) [12]. In Line 6, matrix $\mathbf{W}_1^{\text{opt}}$ is obtained. In

Algorithm 1 ac-dc OPF algorithm.

1: Solve problem (36).
2: **If** $\mathbf{W}^{\text{opt}}$ and $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are all rank one
3: Calculate the nonzero eigenvalue φ with eigenvector $\boldsymbol{\psi}$ of $\mathbf{W}^{\text{opt}}$.
4: **Else if** $\mathbf{W}^{\text{opt}}$ is rank two and $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are rank one
5: Calculate two nonzero eigenvalues ϕ_1 and ϕ_2 with eigenvectors $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ of $\mathbf{W}^{\text{opt}}$.
6: Calculate the rank one matrix $\mathbf{W}_1^{\text{opt}} = (\phi_1 + \phi_2)\boldsymbol{\nu}_1\boldsymbol{\nu}_1^T$.
7: Calculate the nonzero eigenvalue φ with eigenvector $\boldsymbol{\psi}$ of $\mathbf{W}_1^{\text{opt}}$.
8: **End if**
9: Calculate the solution vectors $\mathbf{x}^{\text{opt}}$ and $\mathbf{i}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ from $\mathbf{x}^{\text{opt}} = \sqrt{\varphi}\boldsymbol{\psi}$ and $\mathbf{I}_k^{\text{opt}} = \mathbf{i}_k^{\text{opt}}(\mathbf{i}_k^{\text{opt}})^T$.

Line 7, we calculate the nonzero eigenvalue φ of $\mathbf{W}_1^{\text{opt}}$ with its corresponding eigenvector $\boldsymbol{\psi}$. Then in Line 9, the solution vector $\mathbf{x}^{\text{opt}}$ can be obtained from $\mathbf{x}^{\text{opt}} = \sqrt{\varphi}\boldsymbol{\psi}$. If the rank of $\mathbf{W}^{\text{opt}}$ is greater than two, or at least one of the $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ is not rank one, then the relaxation gap may not be zero and the proposed approach does not return the global solution to the ac-dc OPF problem. Similar to [16], one may use a heuristic method to enforce the low-rank solution of problem (36) to become rank one or rank two. However, this is beyond the scope of this paper.

IV. Performance Evaluation

In this section, we illustrate the performance of the SDP approach for solving the ac-dc OPF problem. The test system is shown in Fig. 4. The IEEE 118-bus test system is connected to five off-shore wind farms at buses 7 and 9 via HVDC lines. An ac-dc microgrid consisting of three PV systems is connected to bus 5 via an ac-dc converter. Buses 5, 7, and 9 in the IEEE test system are connected to each other via dc cables. The base power of the system is 100 MVA. The data for the IEEE 118-bus test system can be found in [33]. The generators' cost function coefficients can be found in Matpower's library [34]. The rated power of each VSC is 50 MVA. The data for the VSC stations in the grid's per unit system are given in Table I. The VSC losses parameters in Table I are from [32] and converted to the grid's per unit system. The resistance of all HVDC lines is 0.06 pu. The resistance of all dc cables is 0.001 pu. The maximum apparent power flow through the HVDC lines and other transmission lines is 1.1 pu. Unless stated otherwise, parameter m_b is set to 0.5 and the maximum modulation factor m_a is set to 1.05 [17]. The lower and upper bounds for voltage magnitudes are 0.9 pu and 1.1 pu, respectively. The active output power of the wind farms connected to buses 132 to 135 are 10 MW. The active output power of the wind farm connected to bus 125 is 50 MW. Wind farms can control the reactive power at its grid connection point. In this study, the wind farms are operating at unity power factor. For the ac-dc microgrid, the output active power of each PV system is 5 MW, and the active load in each dc bus is 20 MW. The active and reactive loads in ac bus 164 are 20 MW and 20 MVAR, respectively. Total active and reactive loads of the system are 4302 MW and 1448 MVAR, respectively. The scaling coefficient ω in the objective function (10) is set to 10 to jointly minimize the total generation cost and total system losses. There are

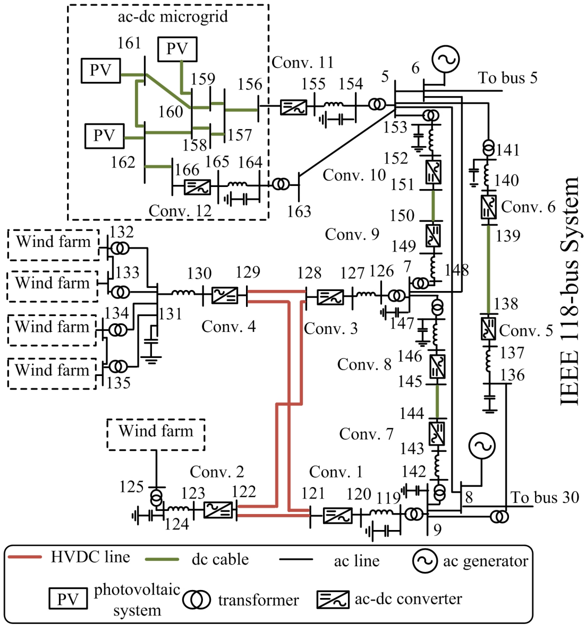

Fig. 4. The IEEE 118-bus test system connected to five wind farms and one ac-dc microgrid.

TABLE I
VSC STATION PARAMETERS WITH CONVERTER BUS k AND FILTER BUS f

VSC parameters (pu)			
$R_{T_k} = 0.0005$	$X_{T_k} = 0.0125$	$B_f = 0.2$	$S_{C_k}^{\text{nom}} = 0.5$
$R_{C_k} = 0.00025$	$X_{C_k} = 0.04$	$V_k^{\max} = 1.05$	$I_k^{\max} = 1.0526$
VSC losses data (pu)			
$a_k = 0.00265$	$b_k = 0.0037$	$c_k = 0.0036$	

12 converters and the set of converter ac buses is $\mathcal{N}_{\text{ac}}^{\text{conv}} = \{120, 123, 127, 130, 137, 140, 143, 146, 149, 152, 155, 165\}$.

We discuss the results obtained from Theorems 1 to 4 in detail. To check the result of Theorem 1, we first solve problem (34) by using CVX with SeDuMi solver in Matlab. As Theorem 1 states, the matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are all symmetric and rank two with zero values for $\mathbf{I}_k^{12,\text{opt}}$. Hence, problem (34) does not return a correct solution to the ac-dc OPF problem. Then, we use a penalty function to obtain problems (35) and (36). We solve problem (36). According to Theorem 2, the appropriate value for the penalty factor ε can be determined from (39). In the grid's per unit system, we have $b^{\max} = 0.0037$ pu, $c_1^{\max} = 40$ pu, $c_2^{\max} = 0.02$ pu, and $P_G^{\max} = 5$ pu. Hence, $\varepsilon = 0.1857$ is obtained from (39). It returns symmetric rank one matrices $\mathbf{I}_k^{\text{opt}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ with positive values for $\mathbf{I}_k^{12,\text{opt}}$. The optimal value of problem (36) is 1093.28 pu (\$128,171 per hour). If the relaxation gap between problems (35) and (36) is zero, then the optimal value of problems (35) and (36) are equal. We use inequality (37) to determine the upper bound for $f_{\text{obj}}^{\text{SDP},35} - f_{\text{obj}}^{\text{SDP},33}$. By solving problem (34), we obtain $\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \left(\mathbf{I}_k^{12,35} - \sqrt{\mathbf{I}_k^{22,34}}\right) = 0.11$ pu. According to (37), for $\varepsilon = 0.1857$ and $I_k^{\max} = 1.0526$ pu, the difference between the optimal values $f_{\text{obj}}^{\text{SDP},33}$ and $f_{\text{obj}}^{\text{SDP},35}$ is at most 0.973 pu, which is negligible compared with 1093.28 pu, the optimal value of problem (35). Therefore, problems (33) and (35) are

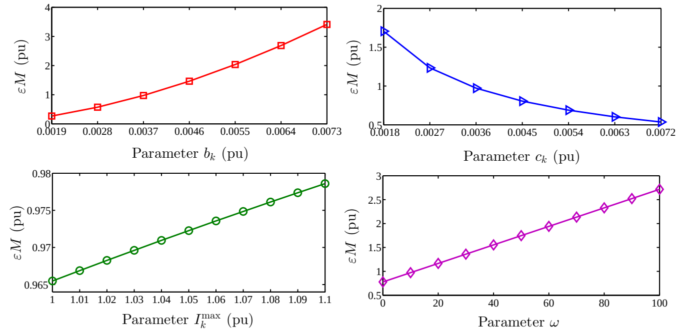

Fig. 5. The value of the upper bound εM in (37) for different values of parameters b_k, c_k, $I_k^{\max}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, and the scaling coefficient ω.

