Optimal Power Flow for AC-DC Networks

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***Abstract*—The presence of distributed generators with DC output power and the advancement in power electronics devices have motivated system planners and grid operators to move towards integration of DC microgrids into conventional AC grid. In this paper, we address the optimal power flow (OPF) problem in AC-DC networks. The goal of the AC-DC OPF problem is to jointly minimize the total electricity generation cost of the network and the cost of transferring active power from the AC grid to the DC microgrids. The optimization problem is subject to the power flow constraints, voltage magnitude limits, the limits of the network power lines, and the limits imposed by the power ratings of AC-DC power electronic converters. The formulated AC-DC OPF problem is shown to be nonlinear. We propose an approach to reformulate the AC-DC OPF problem as an equivalent traditional AC OPF problem. Due to the non-convexity of the AC OPF problem, we use convex relaxation techniques and transform the problem to a semidefinite program (SDP). We show that the relaxation gap is zero. That is the optimal solution of the non-convex and the transformed convex problems are equal. Simulation studies are performed on an IEEE 14-bus system connected to two 9-bus DC microgrids. We show that the sufficient condition for the zero relaxation gap is satisfied, and the proposed SDP approach enables us to find the global optimal solution efficiently.**

I. Introduction

Recent advancement in power electronics technology and the proliferation of devices that can generate or operate on DC are raising the debate over AC vs. DC power. Solar photovoltaic panels and small-scale wind turbines generate DC power. Batteries, super capacitors, and fuel cells store energy as DC. Furthermore, large amount of energy delivered as AC is now consumed as DC [1]. Numerous electrical systems in residential and commercial sectors, such as HVAC (heating, ventilation, and air conditioning) systems, motor loads, pumps, lighting rely on standard AC to be internally converted to DC for their operation. Eliminating conversions from AC to DC (and vice versa) as well as having directly available DC power can greatly improve the efficiency of the grid by reducing losses associated with the conversion. Consequently, having DC microgrids incorporated into the AC grid can provide flexibility and efficiency for the power system.

Opportunities exist to capitalize on the benefits of DC microgrids. DC microgrids are well-suited to connect DC output types of distributed renewable resources, and are appropriate to protect sensitive loads from power outages and disturbances such as voltage dips [2]. Moreover, DC microgrids have simpler power electronic interfaces and fewer points of failure [3].

In a DC microgrid, energy storage and a large portion of the sources and loads are interconnected through one or more DC buses. However, an AC grid is still necessary since some sources and loads cannot be directly connected to DC buses [4]. Therefore, in the near future, DC microgrids are considered as part of the main AC grid [5], where these two networks are connected to each other using the AC-DC converters to transfer power between them [6], [7].

By integrating DC microgrids and conventional AC grids, power network management becomes a challenge for system planners and operators. *Optimal power flow* (OPF) is a useful tool for planning and decision making to ensure reliable operation and to manage power grids. When an AC grid is connected to one or more DC microgrids, the OPF problem of the AC-DC network takes the form of a non-convex optimization problem consisting of the traditional AC network and DC microgrid power flow equations, in addition to the constraints imposed by the AC-DC converters equations [8]–[10].

The non-convexity of the problem arises from the nonlinear power flow equations and quadratic dependency on the set of bus voltages. The problem may have multiple local optimal solutions [11]. Recently, semidefinite programming (SDP) and convex relaxation of the OPF problem have attracted significant research attention since they are guaranteed to find the global optimal solution when the relaxation gap is zero [12]–[15]. In [12], it is shown that SDP relaxation for the DC grids has zero relaxation gap for practical power grids including IEEE test power systems. The work in [13] presents a branch flow model for the analysis and optimization of mesh and radial networks. The proposed convex relaxation method is exact for radial networks provided there are no upper bounds on loads or voltage magnitudes. In [14], the non-convex OPF formulation for unbalanced microgrids is considered. SDP relaxation technique is used to transform the problem to a convex problem. In [15], a model for the power lines capacity is presented. The zero relaxation gap for weakly-cyclic networks is studied. The upper bound on the rank of the minimum-rank solution of the SDP relaxation is provided.

Most of the previous works on OPF in AC-DC power grids (e.g., [8]–[10]) focus on determining local optimal solutions. In this paper, our goal is to determine the global solution of AC-DC OPF problem. The efficiency of SDP approach in solving OPF problem in traditional AC and DC networks has motivated us to study the performance of this approach in solving OPF in AC-DC power grids.

The contributions of this paper are as follows:

- We propose a novel approach to formulate AC-DC OPF problem as a traditional AC OPF appropriate for solving by the SDP method. In our approach, to tackle the converter equations, we merge AC and DC side buses of each converter to have an equivalent AC bus in the system. We replace the DC microgrids with AC microgrids with the same operating points. As a result, we can effectively model an AC-DC network as an equivalent AC grid.