approximately equivalent, and the approximation is negligible. To assess the approximation in solving the ac-dc OPF problem, we obtain the value of the upper bound εM given in (37) for different values of parameters b_k, c_k, $I_k^{\max}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, and the scaling coefficient ω. The results are given in Fig. 5. The value of parameters b_k and c_k, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ are changing from 50% to 200% of their assumed values in the simulation setup. The value of $I_k^{\max}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ is changing from 1 pu to 1.1 pu. The value of the scaling coefficient ω is changing from 0 to 100, which is practical for the case study since coefficients c_{k1}, $k \in \mathcal{G}$ in (8) are about 40 pu. The results show that for different values of b_k, c_k, $I_k^{\max}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, and ω, the value of εM is less than 3.5 pu, which is negligible compared with the optimal value of problem (35). Therefore, the solutions to problem (35) (or problem (36)) and the original ac-dc OPF problem (33) can be treated as "approximately equal" for the purpose of this paper.

By solving problem (36), we obtain a rank two matrix $\mathbf{W}^{\text{opt}}$ with nonzero eigenvalues $\phi_1 = \phi_2 = 8.831$ and the corresponding eigenvectors $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$. According to Theorem 3, when matrix $\mathbf{W}^{\text{opt}}$ is rank two, then the relaxation gap between problems (35) and (36) is zero, and the global optimal solution to the ac-dc OPF problem can be obtained by using Algorithm 1. From Algorithm 1, the rank one matrix $\mathbf{W}_1^{\text{opt}} = (\phi_1 + \phi_2)\boldsymbol{\nu}_1\boldsymbol{\nu}_1^T$ is also the solution of problem (36). Matrix $\mathbf{W}_1^{\text{opt}}$ has one nonzero eigenvalue $\varphi = 17.662$ with corresponding eigenvector $\boldsymbol{\psi}$. The solution vector $\mathbf{x}^{\text{opt}}$ is obtained from $\mathbf{x}^{\text{opt}} = \sqrt{\varphi}\boldsymbol{\psi}$. Fig. 6 shows the voltage profile obtained by the proposed approach and MATPOWER. For the purpose of comparison, we use the primal-dual interior point (PDIP) algorithm using the Matlab interior point method solver (MIPS) available in MATPOWER to obtain a solution to the ac-dc OPF problem. We use the ac-dc OPF formulation proposed in [32] to include constraints of the ac-dc network into the MATPOWER. The objective function given in [32] for the ac-dc OPF problem is replaced by (10). The constraint for the converter dc voltage lower limit in [32] is replaced by (5). The constraints given in [32] for the grid code of wind farms' connection are modified for unity power factor in our simulation. The Jacobian and Hessian matrices in MATPOWER are modified by adding the new variables for the VSCs and the set of equality and inequality constraints imposed by the converters. Unity voltage for all buses is assumed as the initial point. Although MATPOWER can obtain a solution to the ac-dc

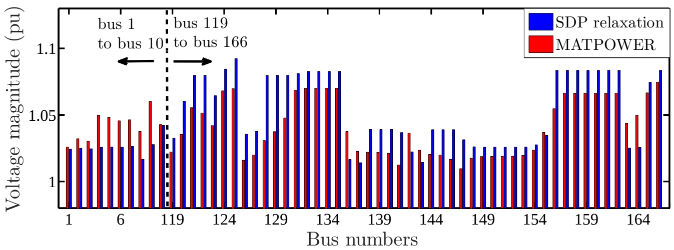

Fig. 6. Voltage profile obtained from the SDP relaxation approach and the approach proposed in [32] using MATPOWER with MIPS solver.

TABLE II
THE GENERATION COST AND SYSTEM LOSSES OBTAINED FROM SDP RELAXATION APPROACH AND THE APPROACH PROPOSED IN [32] USING MATPOWER WITH DIFFERENT SOLVERS.

	Generation cost (\$/hr)	Total losses (MW)	Converter losses (MW)	Line losses (MW)
SDP relaxation	128,171	73.12	6.28	66.84
MATPOWER with MIPS	129,085	87.26	9.12	78.14
MATPOWER with IPOPT	128,641	78.46	7.08	71.38
MATPOWER with SNOPT	128,905	83.05	7.19	75.86
MATPOWER with TSPOPF	129,516	91.38	9.41	81.97

OPF problem using the PDIP algorithm, it does not guarantee the solution to be the global optimal. MATPOWER can also use other OPF solvers such as TSPOPF [35] and a number of modern solvers for convex optimization problems such as IPOPT [36], SNOPT [37], [38]. Although these solvers are well implemented, they can only guarantee convergence to a local optimal solution of the ac-dc OPF problem. The system generation cost and total system losses obtained from the SDP relaxation approach and MATPOWER are given in Table II. IPOPT returns the best sub-optimal solution. All other solvers did not return a global optimal solution (the solution obtained from the SDP relaxation technique). The total system losses consist of two components: the transmission line and VSC losses. From (3), the transmission line losses can be obtained as $P_{\text{loss}} - \sum_{k \in \mathcal{N}_{\text{ac}}^{\text{conv}}} \left(a_k + b_k|I_k| + c_k|I_k|^2\right)$, where the second term is the converter losses.

The converter and transmission line losses obtained from the SDP relaxation approach are given in Table II. Total system losses are about 1.7% of the total load of the system. Moreover, we can observe that the generation cost and total system losses obtained from the SDP relaxation approach are lower since the solution obtained from MATPOWER is a local optimum. The VSC losses are about 8.6% of the total system losses. It confirms that the VSC losses can add up to a significant fraction of total system losses and have to be included in the OPF problem. The Q-P characteristics of the converters 1, 2, 3, 4, 11, and 12 in the VSC per unit system are shown in Fig. 7. The nominal apparent power for the converters is $S_{C_k}^{\text{nom}} = 1$ pu in the VSC's per unit system. From (5), the maximum reactive power that the converters can absorb is 0.5 pu. Constraint (7) for the upper limit of the injected reactive power is active for converters 1 and

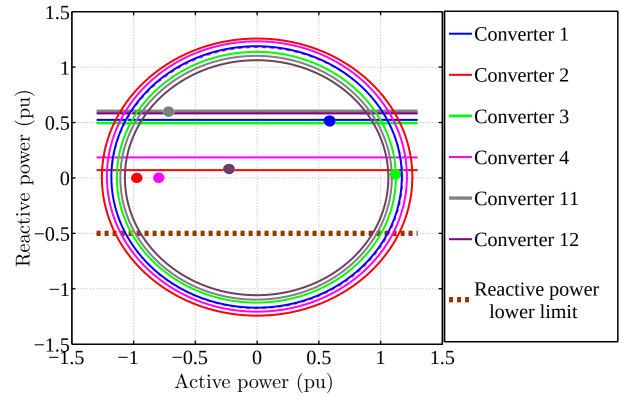

Fig. 7. Q-P characteristics of the converters 1, 2, 3, 4, 11, and 12.

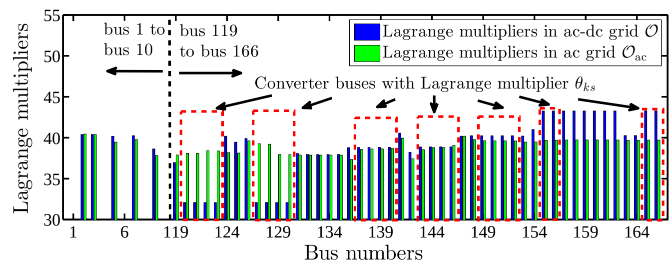

Fig. 8. Lagrange multipliers for ac-dc grid $\mathcal{O}$ and ac grid $\mathcal{O}_{\text{ac}}$.

11. That is, converters 1 and 11 are injecting reactive power to the ac grid with maximum rate for voltage regulation. Constraint (4) for the upper limit of the apparent power is active for converter 3, since the generated active power by the wind farms is flowing through this converter. Converters 2 and 4 are also absorbing the generated active power by the wind farms. Converter 12 is injecting reactive power to meet the load at bus 164.