- We apply the SDP method in the AC OPF problem of the transformed AC grid, and study the zero relaxation gap condition for achieving effective numerical solution. We describe how the solution of the original AC-DC OPF problem can be determined from the solution of the transformed AC OPF problem.
- We perform simulations on an IEEE 14-bus system connected to two 9-bus DC microgrids to evaluate the performance of our approach. We show that the SDP method has zero relaxation gap and it provides the global optimal solution. The proposed approach is computationally feasible for large-scale AC-DC power networks.

Our method can be partly compared with [12]. Our work is different from that in [12] in two aspects. First, in [12], the traditional AC OPF problem is studied, whereas in this paper, we tackle the AC-DC OPF problem that contains AC-DC converters and DC microgrids power flow equations in addition to AC grid operating constraints. Second, the objective in [12] is to determine zero duality gap condition for the SDP form of the AC OPF problem, whereas we focus on formulating an AC-DC OPF problem which can be solved by using the SDP technique as in [12].

The rest of this paper is organized as follows. The AC-DC network and the converter model are presented in Section II. The approach to address AC-DC OPF is described in Section III. In Section IV, the OPF problem is formulated and transformed as an SDP. The sufficient condition for zero relaxation gap is stated. Simulation results are presented in Section V. Finally, the paper is concluded in Section VI.

II. System Model

Consider an AC-DC grid which consists of an AC grid connected to a set of DC microgrids denoted by $\mathcal{H} = \{1,\dots,|\mathcal{H}|\}$. We represent the AC grid by a tuple $\mathcal{O}_{ac}(\mathcal{N}_{ac},\mathcal{L}_{ac})$, where $\mathcal{N}_{ac} = \{1,\dots,|\mathcal{N}_{ac}|\}$ and $\mathcal{L}_{ac}$ denote the sets of AC grid buses and transmission lines, respectively. We represent the DC microgrid $h \in \mathcal{H}$ by a tuple $\mathcal{O}^h_{dc}(\mathcal{N}^h_{dc},\mathcal{L}^h_{dc})$, where $\mathcal{N}^h_{dc} = \{1,\dots,|\mathcal{N}^h_{dc}|\}$ and $\mathcal{L}^h_{dc}$ denote the sets of buses and lines in DC microgrid h, respectively.

The converter between AC bus $r \in \mathcal{N}_{ac}$ and DC bus $s \in \mathcal{N}^h_{dc}$ of microgrid h operates with a power factor angle of $\phi^h_{r,s}$. The converter is used to convert AC voltage V_r to DC voltage V^h_s based on the following equation [16]

$$V^h_s = k_1 a^h_{r,s}|V_r|\cos(\phi^h_{r,s}), \tag{1}$$

where $k_1 = \frac{3\sqrt{2}}{\pi}$ is a constant, and $|\cdot|$ denotes the voltage magnitude. The parameter $a^h_{r,s}$ denotes the tap of the transformer used for controlling the DC voltage level V^h_s.

Typically, the high-power converters operate with efficiencies in the high 90% range. Therefore, for the purpose of this paper, the losses of the AC-DC converter are neglected. The active power $P^h_{r,s}$ flows from AC bus $r \in \mathcal{N}_{ac}$ to DC bus $s \in \mathcal{N}^h_{dc}$ in DC microgrid h through the converter. The direction of the power flow depends on the operating point of the AC grid and the DC microgrid, where we assume a four quadrant AC-DC type converter, e.g. pulse-width-modulation controlled voltage-source converter. Furthermore, the converter acts as a controllable reactive power compensator on its AC side. It can inject or absorb reactive power $Q^h_{r,s}$ to increase voltage regulation, stability, and power factor in the AC grid [16]. The converter power factor in terms of its active and reactive powers can be represented as

$$\cos(\phi^h_{r,s}) = \frac{P^h_{r,s}}{\sqrt{(P^h_{r,s})^2 + (Q^h_{r,s})^2}}. \tag{2}$$

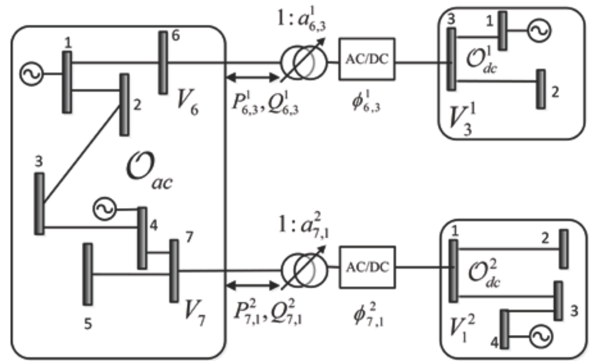

Fig. 1. An AC-DC network consisting of one AC grid and two DC microgrids.