To illustrate the results of Theorem 4, we have solved the dual OPF problem defined in the proof of Theorem 4 in Appendix F for $\mathcal{O}$ and $\mathcal{O}_{\text{ac}}$. The Lagrange multipliers are shown in Fig. 8. As shown by dashed rectangles, instead of λ_k, we have Lagrange multiplier θ_{ks} associated with equality constraint (30) for converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and converter dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ in the same VSC station. We observe that the Lagrange multipliers λ_k in the ac-dc grid $\mathcal{O}$ are all positive and greater than their corresponding Lagrange multiplier λ_k in the ac grid $\mathcal{O}_{\text{ac}}$. It confirms Proposition F.2. Besides, Lagrange multipliers θ_{ks} are positive though they can be lower (e.g., in buses 120 to 123) or greater (e.g., in buses 155 and 156) than their corresponding Lagrange multipliers in $\mathcal{O}_{\text{ac}}$. Hence, the conditions given in the proof of Theorem 4 are satisfied. For this case study, the running time of algorithm 1 was about 40 seconds which is higher than the few seconds of the running time for the well-implemented algorithms used in MATPOWER. The number of variables of problem (36) is quadratic with respect to the number of buses. Hence, the computational cost of solving the SDP relaxation of the ac-dc OPF problem grows rapidly with the size of the system. To reduce the computational burden of SDP relaxation of large OPF problems, different techniques have been proposed such as exploiting sparsity in SDP relaxation [39], [40] and matrix combination algorithm [41]. In our proposed approach, the number of variables of the dual of ac-dc OPF formulated in Appendix F is linear with respect to the number of buses. Thus, solving the dual OPF can save computation time to determine the solution of the OPF problem in large ac-dc grids.

V. Conclusion

In this paper, we studied the OPF problem for ac-dc grids. The converter losses and the operating limits on the voltage and current of the converters were modeled in the OPF problem. The original problem was non-convex, which does not guarantee a global optimal solution. Convex relaxation techniques were used to obtain the SDP form of the ac-dc OPF problem. An algorithm was given to determine the solution to the original ac-dc OPF problem from its SDP form. We provided a sufficient condition for zero relaxation gap that guarantees the OPF algorithm to return the global optimal solution. We also showed that the sufficient condition holds for practical ac-dc grids. Simulation results on a modified IEEE 118-bus test system confirmed that the zero relaxation gap condition holds for the case study, and the SDP approach enabled us to obtain the global optimal solution to the bus voltages and the converters operating points in polynomial time. For future work, we plan to extend the model by considering other power electronic devices such as flexible ac transmission system controllers. We also plan to address the challenges of applying the proposed OPF solution approach to large-scale ac-dc systems through the use of decomposition techniques and distributed computations.

Appendix

A. The Proof of Constraint (28)

We first combine (12b) with constraint (7) and obtain

$$\mathrm{Im}\{V_k I_k^*\} + Q_{D_k} \leq |B_{C_k}| \left((V_k^{\max})^2 - V_k^{\max}|V_f|\right). \quad (40)$$

Substituting (16b) into (40), we obtain

$$\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + Q_{D_k} \leq |B_{C_k}| \left((V_k^{\max})^2 - V_k^{\max}|V_f|\right). \quad (41)$$

We substitute $\rho_k = -|B_{C_k}|\,(V_k^{\max})^2 + Q_{D_k}$ and $\xi_k = (B_{C_k} V_k^{\max})^2$ into (41). Taking the square of both sides, we obtain

$$\left(\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + \rho_k\right)^2 \geq \xi_k |V_f|^2. \quad (42)$$

Substituting (16c) into (42), we obtain

$$\left(\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + \rho_k\right)^2 \geq \xi_k \mathrm{Tr}\{\mathbf{M}_f \mathbf{W}\}. \quad (43)$$

Thus, we have

$$\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^2 + 2\rho_k \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + \rho_k^2 - \xi_k \mathrm{Tr}\{\mathbf{M}_f \mathbf{W}\} \geq 0. \quad (44)$$

Substituting $\mathbf{C}_k = (2\rho_k + 1)\bar{\mathbf{Y}}_k - \xi_k \mathbf{M}_f$ into (44), we obtain

$$\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^2 - \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + \rho_k^2 + \mathrm{Tr}\{\mathbf{C}_k \mathbf{W}\} \geq 0. \quad (45)$$

We multiply (45) by positive number $\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^2$ to obtain

$$\begin{aligned}\mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^4 - \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^3 \\ + \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\}^2 \left(\rho_k^2 + \mathrm{Tr}\{\mathbf{C}_k \mathbf{W}\}\right) \geq 0.\end{aligned} \quad (46)$$

The matrix form of (46) is given in (28). The proof is completed. ■

B. The Proof of Theorem 1

The solution matrices $\mathbf{I}_k^{\mathrm{opt}}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ to problem (34) are symmetric positive semidefinite. Hence, $\mathbf{I}_k^{12,\mathrm{opt}} = \mathbf{I}_k^{21,\mathrm{opt}}$, $\mathbf{I}_k^{22,\mathrm{opt}} \geq 0$ for $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$. However, we have $\mathbf{I}_k^{12,\mathrm{opt}} = 0$. In fact, for each $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, the coefficient b_k in (3) is positive. Thus, a zero value for $\mathbf{I}_k^{12,\mathrm{opt}}$ reduces the VSC losses. Therefore, it reduces the objective value of problem (34). By solving problem (34), the value of $\mathbf{I}_k^{12,\mathrm{opt}}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ becomes as low as possible to minimize the objective function. Matrix $\mathbf{I}_k^{\mathrm{opt}}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ is positive semidefinite and constraint (32) holds. Thus, for each $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$, the lowest value that $\mathbf{I}_k^{12,\mathrm{opt}}$ can take is $\mathbf{I}_k^{12,\mathrm{opt}} = 0$. It is clear that the matrix $\mathbf{I}_k^{\mathrm{opt}}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ is rank two when $\mathbf{I}_k^{12,\mathrm{opt}} = 0$. The proof is completed. ■

C. The Proof of Inequality (37)

Let β_k^{33}, $k \in \mathcal{G}$, $\mathbf{W}^{33}$ and $\mathbf{I}_k^{33}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ denote the optimal solutions to problem (33). Also, let β_k^{35}, $k \in \mathcal{G}$, $\mathbf{W}^{35}$ and $\mathbf{I}_k^{35}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ denote the optimal solutions to problem (35). Since β_k^{35}, $k \in \mathcal{G}$, $\mathbf{W}^{35}$ and $\mathbf{I}_k^{12,35}$, $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ minimize the objective function of problem (35), we have

$$\begin{aligned}&\sum_{k\in\mathcal{G}} \beta_k^{35} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{35}\} - \varepsilon \sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \mathbf{I}_k^{12,35} \\ &\leq \sum_{k\in\mathcal{G}} \beta_k^{33} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{33}\} - \varepsilon \sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \mathbf{I}_k^{12,33}.\end{aligned} \quad (47)$$

Moreover, β_k^{33}, $k \in \mathcal{G}$ and $\mathbf{W}^{33}$ minimize the objective function of problem (33). Hence, we have

$$\begin{aligned}&\sum_{k\in\mathcal{G}} \beta_k^{33} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{33}\} \\ &\leq \sum_{k\in\mathcal{G}} \beta_k^{35} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{35}\}.\end{aligned} \quad (48)$$

According to equation (20), we have $f_{\mathrm{obj}}^{\mathrm{SDP},33} = \sum_{k\in\mathcal{G}} \beta_k^{33} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{33}\}$ and $f_{\mathrm{obj}}^{\mathrm{SDP},35} = \sum_{k\in\mathcal{G}} \beta_k^{35} + \omega \sum_{k\in\mathcal{N}} \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}^{35}\}$. Substituting $f_{\mathrm{obj}}^{\mathrm{SDP},33}$ and $f_{\mathrm{obj}}^{\mathrm{SDP},35}$ into inequalities (47) and (48), we obtain

$$f_{\mathrm{obj}}^{\mathrm{SDP},35} - \varepsilon \sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \mathbf{I}_k^{12,35} \leq f_{\mathrm{obj}}^{\mathrm{SDP},33} - \varepsilon \sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \mathbf{I}_k^{12,33}, \quad (49a)$$

$$f_{\mathrm{obj}}^{\mathrm{SDP},33} \leq f_{\mathrm{obj}}^{\mathrm{SDP},35}. \quad (49b)$$