In general, the current amplitude of a converter should not exceed a specific upper limit [16]. In power grids, the changes in the voltage magnitudes are negligible [17]. Hence, the upper bound of the current amplitude can be replaced by the maximum apparent power flow $S^{h,\max}_{r,s}$ as the operation constraint. Let $S^h_{r,s}$ denote the apparent power flow from AC bus $r \in \mathcal{N}_{ac}$ to DC bus $s \in \mathcal{N}^h_{dc}$ of microgrid h. We have

$$|S^h_{r,s}| = \sqrt{(P^h_{r,s})^2 + (Q^h_{r,s})^2} \le S^{h,\max}_{r,s}. \tag{3}$$

Fig. 1 shows an AC grid connecting to two DC microgrids by AC-DC converters between AC buses 6 and 7 and DC buses 3 and 1 in microgrids $\mathcal{O}^1_{dc}$ and $\mathcal{O}^2_{dc}$, respectively. The parameters $\phi^h_{r,s}, P^h_{r,s}, Q^h_{r,s}$ are the converter variables and need to be calculated in the AC-DC OPF. The parameter $a^h_{r,s}$ is known by the system. By computing $P^h_{r,s}$ and $Q^h_{r,s}$, the power factor $\phi^h_{r,s}$ can be determined from (2).

III. AC-DC OPF

In the AC-DC OPF problem, we aim to minimize a cost function subject to both AC grid and DC microgrids power flow equality constraints and the converters equations [16]. In addition to the converters variables, we need to determine the voltage phase and magnitude of the AC buses, the active and reactive output powers of the generators in the AC grid, the voltage magnitude of the DC buses, and the active output power of the generators in the DC microgrids.

The OPF problem for AC-DC grids is nonlinear since the voltage of buses in the AC grid are coupled with the voltage of the buses in the DC microgrids according to (1). Furthermore, the converter variables are necessary to be included in the traditional OPF problem formulation (see [17] for traditional AC OPF formulation). Hence, it is imperative to find an appropriate model for the converters in the AC-DC OPF problem. In this section, we propose a model for the converters to solve the AC-DC OPF problem using SDP method.

Consider a converter between AC bus $r \in \mathcal{N}_{ac}$ and DC bus $s \in \mathcal{N}_{dc}^h$ of the DC microgrid $h \in \mathcal{H}$. We model the converter in three steps as follows. First, we merge the buses r and s to create a new bus in order to eliminate the dependency of the AC and DC voltage magnitudes. Then, we add a generator with only reactive output power to the new buses to model the compensation ability of the converter. Finally, we replace all lines in DC microgrid with only resistive elements (since there is no voltage drop across inductors in DC), and all DC sources and loads only with real power. Without loss of generality, we can assume that the DC microgrids are AC microgrids with resistive transmission lines, active output power generators, and active loads. Let $\mathcal{O}_{ac}^h$ denote an AC microgrid that corresponds to the DC microgrid h. After performing these steps for all the converters, the DC microgrids can be connected to the AC grid directly in the model. Hence, we have an equivalent AC grid consisting of the AC grid $\mathcal{O}_{ac}$ and the AC microgrids $\mathcal{O}_{ac}^h$, for $h \in \mathcal{H}$.

We represent this equivalent AC grid by tuple $\mathcal{C}_{ac}(\mathcal{N}_{equ}, \mathcal{J}_{Conv}, \mathcal{L}_{equ})$, where $\mathcal{N}_{equ} = \{1, \ldots, |\mathcal{N}_{equ}|\}$ is the set of buses with new numbering in $\mathcal{C}_{ac}$ to prevent repeated bus numbers, $\mathcal{J}_{Conv}$ denotes the set of equivalent converter buses, and $\mathcal{L}_{equ}$ is the set of transmission lines.

Fig. 2 shows the equivalent AC grid with 12 buses after performing the above steps for the AC-DC grid shown in Fig. 1. Buses 6 and 7 are the equivalent converter buses. Q_6 and Q_7 are the reactive output power of the generators connected to the equivalent converter buses in the model. These parameters are equal to $Q_{6,3}^1$ and $Q_{7,1}^2$ in Fig. 1, respectively. Furthermore, $P_{6,trans}$ and $P_{7,trans}$ are the active transferred powers from $\mathcal{O}_{ac}$ to $\mathcal{O}_{ac}^1$ and $\mathcal{O}_{ac}^2$, respectively. These parameters are equal to $P_{6,3}^1$ and $P_{7,1}^2$ in Fig. 1, respectively.

The OPF problem in $\mathcal{C}_{ac}$ is a traditional AC OPF, which can be transformed to an SDP as in [12]. The voltage phase and magnitude of all buses and the generators output powers can then be determined. We can recover the AC network $\mathcal{O}_{ac}$, DC microgrids $\mathcal{O}_{dc}^h$ and the converters variables as follows. The voltage phase and magnitude of the buses in $\mathcal{O}_{ac}$ is equal to the voltage phase and magnitude of the buses in $\mathcal{C}_{ac}$. The voltage phase and magnitude of the equivalent converter bus in $\mathcal{C}_{ac}$ is equal to the voltage phase and magnitude of the AC bus connected to that converter in $\mathcal{O}_{ac}$. The active power flowing through the equivalent converter bus in $\mathcal{C}_{ac}$ is equal to the active power flowing through the converter from $\mathcal{O}_{ac}$ to its corresponding DC microgrid. The reactive output power of the generator connected to the equivalent converter bus in $\mathcal{C}_{ac}$ is equal to the reactive power injected or absorbed by the converter in $\mathcal{O}_{ac}$. The voltage magnitudes in the DC microgrids are proportional to the voltage magnitudes in $\mathcal{C}_{ac}$. The voltage magnitude of all buses in each DC microgrid should be scaled by $k_1 a_{r,s}^h \cos(\phi_{r,s}^h)$ according to (1). The computed phase angles can be ignored because the lines in DC microgrids are resistive and all powers are real. In Fig. 3, the proposed approach to solve OPF in AC-DC networks is presented. In the following section, the OPF problem in $\mathcal{C}_{ac}$ is formulated and solved using SDP.