After rearranging the terms, inequality (49a) becomes

$$f_{\mathrm{obj}}^{\mathrm{SDP},35} - f_{\mathrm{obj}}^{\mathrm{SDP},33} \leq \varepsilon \sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \left(\mathbf{I}_k^{12,35} - \mathbf{I}_k^{12,33}\right). \quad (50)$$

Inequality (49b) is equivalent to

$$0 \leq f_{\mathrm{obj}}^{\mathrm{SDP},35} - f_{\mathrm{obj}}^{\mathrm{SDP},33}, \quad (51)$$

which proves the left-hand side of (37). Moreover, converter currents $\mathbf{I}_k^{12,33}$ and $\mathbf{I}_k^{12,35}$ for $k \in \mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}$ are non-negative and upper bounded by $\mathbf{I}_k^{\max}$. Therefore, $\sum_{k\in\mathcal{N}_{\mathrm{ac}}^{\mathrm{conv}}} \mathbf{I}_k^{12,35} -$

$\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\mathbf{I}_k^{12,33}$ in the right-hand side of (50) is bounded by

$$\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\left(\mathbf{I}_k^{12,35}-\mathbf{I}_k^{12,33}\right)\le\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\mathbf{I}_k^{\max}. \tag{52}$$

By multiplying both sides of (52) by ε and substituting into (50), we obtain

$$f_{\text{obj}}^{\text{SDP},35}-f_{\text{obj}}^{\text{SDP},33}\le\varepsilon\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\mathbf{I}_k^{\max}. \tag{53}$$

The right-hand side of (52) is generally not a tight upper bound for $\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\left(\mathbf{I}_k^{12,35}-\mathbf{I}_k^{12,33}\right)$. The exact value of $\mathbf{I}_k^{12,33}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ cannot be obtained since problem (33) is difficult to be solved. However, one can solve problem (34) and approximate $\mathbf{I}_k^{12,33}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ by the value of $\sqrt{\mathbf{I}_k^{22,34}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, where $\mathbf{I}_k^{22,34}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ is the solution to problem (34). We can show that the relaxation gap between problem (33) with $b_k=0$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ and problem (34) is zero. Condition $b_k=0$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ implies that the linear losses (e.g., switching losses) in the converters are zero. Thus, the losses in a VSC become lower. The losses in a VSC with $b_k=0$ and current $\mathbf{I}_k^{22,34}$ is $P_{\text{loss},k}^{\text{conv}}=c_k\mathbf{I}_k^{22,34}+a_k$. It should be lower than the losses in a VSC with $b_k>0$ and current $\mathbf{I}_k^{12,33}$ obtained from (3). Therefore, we have

$$c_k\mathbf{I}_k^{22,34}+a_k\le c_k(\mathbf{I}_k^{12,33})^2+b_k\mathbf{I}_k^{12,33}+a_k,\quad\forall\,k\in\mathcal{N}_{\text{ac}}^{\text{conv}}. \tag{54}$$

Hence, we obtain

$$\mathbf{I}_k^{22,34}\le(\mathbf{I}_k^{12,33})^2+\frac{b_k}{c_k}\mathbf{I}_k^{12,33},\quad\forall\,k\in\mathcal{N}_{\text{ac}}^{\text{conv}}. \tag{55}$$

We take the square root of both sides of (55). Then, we subtract both sides by $\mathbf{I}_k^{12,33}$. For all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, we obtain

$$\sqrt{\mathbf{I}_k^{22,34}}-\mathbf{I}_k^{12,33}\le\sqrt{(\mathbf{I}_k^{12,33})^2+\frac{b_k}{c_k}\mathbf{I}_k^{12,33}}-\mathbf{I}_k^{12,33}. \tag{56}$$

The right-hand side of (56), $\sqrt{(\mathbf{I}_k^{12,33})^2+\frac{b_k}{c_k}\mathbf{I}_k^{12,33}}-\mathbf{I}_k^{12,33}$, is an increasing function of $\mathbf{I}_k^{12,33}$. For all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, we have

$$\sqrt{(\mathbf{I}_k^{12,33})^2+\frac{b_k}{c_k}\mathbf{I}_k^{12,33}}-\mathbf{I}_k^{12,33}\le I_k^{\max}\left(\sqrt{1+\frac{b_k}{c_kI_k^{\max}}}-1\right). \tag{57}$$

Substituting (57) into (56), we obtain

$$\sqrt{\mathbf{I}_k^{22,34}}-\mathbf{I}_k^{12,33}\le I_k^{\max}\left(\sqrt{1+\frac{b_k}{c_kI_k^{\max}}}-1\right),\forall\,k\in\mathcal{N}_{\text{ac}}^{\text{conv}}. \tag{58}$$

We add $\mathbf{I}_k^{12,35}$ to the both sides of (58). After rearranging the terms, for all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, we obtain

$$\mathbf{I}_k^{12,35}-\mathbf{I}_k^{12,33}\le\mathbf{I}_k^{12,35}-\sqrt{\mathbf{I}_k^{22,34}}+I_k^{\max}\left(\sqrt{1+\frac{b_k}{c_kI_k^{\max}}}-1\right). \tag{59}$$

Substituting (59) into (50), we have

$$f_{\text{obj}}^{\text{SDP},35}-f_{\text{obj}}^{\text{SDP},33}\le\varepsilon\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\Bigg(\mathbf{I}_k^{12,35}-\sqrt{\mathbf{I}_k^{22,34}}+I_k^{\max}\left(\sqrt{1+\frac{b_k}{c_kI_k^{\max}}}-1\right)\Bigg). \tag{60}$$

The upper bound in (37) is obtained from (53) and (60). The proof is completed. ■

D. The Proof of Theorem 2

We use the penalty function in the objective function of problem (36) to obtain rank one solution matrices $\mathbf{I}_k^{\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ with positive value for $\mathbf{I}_k^{12,\text{opt}}$. By changing the value of $\mathbf{I}_k^{12,\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ from zero to positive, the total losses of the system will increase by approximately $\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}b_k\mathbf{I}_k^{12,\text{opt}}$. The increase in the losses results in increasing the total generation level. Let $P_{G_k}^0$ denote the generated power of the generator in bus $k\in\mathcal{G}$ when $\mathbf{I}_k^{12,\text{opt}}$ is zero for all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$. Let $P_{G_k}^+$ denote the generated power of the generator in bus $k\in\mathcal{G}$ when $\mathbf{I}_k^{12,\text{opt}}$ is positive for all $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$. Let Δf_k and ΔC_{gen} denote the change in the generation cost of the generator in bus $k\in\mathcal{G}$ and the total generation cost of the system, respectively. If the generation level for generator $k\in\mathcal{G}$ increases from $P_{G_k}^0$ to $P_{G_k}^+$, then the value of Δf_k can be approximated as

$$\Delta f_k\approx\left(2c_{k2}P_{G_k}^0+c_{k1}\right)\left(P_{G_k}^+-P_{G_k}^0\right). \tag{61}$$

The term $2c_{k2}P_{G_k}^0+c_{k1}$ in (61) is the marginal cost for the generator in bus $k\in\mathcal{G}$ with generation level $P_{G_k}^0$. The value of ΔC_{gen} is

$$\begin{aligned}\Delta C_{\text{gen}}&=\sum_{k\in\mathcal{G}}\Delta f_k\\&\approx\sum_{k\in\mathcal{G}}\left(2c_{k2}P_{G_k}^0+c_{k1}\right)\left(P_{G_k}^+-P_{G_k}^0\right).\end{aligned} \tag{62}$$

Substituting $c_1^{\max}$, $c_2^{\max}$, and $P_G^{\max}$ into (62) for all generators $k\in\mathcal{G}$, we obtain

$$\Delta C_{\text{gen}}\le(2c_2^{\max}P_G^{\max}+c_1^{\max})\sum_{k\in\mathcal{G}}\left(P_{G_k}^+-P_{G_k}^0\right). \tag{63}$$

The increase in the generation level of the generators is due to the increase in the losses. Hence, we have $\sum_{k\in\mathcal{G}}\left(P_{G_k}^+-P_{G_k}^0\right)=\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}b_k\mathbf{I}_k^{12,\text{opt}}$ and we obtain

$$\Delta C_{\text{gen}}\le(2c_2^{\max}P_G^{\max}+c_1^{\max})\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}b_k\mathbf{I}_k^{12,\text{opt}}. \tag{64}$$