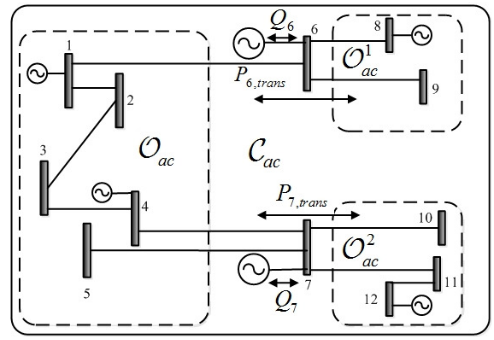

Fig. 2. The equivalent AC grid of the original AC-DC network, where $\mathcal{N}_{equ} = \{1, \ldots, 12\}$, $\mathcal{J}_{Conv} = \{6, 7\}$, and $\mathcal{L}_{equ} = \{(1,2), \ldots, (11,12)\}$.

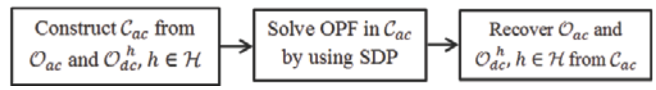

Fig. 3. The proposed approach to solve the AC-DC OPF problem.

IV. AC-DC OPF Formulation and SDP Method

The objective function of the OPF problem in the equivalent AC grid $\mathcal{C}_{ac}$ includes the generation cost in $\mathcal{C}_{ac}$ and the cost of transferring power from $\mathcal{O}_{ac}$ to $\mathcal{O}_{ac}^h$. We denote the set of generator buses in $\mathcal{C}_{ac}$ as $\mathcal{G}_{ac}$. Let P_{G_k} and Q_{G_k} denote the generated active and reactive powers in bus $k \in \mathcal{N}_{equ}$, respectively. The generation cost function for $k \in \mathcal{G}_{ac}$ is denoted by $f_k(P_{G_k})$. Let $P_{j,trans}$ and υ_j denote the transferred power and its price from equivalent converter bus $j \in \mathcal{J}_{Conv}$, respectively. Let $\mathcal{M}_{dc}$ denote the set of buses in $\mathcal{C}_{ac}$ that are not in $\mathcal{O}_{ac}$. ω_j, $j \in \mathcal{J}_{Conv}$ are the weighting coefficients. Let P_{D_k}, Q_{D_k}, V_k, and I_k denote the active load, reactive load, voltage, and injected current in bus $k \in \mathcal{N}_{equ}$, respectively. The AC OPF problem in $\mathcal{C}_{ac}$ can be formulated as

$$\text{minimize} \sum_{k \in \mathcal{G}_{ac}} f_k(P_{G_k}) + \sum_{j \in \mathcal{J}_{Conv}} \omega_j \upsilon_j P_{j,trans} \tag{4a}$$

$$\text{subject to} \quad P_{G_k} - P_{D_k} = \text{Re}\{V_k I_k^*\}, \quad \forall k \in \mathcal{N}_{equ} \tag{4b}$$

$$Q_{G_k} - Q_{D_k} = \text{Im}\{V_k I_k^*\}, \quad \forall k \in \mathcal{N}_{equ} \tag{4c}$$

$$P_{j,trans} = \sum_{(j,l) \in \mathcal{L}_{equ},\, l \in \mathcal{M}_{dc}} P_{s,j}^h, \quad \forall j \in \mathcal{J}_{Conv} \tag{4d}$$

$$P_k^{\min} \leq P_{G_k} \leq P_k^{\max}, \quad \forall k \in \mathcal{N}_{equ} \tag{4e}$$

$$Q_k^{\min} \leq Q_{G_k} \leq Q_k^{\max}, \quad \forall k \in \mathcal{N}_{equ} \tag{4f}$$

$$V_k^{\min} \leq |V_k| \leq V_k^{\max}, \quad \forall k \in \mathcal{N}_{equ} \tag{4g}$$

$$|S_{lm}| \leq S_{lm}^{\max}, \quad \forall (l,m) \in \mathcal{L}_{equ} \tag{4h}$$

$$|S_{j,trans}| \leq S_{j,trans}^{\max}, \quad \forall j \in \mathcal{J}_{Conv}. \tag{4i}$$

The minimization is performed over all voltage phases and magnitudes, active and reactive output powers of the generators, and the flowing active powers through the converters. Constraints (4b) and (4c) show the power balance equations in bus $k \in \mathcal{N}_{equ}$. Constraint (4d) shows that the input power to the equivalent converter bus is equal to the flowing power through that bus. Constraints (4e)-(4i) are the operation constraints. The generators output powers, the voltage magnitudes, the apparent power flowing through the transmission lines S_{lm} and the equivalent converter buses $S_{j,trans}$ are bounded. We use $P_k^{\min}$, $P_k^{\max}$, $Q_k^{\min}$, $Q_k^{\max}$, $V_k^{\min}$, and $V_k^{\max}$ to represent the lower and upper bounds on generator active power, reactive power, and bus voltage at bus k, respectively. If bus k is not a generator bus, then $P_k^{\max} = P_k^{\min} = Q_k^{\min} = Q_k^{\max} = 0$. $S_{lm}^{\max}$ is the maximum apparent power of the line $(l,m) \in \mathcal{L}_{equ}$. Constraint (4i) is equivalent to constraint (3) since $S_{j,trans}$ and $S_{j,trans}^{\max}$ are replaced by $S_{r,s}^{h}$ and $S_{r,s}^{h,\max}$, respectively.

We now introduce a semidefinite relaxation of the AC-DC OPF problem for the equivalent AC grid $\mathcal{C}_{ac}$. Our approach and notations are similar to [12] except for the formulation given in (6)-(8) for the power transferred through the converters. For $k \in \mathcal{N}_{equ}$ and $(l,m) \in \mathcal{L}_{equ}$, e_k is the k^{th} basis vector in $\mathbb{R}^{|\mathcal{N}_{equ}|}$, e_k^T is its transposed vector, and $Y_k := e_k e_k^T Y$. Matrix Y is the admittance matrix of $\mathcal{C}_{ac}$. The entry (k,k) of matrix Y_k is equal to the entry (k,k) of Y. All other elements of Y_k are zero.

We use the Π model of the transmission lines (l,m) [17]. Let y_{lm} and $\bar{y}_{lm}$ denote the value of the shunt and series element at bus l connected to bus m, respectively. We define $Y_{lm} := (\bar{y}_{lm} + y_{lm}) e_l e_l^T - (y_{lm}) e_l e_m^T$, where the entries (l,l) and (l,m) of Y_{lm} are equal to $\bar{y}_{lm} + y_{lm}$ and $-y_{lm}$, respectively. The other entries of Y_{lm} are zero.

We define matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$, $\mathbf{Y}_{lm}$, $\bar{\mathbf{Y}}_{lm}$, and $\mathbf{M}_k$ as follows. These matrices will be used to simplify the OPF formulation.

$$\mathbf{Y}_k := \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_k := -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\{Y_k + Y_k^T\} & \mathrm{Re}\{Y_k - Y_k^T\} \\ \mathrm{Re}\{Y_k^T - Y_k\} & \mathrm{Im}\{Y_k + Y_k^T\} \end{bmatrix},$$

$$\mathbf{Y}_{lm} := \frac{1}{2}\begin{bmatrix} \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} & \mathrm{Im}\{Y_{lm}^T - Y_{lm}\} \\ \mathrm{Im}\{Y_{lm} - Y_{lm}^T\} & \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix},$$

$$\bar{\mathbf{Y}}_{lm} := -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\{Y_{lm} + Y_{lm}^T\} & \mathrm{Re}\{Y_{lm} - Y_{lm}^T\} \\ \mathrm{Re}\{Y_{lm}^T - Y_{lm}\} & \mathrm{Im}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix},$$

$$\mathbf{M}_k := \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix}.$$

We define the variable vector $\mathbf{X}$ as the real and imaginary values of the vector of the bus voltages $\mathbf{V} = (V_1, \dots, V_{|\mathcal{N}_{equ}|})$.

$$\mathbf{X} := \begin{bmatrix} \mathrm{Re}\{\mathbf{V}\}^T & \mathrm{Im}\{\mathbf{V}\}^T \end{bmatrix}^T.$$

We also define variable matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^T$.

In [12], it is proved that for $k \in \mathcal{N}_{equ}$ and $(l,m) \in \mathcal{L}_{equ}$,

$$P_{G_k} = \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}\} + P_{D_k}, \tag{5a}$$

$$Q_{G_k} = \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} + Q_{D_k}, \tag{5b}$$

$$|V_k|^2 = \mathrm{Tr}\{\mathbf{M}_k \mathbf{W}\}, \tag{5c}$$

$$|S_{lm}|^2 = \mathrm{Tr}\{\mathbf{Y}_{lm} \mathbf{W}\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} \mathbf{W}\}^2, \tag{5d}$$

$$P_{lm} = \mathrm{Tr}\{\mathbf{Y}_{lm} \mathbf{W}\}, \tag{5e}$$

where P_{lm} is the active power flowing through line $(l,m) \in \mathcal{L}_{equ}$. In our model, we connect a generator with reactive output power to the converter bus $j \in \mathcal{J}_{Conv}$. Let Q_j denote the reactive power of the generator. From (5b), we have