Substituting $b^{\max}$ into (64), we obtain

$$\Delta C_{\text{gen}}\le b^{\max}(2c_2^{\max}P_G^{\max}+c_1^{\max})\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}}\mathbf{I}_k^{12,\text{opt}}. \tag{65}$$

Let Δf_{obj} denote the change in the objective value of problem (34) when the value of $\mathbf{I}_k^{12,\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ changes from zero to a positive number. The objective function f_{obj} includes the total generation cost and the total system losses. When the value of $\mathbf{I}_k^{12,\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ changes from zero to positive, the system losses will increase by at most $b^{\max} \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12,\text{opt}}$. Therefore, from (65), we have

$$\Delta f_{\text{obj}} \leq b^{\max}\left(2c_2^{\max} P_G^{\max} + c_1^{\max} + \omega\right) \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12,\text{opt}}. \quad (66)$$

To obtain rank one solution matrices $\mathbf{I}_k^{\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ with positive value for $\mathbf{I}_k^{12,\text{opt}}$, it is sufficient that the penalty function $\varepsilon\left(\sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12,\text{opt}}\right)$ be greater than the change in the objective function Δf_{obj}. Therefore, the penalty coefficient ε can be approximated by

$$\varepsilon \approx b^{\max}\left(2c_2^{\max} P_G^{\max} + c_1^{\max} + \omega\right). \quad (67)$$

The proof is completed. ■

E. The Proof of Theorem 3

If the solution matrices $\mathbf{W}^{\text{opt}}$ and $\mathbf{I}_k^{\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ to problem (36) are all rank one, then $\mathbf{W}^{\text{opt}} = \mathbf{x}^{\text{opt}}(\mathbf{x}^{\text{opt}})^T$ and $\mathbf{I}_k^{\text{opt}} = \mathbf{i}_k^{\text{opt}}(\mathbf{i}_k^{\text{opt}})^T$. Thus, the relaxation gap is zero. If $\mathbf{I}_k^{\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ are all rank one, but $\mathbf{W}^{\text{opt}}$ is rank two, then $\mathbf{W}^{\text{opt}}$ has two nonzero eigenvalues ϕ_1 and ϕ_2 with corresponding eigenvectors $\boldsymbol{\nu_1}$ and $\boldsymbol{\nu_2}$. It can be shown that the rank one matrix $\mathbf{W}_1^{\text{opt}} = (\phi_1+\phi_2)\boldsymbol{\nu_1}\boldsymbol{\nu_1}^T$ is also the solution of the OPF problem [12]. Matrix $\mathbf{W}_1^{\text{opt}}$ has only one nonzero eigenvalue φ with corresponding eigenvector $\boldsymbol{\psi}$. Then, the solution vector $\mathbf{x}^{\text{opt}}$ can be obtained from $\mathbf{x}^{\text{opt}} = \sqrt{\varphi}\boldsymbol{\psi}$. If the rank of $\mathbf{W}^{\text{opt}}$ is greater than two, or at least one of the matrices $\mathbf{I}_k^{\text{opt}}$, $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ is not rank one, then the relaxation gap may not be zero. The proof is completed. ■

F. The Proof of Theorem 4

For the sake of convenience, we rewrite the modified ac-dc OPF problem (35) again as follows:

$$\text{minimize} \sum_{k\in\mathcal{G}} \beta_k + \omega \sum_{k\in\mathcal{N}} \text{Tr}\{\mathbf{Y}_k \mathbf{W}\} - \varepsilon \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \mathbf{I}_k^{12} \quad (68\text{a})$$

$$\text{subject to (18), and (21)−(32),} \quad (68\text{b})$$

$$\text{rank}(\mathbf{I}_k) = 1, \quad \forall\, k\in\mathcal{N}_{\text{ac}}^{\text{conv}}, \quad (68\text{c})$$

$$\text{rank}(\mathbf{W}) = 1. \quad (68\text{d})$$

We consider the dual problem of problem (68). We define the dual variables. Let $\underline{\lambda}_k$, $\underline{\gamma}_k$, and $\underline{\mu}_k$ denote the Lagrange multipliers associated with the lower inequalities in (21), (22), and (23), respectively. Let $\overline{\lambda}_k$, $\overline{\gamma}_k$, and $\overline{\mu}_k$ denote the Lagrange multipliers associated with the upper inequalities in (21), (22), and (23), respectively. For each transmission line $(l,m)\in\mathcal{L}$, the matrix

$$R_{lm} = \begin{bmatrix} r_{lm}^1 & r_{lm}^2 & r_{lm}^3 \\ r_{lm}^2 & r_{lm}^4 & r_{lm}^5 \\ r_{lm}^3 & r_{lm}^5 & r_{lm}^6 \end{bmatrix}$$

is the Lagrange multiplier associated with the matrix inequality (25). For each generator bus $k\in\mathcal{G}$, the matrix

$$R_k = \begin{bmatrix} 1 & r_k^1 \\ r_k^1 & r_k^2 \end{bmatrix}$$

is the Lagrange multiplier for the matrix inequality (18).

λ_k, γ_k, and μ_k are defined as follows:

$$\lambda_k = \begin{cases} \overline{\lambda}_k - \underline{\lambda}_k + c_{k1} + 2\sqrt{c_{k2}}\, r_k^1, & \text{if } k\in\mathcal{G} \\ \overline{\lambda}_k - \underline{\lambda}_k, & \text{otherwise,} \end{cases}$$

$$\gamma_k = \overline{\gamma}_k - \underline{\gamma}_k,$$
$$\mu_k = \overline{\mu}_k - \underline{\mu}_k.$$

Let X denote the set of all multipliers $\underline{\lambda}_k$, $\overline{\lambda}_k$, $\underline{\gamma}_k$, $\overline{\gamma}_k$, $\underline{\mu}_k$, and $\overline{\mu}_k$. Also, let R denote the set of all Lagrange multipliers R_{lm} and R_k.

For each converter connected to ac bus $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s\in\mathcal{N}_{\text{dc}}^{\text{conv}}$, let η_{ks} and θ_{ks} denote the Lagrange multipliers associated with inequality (26) and equality (30), respectively. For each bus $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, let ϑ_k and υ_k denote the Lagrange multipliers associated with inequalities (27) and (32), respectively. For each ac bus $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$ and filter bus $f\in\mathcal{N}_{\text{ac}}$ in a VSC station, let σ_{kf} denote the Lagrange multiplier associated with equality (31). Also, the matrix

$$T_{kf} = \begin{bmatrix} t_{kf}^1 & t_{kf}^2 & t_{kf}^3 & t_{kf}^4 \\ t_{kf}^2 & t_{kf}^5 & t_{kf}^6 & t_{kf}^7 \\ t_{kf}^3 & t_{kf}^6 & t_{kf}^8 & t_{kf}^9 \\ t_{kf}^4 & t_{kf}^7 & t_{kf}^9 & t_{kf}^{10} \end{bmatrix}$$

is the Lagrange multiplier associated with the matrix inequality (28). For each converter ac bus $k\in\mathcal{N}_{\text{ac}}^{\text{conv}}$, the matrix

$$T_k = \begin{bmatrix} t_k^1 & t_k^2 & t_k^3 \\ t_k^2 & t_k^4 & t_k^5 \\ t_k^3 & t_k^5 & t_k^6 \end{bmatrix}$$

is the Lagrange multiplier associated with the matrix inequality (29). Let T denote the set of all Lagrange multipliers η_{ks}, θ_{ks}, ϑ_k, υ_k, σ_{kf}, T_{kf}, and T_k.