$$Q_j = \mathrm{Tr}\{\bar{\mathbf{Y}}_j \mathbf{W}\}. \tag{6}$$

Furthermore, from (4d) and (5e), we have

$$\begin{aligned} P_{j,trans} &= \sum_{(j,l)\in\mathcal{L}_{equ},\, l\in\mathcal{M}_{dc}} \mathrm{Tr}\{\mathbf{Y}_{jl}\mathbf{W}\} \\ &= \mathrm{Tr}\{\sum_{(j,l)\in\mathcal{L}_{equ},\, l\in\mathcal{M}_{dc}} \mathbf{Y}_{jl}\mathbf{W}\} \\ &= \mathrm{Tr}\{\mathbf{Y}_{j,Conv}\mathbf{W}\}, \end{aligned} \tag{7}$$

where $\mathbf{Y}_{j,Conv} = \sum_{(j,l)\in\mathcal{L}_{equ},\, l\in\mathcal{M}_{dc}} \mathbf{Y}_{jl}$. Consequently, we have

$$|S_{j,trans}|^2 = \mathrm{Tr}\{\mathbf{Y}_{j,Conv}\mathbf{W}\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_j \mathbf{W}\}^2. \tag{8}$$

Consider the quadratic generation cost $f_k(P_{G_k}) = c_{k2} P_{G_k}^2 + c_{k1} P_{G_k} + c_{k0}$ for each generation bus k. By using (5a), we can represent the generation cost as a quadratic function of $\mathbf{W}$. However, in the SDP form, the objective function should be linear. Hence, we can rewrite it as a linear function of new variables β_k, and include the matrix form of inequality $f_k(P_{G_k}) \le \beta_k$ in the constraints set as shown in (9g). Problem (4) can be written as

$$\text{minimize} \sum_{k\in\mathcal{G}_{ac}} \beta_k + \sum_{j\in\mathcal{J}_{Conv}} \omega_j \upsilon_j \mathrm{Tr}\{\mathbf{Y}_{j,Conv}\mathbf{W}\} \tag{9a}$$

subject to the following constraints for $k \in \mathcal{N}_{equ}$, $j \in \mathcal{J}_{Conv}$ and $(l,m) \in \mathcal{L}_{equ}$

$$P_k^{\min} - P_{D_k} \le \mathrm{Tr}\{\mathbf{Y}_k \mathbf{W}\} \le P_k^{\max} - P_{D_k}, \tag{9b}$$

$$Q_k^{\min} - Q_{D_k} \le \mathrm{Tr}\{\bar{\mathbf{Y}}_k \mathbf{W}\} \le Q_k^{\max} - Q_{D_k}, \tag{9c}$$

$$(V_k^{\min})^2 \le \mathrm{Tr}\{\mathbf{M}_k \mathbf{W}\} \le (V_k^{\max})^2, \tag{9d}$$

$$\begin{bmatrix} -(S_{lm}^{\max})^2 & \mathrm{Tr}\{\mathbf{Y}_{lm}\mathbf{W}\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}\} \\ \mathrm{Tr}\{\mathbf{Y}_{lm}\mathbf{W}\} & -1 & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}\mathbf{W}\} & 0 & -1 \end{bmatrix} \preceq 0, \tag{9e}$$

$$\begin{bmatrix} -(S_{j,trans}^{\max})^2 & \mathrm{Tr}\{\mathbf{Y}_{j,Conv}\mathbf{W}\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_j\mathbf{W}\} \\ \mathrm{Tr}\{\mathbf{Y}_{j,Conv}\mathbf{W}\} & -1 & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_j\mathbf{W}\} & 0 & -1 \end{bmatrix} \preceq 0, \tag{9f}$$

$$\begin{bmatrix} -\beta_k + c_{k1}\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}\} + \tau_k & \sqrt{c_{k2}}(\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}\} + P_{D_k}) \\ \sqrt{c_{k2}}(\mathrm{Tr}\{\mathbf{Y}_k\mathbf{W}\} + P_{D_k}) & -1 \end{bmatrix} \preceq 0, \tag{9g}$$

$$\mathbf{W} = \mathbf{X}\mathbf{X}^T, \tag{9h}$$

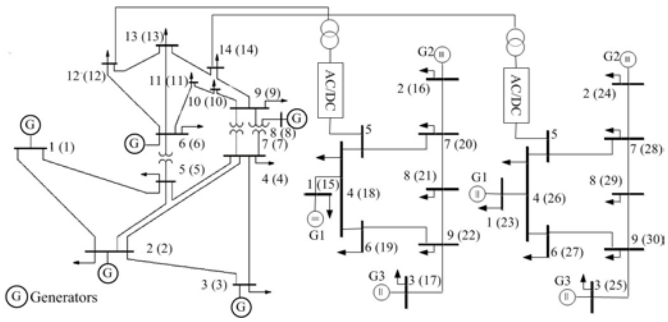

Fig. 4. An IEEE 14-bus system model connected to two 9-bus DC microgrids.

where $\tau_k = c_{k1} P_{D_k} + c_{k0}$ in constraint (9g). Constraints (9e) and (9f) are equivalent matrix form of (4h) and (4i) after substituting (5d) and (8). Constraint (9d) is equivalent to (4g). The constraint (9h) is equivalent to the constraints $\mathbf{W} \succeq 0$ and $\text{rank}(\mathbf{W}) = 1$. The SDP relaxation form of problem (4) can be obtained by removing the rank one constraint. Therefore, replacing (9h) with $\mathbf{W} \succeq 0$ gives the *SDP relaxation* of the AC network OPF problem (4). From now, we consider problem (9) with constraint $\mathbf{W} \succeq 0$ as the SDP form of AC-DC OPF.