We define an affine function $h(X,R,T)$ as follows:

$$\begin{aligned} &h(X,R,T) = \\ &\sum_{k\in\mathcal{N}\setminus\mathcal{N}^{\text{conv}}} \Big(\underline{\lambda}_k P_{G_k}^{\min} - \overline{\lambda}_k P_{G_k}^{\max} + \lambda_k P_{D_k} + \underline{\gamma}_k Q_{G_k}^{\min} - \overline{\gamma}_k Q_{G_k}^{\max} \\ &+ \gamma_k Q_{D_k} + \underline{\mu}_k \left(V_k^{\min}\right)^2 - \overline{\mu}_k \left(V_k^{\max}\right)^2\Big) + \sum_{k\in\mathcal{G}} \left(c_{k0} - r_k^2\right) \\ &- \sum_{(l,m)\in\mathcal{L}} \left(S_{lm}^{\max} r_{lm}^1 + r_{lm}^4 + r_{lm}^6\right) + \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \Big((2t_k^2 + \theta_{ks}) P_{D_k} \\ &+ 2t_k^3 Q_{D_k} - \rho_k^2 t_{kf}^1 - t_{kf}^{10} - t_k^4 - t_k^6\Big) - \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \vartheta_k m_b S_{C_k}^{\text{nom}} \\ &+ \sum_{s\in\mathcal{N}_{\text{dc}}^{\text{conv}}} \left(\theta_{ks} P_{D_s}\right). \end{aligned} \quad (69)$$

Furthermore, we define the function

$$\begin{aligned} &A(X,R,T) = \\ &\sum_{k\in\mathcal{N}\setminus\mathcal{N}^{\text{conv}}} \left(\lambda_k \mathbf{Y}_k + \gamma_k \bar{\mathbf{Y}}_k + \mu_k \mathbf{M}_k\right) + \sum_{(l,m)\in\mathcal{L}} \Big(2r_{lm}^2 \mathbf{Y}_{lm} \\ &+ 2r_{lm}^3 \bar{\mathbf{Y}}_{lm}\Big) + \sum_{k\in\mathcal{N}_{\text{ac}}^{\text{conv}}} \Big((2t_k^2 + \theta_{ks})\mathbf{Y}_k + (\sqrt{2}t_{kf}^2 + \sqrt{2}t_{kf}^3 \\ &+ 2t_{kf}^4 + t_{kf}^5 + 2\sqrt{2}t_{kf}^7 + t_{kf}^8 + 2t_k^3 - \vartheta_k)\bar{\mathbf{Y}}_k - t_{kf}^1 \mathbf{C}_k \\ &+ \left(\eta_{ks} - \left(I_k^{\max}\right)^2 t_k^1\right)\mathbf{M}_k\Big) + \sum_{s\in\mathcal{N}_{\text{dc}}^{\text{conv}}} \left(\theta_{ks} \mathbf{Y}_s - m_a^2 \eta_{ks} \mathbf{M}_s\right). \end{aligned} \quad (70)$$

In (69) and (70), the converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, the converter dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ and the filter bus $f \in \mathcal{N}_{\text{ac}}$ are in the same VSC station. Hence, we only use index k in the fifth summation of (69) and the third summation of (70).

Also, we define the following affine function for converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, converter dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$, and filter bus $f \in \mathcal{N}_{\text{ac}}$ in the same VSC station

$$B_k(X) = \frac{1}{2}\begin{bmatrix} 0 & \varepsilon + \upsilon_k \\ \varepsilon + \upsilon_k & 2\sigma_{kf} \end{bmatrix} + \theta_{ks}\mathbf{S}_k. \tag{71}$$

Consider the dual of the ac-dc OPF problem as follows:

$$\begin{aligned}
&\underset{X,R,T}{\text{maximize}} && h(X,R,T) && \text{(72a)}\\
&\text{subject to} && A(X,R,T) \succeq 0, && \text{(72b)}\\
& && B_k(X) \succeq 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}} && \text{(72c)}\\
& && R_{lm} \succeq 0, \qquad \forall\, (l,m) \in \mathcal{L} && \text{(72d)}\\
& && R_k \succeq 0, \qquad \forall\, k \in \mathcal{G} && \text{(72e)}\\
& && T_{kf} \succeq 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \forall f \in \mathcal{N}_{\text{ac}} && \text{(72f)}\\
& && T_k \succeq 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}} && \text{(72g)}\\
& && \underline{\lambda}_k, \overline{\lambda}_k, \underline{\gamma}_k, \overline{\gamma}_k, \underline{\mu}_k, \overline{\mu}_k \geq 0 \qquad \forall\, k \in \mathcal{N} && \text{(72h)}\\
& && \eta_{ks}, \vartheta_k \geq 0 \quad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}, \forall s \in \mathcal{N}_{\text{dc}}^{\text{conv}}. && \text{(72i)}
\end{aligned}$$

Problem (72) is the dual of problem (68). We can show that problem (72) is also the dual of problem (36) and strong duality holds between these optimization problems. Furthermore, the matrix $\mathbf{W}$ in problem (36) is the Lagrange multiplier associated with the inequality constraint (72b). Besides, for every $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, the matrix $\mathbf{I}_k$ is the Lagrange multiplier associated with inequality (72c). Let $(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ denote the solution to problem (72). From the complementary slackness in Karush-Kuhn-Tucker (KKT) conditions, we obtain

$$\begin{aligned}
&\text{Tr}\{A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})\mathbf{W}^{\text{opt}}\} = 0, && \text{(73a)}\\
&\text{Tr}\{B_k(X^{\text{opt}})\mathbf{I}_k^{\text{opt}}\} = 0, \qquad \forall\, k \in \mathcal{N}_{\text{ac}}^{\text{conv}}. && \text{(73b)}
\end{aligned}$$

From (73a) and (73b), the orthogonal eigenvectors of $\mathbf{W}^{\text{opt}}$ and $\mathbf{I}_k^{\text{opt}},\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ belong to the null space of $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ and $B_k(X^{\text{opt}}),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, respectively [12]. Thus, if matrix $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ has a zero eigenvalue of multiplicity two, then matrix $\mathbf{W}^{\text{opt}}$ is at most rank 2. If matrix $B_k(X^{\text{opt}}),\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ has one zero eigenvalue, then matrix $\mathbf{I}_k^{\text{opt}},\ k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ is also rank one. Proposition F.1 summarizes the obtained result for an ac-dc grid $\mathcal{O}$.

Proposition F.1 *The solution matrix $\mathbf{W}^{\text{opt}}$ to problem (36) is at most rank 2 if the solution matrix $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ to problem (72) has a zero eigenvalue of multiplicity two.*

In [12], it is shown that the Proposition F.1 remains valid for the ac grid $\mathcal{O}_{\text{ac}}$. In the ac grid $\mathcal{O}_{\text{ac}}$, the constraints imposed by the converters are removed. Hence, the Lagrange multipliers in set T will be removed from the affine functions defined in (69) and (70). Besides, we do not require to define functions $B_k(X)$ in (71). Consequently, for the ac grid $\mathcal{O}_{\text{ac}}$, function $h(X,R,T)$ and matrix $A(X,R,T)$ will be replaced by function $h(X,R)$ and matrix $A(X,R)$, respectively. The dual OPF problem will be simplified to an optimization problem with objective function $h(X,R)$ and constraints (72b), (72d), (72e), and (72h). Details can be found in [12]. Let $(X^{\text{opt}}, R^{\text{opt}})$ denote the solution to the dual OPF problem in the ac grid $\mathcal{O}_{\text{ac}}$. Then, the solution matrix $\mathbf{W}^{\text{opt}}$ to the SDP relaxation form of the ac OPF problem is at most rank 2 if the solution matrix $A(X^{\text{opt}}, R^{\text{opt}})$ to the dual of the ac OPF problem has a zero eigenvalue of multiplicity two [12]. We use this fact to prove Theorem 4.

It is sufficient to show that the solution matrix $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ to the dual OPF problem in the ac-dc grid $\mathcal{O}$ has a zero eigenvalue of multiplicity two if the solution matrix $A(X^{\text{opt}}, R^{\text{opt}})$ to the dual OPF problem in the ac grid $\mathcal{O}_{\text{ac}}$ has a zero eigenvalue of multiplicity two.