In [12], it is proved that under sufficient condition, the relaxation gap is zero and SDP method provides global optimal solution for the OPF problem. This condition is as follows.

Theorem 1 *If the rank of the solution of SDP form of AC OPF problem $\mathbf{W}^{opt}$, be less than or equal to 2, then the SDP relaxation gap will be zero (the reverse is not true).*

Proof. If $\mathbf{W}^{opt}$ is rank one, then $\mathbf{W}^{opt} = \mathbf{X}^{opt}(\mathbf{X}^{opt})^T$ and the relaxation gap is zero. If $\mathbf{W}^{opt}$ is rank two, then it has two nonzero eigenvalues ρ_1 and ρ_2 with corresponding eigenvectors ν_1 and ν_2. In [12], it is shown that rank one matrix $\mathbf{W}_1^{opt} = (\rho_1 + \rho_2)\nu_1\nu_1^T$ is also the solution of the OPF problem. Matrix $\mathbf{W}_1^{opt}$ has only one nonzero eigenvalue ϱ with corresponding eigenvector ζ. Thus, the solution vector $\mathbf{X}^{opt}$ can be obtained from $\mathbf{X}^{opt} = \zeta\sqrt{\varrho}$. If the rank of $\mathbf{W}^{opt}$ is greater than two, then the relaxation gap may not be zero. ■

V. Performance Evaluation

In this section, we perform simulations on an IEEE 14-bus system connected to two IEEE 9-bus with only resistive lines as the DC microgrids. The system is shown in Fig. 4. AC network $\mathcal{O}_{ac}$ is connected to the DC microgrids through a converter between AC bus 12 and DC bus 5 of DC microgrid $\mathcal{O}_{dc}^1$, and through a converter between AC bus 14 and DC bus 5 of DC microgrid $\mathcal{O}_{dc}^2$. The line data and the limits of the parameters associated with the IEEE test systems are from [18]. Since we assumed that the converter bus is not a load bus, the loads connected to buses 12 and 14 are set to be zero. The minimum and maximum voltage limits of the buses are 0.9 p.u. and 1.15 p.u., respectively. The tap of the converter tap changers are 0.9. The minimum and maximum generation limits of the generators in the DC microgrids are 1 MW and 30 MW, respectively. The maximum transmission lines and the converters apparent power flows are 25 MW.

TABLE I
The Generation Buses Data

Bus	c_{k2}	c_{k1}	c_{k0}	P_{G_k}(MW)	Q_{G_k}(MVAr)
1	50	245	105	37.82	0
2	50	351	44	36.82	12.37
3	50	389	40	102.11	18.60
6	50	350	40	58.26	23.72
8	50	340	45	6.88	2.21
15	50	345	40	30.0	0
16	50	345	40	28.18	0
17	50	345	40	5.0	0
23	50	345	40	12.24	0
24	50	345	40	15.25	0
25	50	345	40	14.53	0

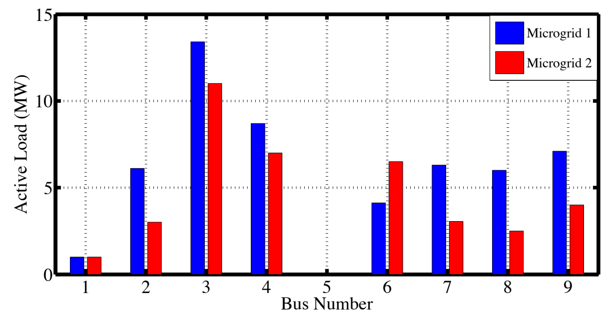

Fig. 5. The active load data of microgrids 1 and 2.

In the first stage, we construct an equivalent AC grid $\mathcal{C}_{ac}$. First, we merge AC buses 12 and 14 with bus 5 in both DC microgrids. Second, we add a reactive output power generator to the equivalent converter buses. Third, we replace the DC microgrids by AC microgrids $\mathcal{O}_{ac}^1$ and $\mathcal{O}_{ac}^2$. After these steps, the equivalent AC grid $\mathcal{C}_{ac}$ with $|\mathcal{N}_{equ}| = 30$ buses is constructed. We renumber the buses. In Fig. 4, the new numbering of each bus is shown inside the parentheses next to its original numbering. The parameters c_{k0}, c_{k1} and c_{k2} of the generators cost functions are given in Table I. The weighting coefficient $\omega_{12} = \omega_{14} = 50$. The load data of the DC microgrids is given in Fig. 5. In the next step, the OPF in $\mathcal{C}_{ac}$ is simulated. The SDP relaxation optimization problem (9) is solved to check the relaxation gap. We obtain rank-two matrix $\mathbf{W}^{opt}$ with nonzero eigenvalues $\rho_1 = \rho_2 = 6.73$ with corresponding eigenvectors ν_1 and ν_2. According to Theorem 1, the relaxation gap is zero, and the global optimal solution can be obtained. The rank one matrix $\mathbf{W}_1^{opt} = (\rho_1 + \rho_2)\nu_1\nu_1^T$ is also the solution of problem (9). Matrix $\mathbf{W}_1^{opt}$ has one nonzero eigenvalue $\varrho = 9.45$ with corresponding eigenvector ζ, and the solution vector $\mathbf{X}^{opt}$ is obtained from $\mathbf{X} = \sqrt{\varrho}\zeta$. Fig. 6 presents the voltage profile of the grid after performing the simulation when the electricity prices are $\upsilon_{12} = \upsilon_{14} = \$50$/MWh [19]. After obtaining $\mathbf{W}^{opt}$, the output generator powers can be computed from (5a) and (5b) and are shown in Table I.