Consider the dual OPF problem in the ac grid $\mathcal{O}_{\text{ac}}$. The solution matrix $A(X^{\text{opt}}, R^{\text{opt}})$ has a simple structure as follows:

$$A(X^{\text{opt}}, R^{\text{opt}}) = \frac{1}{2}\begin{bmatrix} H_1(X^{\text{opt}}, R^{\text{opt}}) & H_2(X^{\text{opt}}, R^{\text{opt}}) \\ -H_2(X^{\text{opt}}, R^{\text{opt}}) & H_1(X^{\text{opt}}, R^{\text{opt}}) \end{bmatrix}, \tag{74}$$

where $H_1(X^{\text{opt}}, R^{\text{opt}})$ and $H_2(X^{\text{opt}}, R^{\text{opt}})$ are symmetric real matrices. Consider matrix $H_1(X^{\text{opt}}, R^{\text{opt}})$. Let Y_{lm} denote the entry (l,m) of admittance matrix Y. The off-diagonal entry $(l,m) \in \mathcal{L}$ of $H_1(X^{\text{opt}}, R^{\text{opt}})$ can be obtained as

$$\begin{aligned} H_1(X^{\text{opt}}, R^{\text{opt}})_{lm} = {} & \text{Re}\{Y_{lm}\}(\lambda_l + \lambda_m + 2r_{lm}^2) \\ & - \text{Im}\{Y_{lm}\}(\gamma_l + \gamma_m + 2r_{lm}^3). \end{aligned} \tag{75}$$

In [12], it is shown that the smallest eigenvalue of $A(X^{\text{opt}}, R^{\text{opt}})$ is zero if two conditions are satisfied. First, the graph of the resistive part of the grid is connected. That is, there exists a connected path between all two buses in the graph of the resistive part of the grid. Second, the off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}})$ are non-positive and the entries of matrix $H_2(X^{\text{opt}}, R^{\text{opt}})$ are sufficiently smaller than the entries of $H_1(X^{\text{opt}}, R^{\text{opt}})$. For a practical ac grid with connected resistive part, the off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}})$ are non-positive because $\text{Re}\{Y_{lm}\},\ (l,m) \in \mathcal{L}$ is non-positive and $\text{Im}\{Y_{lm}\},\ (l,m) \in \mathcal{L}$ is non-negative. Furthermore, the Lagrange multipliers $\lambda_k,\ k \in \mathcal{N}$ are positive and $\gamma_k,\ k \in \mathcal{N}$ are small as compared with $\lambda_k,\ k \in \mathcal{N}$. Similarly, we can obtain the entries of matrix $H_2(X^{\text{opt}}, R^{\text{opt}})$ and show that the entries of this matrix are sufficiently smaller than the entries of matrix $H_1(X^{\text{opt}}, R^{\text{opt}})$. Consequently, the smallest eigenvalue of $A(X^{\text{opt}}, R^{\text{opt}})$ is zero. The structure of $A(X^{\text{opt}}, R^{\text{opt}})$ guarantees that its smallest eigenvalue has multiplicity of two [12].

Now, consider the dual OPF problem in the ac-dc grid $\mathcal{O}$. Matrix $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ has a similar structure to (74). That is, there exist symmetric real matrices $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ and $H_2(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$, for which we have

$$\begin{aligned} & A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}}) = \\ & \frac{1}{2}\begin{bmatrix} H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}}) & H_2(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}}) \\ -H_2(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}}) & H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}}) \end{bmatrix}. \end{aligned} \tag{76}$$

Again, the smallest eigenvalue of $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ is zero if the graph of the resistive part of the grid is connected and the off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ are non-positive, as well as the entries of matrix $H_2(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ are sufficiently smaller than the entries of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$. In the ac-dc grid $\mathcal{O}$, the converter buses are not connected to each other directly. To make the graph of the resistive part of the ac-dc grid $\mathcal{O}$, we add a large resistance (e.g.,

10^5 pu) between the converter ac bus $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and dc bus $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$ in the same VSC station. Since the added resistance is sufficiently large, the converter buses k and s will have independent voltage magnitudes as before. The grids $\mathcal{O}$ and $\mathcal{O}_{\text{ac}}$ are different in the constraints imposed by the converters in buses $f \in \mathcal{N}_{\text{ac}}$, $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$. Moreover, in the ac-dc grid $\mathcal{O}$, the converter losses are included in the total losses of the system. Therefore, at the global optimal solution to the OPF problem, the generation levels in $\mathcal{O}$ will be higher than the generation levels in $\mathcal{O}_{\text{ac}}$ to compensate the higher losses in the ac-dc grid. Thus, the optimal value of the objective function in $\mathcal{O}$ is greater than the optimal value of the objective function in $\mathcal{O}_{\text{ac}}$. The Lagrange multipliers λ_k, $k \in \mathcal{N} \backslash \mathcal{N}^{\text{conv}}$ measure the rate of increase of the objective function at the optimal point as the corresponding constraint is relaxed. Consequently, we have the following proposition.

Proposition F.2 *The Lagrange multipliers λ_k, $k \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}}$ in the ac-dc grid $\mathcal{O}$ are greater than or equal to their corresponding Lagrange multipliers in $\mathcal{O}_{\text{ac}}$.*

If the off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}})$ for ac grid $\mathcal{O}_{\text{ac}}$ are non-positive, then the off-diagonal entries $(l, m) \in \mathcal{L}$, $l, m \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}}$ of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ for ac-dc grid $\mathcal{O}$ remain non-positive. The other off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ can be obtained as follows:

For $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$ and $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$, we have

$$H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})_{ks} = \text{Re}\{Y_{ks}\}(2\theta_{ks}). \tag{77}$$

For $(f, k) \in \mathcal{L}$, $f \in \mathcal{N}_{\text{ac}}$, and $k \in \mathcal{N}_{\text{ac}}^{\text{conv}}$, we have

$$H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})_{fk} = \tag{78}$$

$$\text{Re}\{Y_{fk}\}(\lambda_f + 2r_{kf}^2 + 2t_k^2 + 2\theta_{ks}) - \text{Im}\{Y_{kf}\}\big(\sqrt{2}t_{kf}^2 + \sqrt{2}t_{kf}^3 + 2t_{kf}^4 + t_{kf}^5 + 2\sqrt{2}t_{kf}^7 + t_{kf}^8 + 2t_k^3 - \vartheta_k + \bar{\lambda}_f + 2r_{kf}^3 - (2\rho_k + 1)t_{kf}^1\big). \tag{79}$$

For $(s, m) \in \mathcal{L}$, $s \in \mathcal{N}_{\text{dc}}^{\text{conv}}$, and $m \in \mathcal{N} \setminus \mathcal{N}^{\text{conv}}$, we have

$$H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})_{sm} = \text{Re}\{Y_{sm}\}(2r_{sm}^2 + 2\theta_{ks} + \lambda_m). \tag{80}$$

For practical ac-dc grids, θ_{ks} is non-negative since over satisfaction of the active loads P_{D_s} and P_{D_k} in converter buses leads to higher losses and generation levels in the system. Furthermore, λ_f, $f \in \mathcal{N}_{\text{ac}}$ and λ_m, $m \in \mathcal{N} \backslash \mathcal{N}^{\text{conv}}$ are positive in the ac-dc grid $\mathcal{O}$ since they are greater than or equal to their corresponding Lagrange multipliers in $\mathcal{O}_{\text{ac}}$ (Proposition F.2) and they are positive in $\mathcal{O}_{\text{ac}}$. Entry t_{kf}^1 is positive since matrix T_k is positive semidefinite. $\rho_k = -|B_{C_k}|\,(V_k^{\max})^2 + Q_{D_k}$ is a large negative number. Therefore, $(2\rho_k + 1)t_{kf}^1$ is a negative number. Other Lagrange multipliers associated with the inequality constraints are positive. Other Lagrange multipliers associated with the equality constraints can be either positive or negative. If they are negative, they have small values compared with the value of $(2\rho_k + 1)t_{kf}^1$, λ_f, λ_m, and θ_{ks}. Consequently, the off-diagonal entries of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ for the ac-dc grid $\mathcal{O}$ are non-positive. Similarly, we can show that the entries of matrix $H_2(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ are sufficiently smaller than the entries of $H_1(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ for the ac-dc grid $\mathcal{O}$. Therefore, matrix $A(X^{\text{opt}}, R^{\text{opt}}, T^{\text{opt}})$ has a zero eigenvalue of multiplicity two for the ac-dc grid $\mathcal{O}$, and $\mathbf{W}^{\text{opt}}$ is at most rank two. The proof is completed. ■

REFERENCES

[1] M. Sechilariu, B. Wang, and F. Locment, "Building integrated photovoltaic system with energy storage and smart grid communication," *IEEE Trans. on Industrial Electronics*, vol. 60, no. 4, pp. 1607–1618, Apr. 2013.

[2] J. Liang, T. Jing, O. Gomis-Bellmunt, J. Ekanayake, and N. Jenkins, "Operation and control of multiterminal HVDC transmission for offshore wind farms," *IEEE Trans. on Power Delivery*, vol. 26, no. 4, pp. 2596–2604, Oct. 2011.

[3] B. T. Patterson, "DC come home: DC microgrids and the birth of the Enernet," *IEEE Power and Energy Magazine*, vol. 10, no. 6, pp. 60–69, Nov. 2012.

[4] D. Salomonsson, L. Soder, and A. Sannino, "An adaptive control system for a DC microgrid for data centers," *IEEE Trans. on Industry Applications*, vol. 44, no. 6, pp. 1910–1917, Nov. 2008.