We perform the simulation for electricity prices in range of \$5/MWh to \$85/MWh. Fig. 7 shows the amount of transferred power to the microgrids. When the prices increase, the transferred power to $\mathcal{O}_{ac}^1$ and $\mathcal{O}_{ac}^2$ increases and decreases, respectively. In fact, we minimize the aggregate cost of the system, and the load demand in $\mathcal{O}_{ac}^1$ is larger than $\mathcal{O}_{ac}^2$. Hence, microgrid $\mathcal{O}_{ac}^1$ purchases more power to meet the load

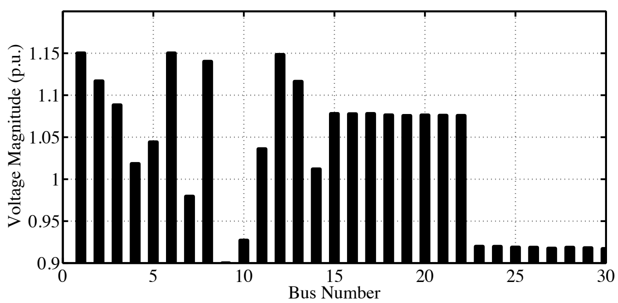

Fig. 6. Voltage profile for the AC grid and two DC microgrids.

TABLE II
THE AC-DC CONVERTERS SOLUTIONS FOR $\upsilon_{12} = \upsilon_{14} = \$50$/MWH.

Converter #	$\phi^h_{r,s}$	$P_{j,trans}$ (MW)	Q_j (MVAr)
1	31°	−12.34	8.08
2	32°	−4.91	3.07

demand even when the electricity prices increase. However, the transferred power to $\mathcal{O}^2_{ac}$ decreases. Thus, the aggregate transferred power to both microgrids decreases as well to reduce the total cost of the system. The reactive compensation and the power factor for the converters can be obtained from (6) and (2), respectively. The injected reactive powers are used to maintain the voltage level in its acceptable limit. The results for $\upsilon_{12} = \upsilon_{14} = \$50$/MWh are given in Table II.

Simulation results show that the SDP method can be used to solve OPF efficiently in the AC-DC network. SDP is a convex optimization approach. It can determine the global optimal solution in polynomial time. Hence, it is efficient in solving OPF in large scale power grids. In [12], it is claimed that the SDP relaxation in practical AC power grids including IEEE benchmarks has zero relaxation gap. The proposed approach of this paper can be used to transform an AC-DC OPF to an AC OPF and to be solved using SDP method.

VI. CONCLUSION

In this paper, we studied optimal power flow problem in the AC-DC system consisting of several DC microgrids and one AC network. The AC-DC converters have been represented by the corresponding real and reactive power conversion and power rating constraint equations in the OPF problem. As a result, the AC-DC OPF is transformed to an AC OPF problem. To compute the global optimal solution, the AC OPF is solved by using SDP relaxation technique. The condition for zero relaxation gap is studied. Simulations are performed on an IEEE 14-bus system connected to two sample 9-bus DC microgrids. By applying SDP approach, we determined the voltage profile, the generation output powers, and transferred power between the DC microgrids and the AC grid. We showed that the sufficient condition for the zero relaxation gap is satisfied, and the SDP approach enables us to find the global optimal solution in polynomial time. For future work, we plan to extend the model by considering other power electronic devices such as flexible AC transmission system controllers.

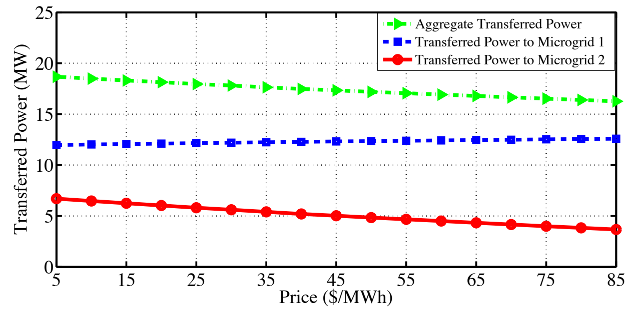

Fig. 7. Transferred power to the microgrids versus the electricity price.

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