[5] Amin, R. T. Bambang, A. S. Rohman, C. J. Dronkers, R. Ortega, and A. Sasongko, "Energy management of fuel cell/battery/supercapacitor hybrid power sources using model predictive control," *IEEE Trans. on Industrial Informatics*, vol. 10, no. 4, pp. 1992–2002, Nov. 2014.

[6] D. Chen, L. Xu, and L. Yao, "DC voltage variation based autonomous control of DC microgrids," *IEEE Trans. on Power Delivery*, vol. 28, no. 2, pp. 637–648, Apr. 2013.

[7] A. Bracale, P. Caramiaa, G. Carpinelli, F. Mottola, and D. Proto, "A hybrid AC/DC smart grid to improve power quality and reliability," in *Proc. of IEEE Int.'l Energy Conference and Exhibition (ENERGYCON)*, Florence, Italy, Sept. 2012.

[8] J. Zhu, *Optimization of Power System Operation*, 1st ed. NJ: Wiley-IEEE Press, 2009.

[9] S. H. Low, "Convex relaxation of optimal power flow - Part I: Formulations and equivalence," *IEEE Trans. on Control of Network Systems*, vol. 1, no. 1, pp. 15–27, Mar. 2014.

[10] ——, "Convex relaxation of optimal power flow - Part II: Exactness," *IEEE Trans. on Control of Network Systems*, vol. 1, no. 2, pp. 177–189, Jun. 2014.

[11] N. Chen, C. W. Tan, and T. Quek, "Electric vehicle charging in smart grid: Optimality and valley-filling algorithms," *IEEE Journal of Selected Topics in Signal Processing*, vol. 8, no. 6, pp. 1073–1083, Dec. 2014.

[12] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Trans. on Power Systems*, vol. 27, no. 1, pp. 92–107, Feb. 2012.

[13] J. Lavaei, D. Tse, and B. Zhang, "Geometry of power flows and optimization in distribution networks," *IEEE Trans. on Power Systems*, vol. 29, no. 2, pp. 572–583, Mar. 2014.

[14] M. Farivar and S. H. Low, "Branch flow model: Relaxations and convexification - Parts I and II," *IEEE Trans. on Power Systems*, vol. 28, no. 3, pp. 2554–2572, Aug. 2013.

[15] C. W. Tan, D.W.H. Cai, and X. Lou, "Resistive network optimal power flow: Uniqueness and algorithms," *IEEE Trans. on Power Systems*, vol. 30, no. 1, pp. 263–273, Jan. 2015.

[16] S. Sojoudi, R. Madani, and J. Lavaei, "Low-rank solution of convex relaxation for optimal power flow problem," in *Proc. of IEEE SmartGridComm*, Vancouver, Canada, Oct. 2013.

[17] W. Feng, A. L. Tuan, L. Tjernberg, A. Mannikoff, and A. Bergman, "A new approach for benefit evaluation of multiterminal VSC-HVDC using a proposed mixed AC/DC optimal power flow," *IEEE Trans. on Power Delivery*, vol. 29, no. 1, pp. 432–443, Feb. 2014.

[18] J. Beerten, S. Cole, and R. Belmans, "A sequential AC-DC power flow algorithm for networks containing multi-terminal VSC HVDC systems," in *Proc. of IEEE Power and Energy Society General Meeting (PES)*, Minneapolis, MN, Jul. 2010.

[19] R. Wiget and G. Andersson, "Optimal power flow for combined AC and multi-terminal HVDC grids based on VSC converters," in *Proc. of IEEE Power and Energy Society General Meeting*, San Diego, CA, Jul. 2012.

[20] J. Yu, W. Yan, W. Li, C. Chung, and K. Wong, "An unfixed piecewise-optimal reactive power-flow model and its algorithm for AC-DC systems," *IEEE Trans. on Power Systems*, vol. 23, no. 1, pp. 170–176, Feb. 2008.

[21] A. Pizano-Martinez, C. Fuerte-Esquivel, H. Ambriz-Perez, and E. Acha, "Modeling of VSC-based HVDC systems for a Newton-Raphson OPF algorithm," *IEEE Trans. on Power Systems*, vol. 22, no. 4, pp. 1794–1803, Nov. 2007.

[22] M. Baradar, M. Hesamzadeh, and M. Ghandhari, “Second-order cone programming for optimal power flow in VSC-type AC-DC grids,” *IEEE Trans. on Power Systems*, vol. 28, no. 4, pp. 4282–4291, Nov. 2013.
[23] C. Liu, B. Zhang, Y. Hou, F. F. Wu, and Y. Liu, “An improved approach for AC-DC power flow calculation with multi-infeed DC systems,” *IEEE Trans. on Power Systems*, vol. 26, no. 2, pp. 862–869, May 2011.
[24] D. O. Neacsu, *Switching Power Converters: Medium and High Power*, 2nd ed. FL: CRC Press, 2013.
[25] S. Bahrami, V.W.S. Wong, and J. Jatskevich, “Optimal power flow for AC-DC networks,” in *Proc. of IEEE SmartGridComm*, Venice, Italy, Nov. 2014.
[26] Z. Zhou, “Co-ordination of converter controls and an analysis of converter operating limits in VSC-HVDC grids,” Ph.D. dissertation, The University of Manitoba, 2013.
[27] R. W. Erickson and D. Maksimovic, *Fundamentals of Power Electronics*, 2nd ed. NY: Springer, 2001.
[28] M. Bierhoff and F. Fuchs, “Semiconductor losses in voltage source and current source IGBT converters based on analytical derivation,” in *Proc. of IEEE Power Electronics Specialists Conference (PESC)*, Aachen, Germany, Jun. 2004.
[29] L. Mestha and P. Evans, “Analysis of on-state losses in PWM inverters,” *IEE Proceedings B, Electric Power Applications*, vol. 136, no. 4, pp. 189–195, 1989.
[30] G. Daelemans, “VSC HVDC in meshed networks,” Master’s thesis, Katholieke Universiteit Leuven, Leuven, Belgium, 2008.
[31] J. D. Glover, M. S. Sarma, and T. Overbye, *Power System Analysis and Design*, 5th ed. CT: Cengage Learning, 2011.
[32] J. Cao, W. Du, H. Wang, and S. Bu, “Minimization of transmission loss in meshed AC/DC grids with VSC-MTDC networks,” *IEEE Trans. on Power Systems*, vol. 28, no. 3, pp. 3047–3055, Aug. 2013.
[33] The University of Washington, power systems test case archive. [Online]. Available: http://www.ee.washington.edu/research/pstca.
[34] R. D. Zimmerman, C. E. Murillo-Sanchez, and R. J. Thomas, “MATPOWER: Steady state operations, planning and analysis tools for power systems research and education,” *IEEE Trans. on Power Systems*, vol. 26, no. 1, pp. 12–19, Feb. 2011.
[35] TSPOPF, high performance AC optimal power flow solvers for MATPOWER. [Online]. Available: http://www.pserc.cornell.edu/tspopf/
[36] A. Wachter and L. T. Biegler, “On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming,” *Mathematical Programming*, vol. 106, no. 1, pp. 25–57, Mar. 2006.
[37] Stanford Business Software, SNOPT. [Online]. Available: http://ampl.com/products/solvers/solvers-we-sell/snopt/
[38] P. E. Gill, W. Murray, M. A. Saunders, and E. Wong, “User’s guide for SNOPT 7.5: Software for large-scale nonlinear programming,” Department of Mathematics, University of California, San Diego, La Jolla, CA, Center for Computational Mathematics Report CCoM 15-3, 2015.
[39] R. A. Jabr, “Exploiting sparsity in SDP relaxations of the OPF problem,” *IEEE Trans. on Power Systems*, vol. 27, no. 2, pp. 1138–1139, May 2011.
[40] D. K. Molzahn and I. A. Hiskens, “Sparsity-exploiting moment-based relaxations of the optimal power flow problem,” *IEEE Trans. on Power Systems*, vol. 30, no. 6, pp. 3168–3180, Nov. 2015.
[41] D. K. Molzahn, J. T. Holzer, B. C. Lesieutre, and C. L. DeMarco, “Implementation of a large-scale optimal power flow solver based on semidefinite programming,” *IEEE Trans. on Power Systems*, vol. 28, no. 4, pp. 3987–3998, Nov. 2013.

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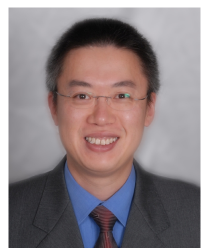

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