A Potential Game Framework for Charging PHEVs in Smart Grid

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***Abstract*—Due to the proliferation of plug-in hybrid electric vehicles (PHEVs), the peak load in the power grid is expected to increase in future. The peak load can be reduced by implementing appropriate load scheduling schemes using advanced metering infrastructure (AMI) and smart chargers. In this paper, we formulate the charging problem of PHEVs as a potential game to jointly optimize the cost of the utility company and payoff of the customers. The potential game approach enables us to study the existence and uniqueness of the pure strategy Nash equilibrium and to design a polynomial time distributed algorithm to achieve that equilibrium. It also enables us to define a Lyapunov function to show that the Nash equilibrium is globally asymptotically stable, i.e., the proposed distributed algorithm converges to the Nash equilibrium from any arbitrary initial conditions. To evaluate the efficiency of our proposed algorithm, we compare its running time with an algorithm based on the customers' best response.**

***Keywords*: Charging PHEVs, potential game, Lyapunov function.**

I. Introduction

When a large number of plug-in hybrid electric vehicles (PHEVs) are integrated into the grid, the aggregate charging demand can further increase the existing peak load demand. It may also introduce new peaks to the daily load profile. To decrease the cost of generating electricity, utility companies need to motivate customers to shift the charging load of PHEVs from peak to off-peak periods [1]. In smart grid, advanced metering infrastructure (AMI) and smart chargers enable utility companies to provide customers with the price data [2]. They also enable utility companies to deploy automated charging scheduling approach to mitigate the charging impact of PHEVs on the load profile [3].

In the literature, there are several studies that address the charging problem of PHEVs. Wu *et al.* in [4] proposed a game theoretic framework to model the interaction between the aggregator and electric vehicles (EVs) to achieve optimal frequency regulation in the power grid. Shi *et al.* in [5] modeled vehicle-to-grid (V2G) control problem as a Markov decision process. They proposed an algorithm that can adapt to the unknown pricing information and make optimal control decisions. Wang *et al.* [6] designed an optimal V2G aggregator to control the charging and frequency regulation processes of a group of PHEVs. Couillet *et al.* in [7] formulated a competitive interaction between PHEVs in a Cournot power market. They modeled the competition market as a mean field game and introduced a set of differential equations to model the actions of the vehicles at the mean field equilibrium. Fan in [8] applied the congestion pricing concept used in communication networks and distributed systems to PHEVs charging and demand response problems in power networks. Nguyen *et al.* in [9] proposed centralized and decentralized optimization models for PHEVs charging. The centralized approach aims to minimize the Euclidean distance between the instantaneous load demand and the average demand. In the decentralized approach, a distributed algorithm is proposed in which users independently determine their charging schedules.

The PHEVs charging problem has been studied in several works [4]–[9]. The key challenge in this paper is to propose a scheduling algorithm to tackle the overload associated with the charging demand of PHEVs. In our system model, each PHEV is equipped with a smart charger that can collect the market clearing price data. We take into account the interaction between the charging strategies of PHEVs. Hence, we use non-cooperative game theoretic approach to model the problem. The contributions of this paper are as follows:

- We propose a novel charging approach for PHEVs. We show that the charging problem is an ordinal potential game with strictly concave potential function. We prove that the proposed potential game has a unique pure strategy Nash equilibrium, and we develop a distributed algorithm to determine that equilibrium.
- We model the game as a dynamical system. We show that there exists a Lyapunov function for this system, and the Nash equilibrium is globally asymptotically stable.
- The proposed scheduling approach is simulated on the system with 1000 households and 800 PHEVs. Simulations show that the proposed approach decreases the peak load, and increases the payoff of the customers. We compare the proposed algorithm with an algorithm based on the best response of the vehicle owners. We show that the running time of our charging algorithm increases linearly with the number of PHEVs.

The rest of this paper is organized as follows. In Section II, the charging problem of PHEVs is formulated as a potential game. The existence, uniqueness and stability of the Nash equilibrium are proven, and a distributed algorithm is proposed to determine the equilibrium. In Section III, simulation results are presented, and the performance of the proposed algorithm is evaluated. Conclusion is given in Section IV.

II. System Model

Consider H households, $N \leq H$ of which are equipped with PHEVs for regular daily usage. There is a utility company that supplies electricity to the customers. The set of the households and the PHEVs are denoted by $\mathcal{H} = \{1, \ldots, H\}$, and $\mathcal{N} = \{1, \ldots, N\}$, respectively. Each day is divided into T equal time slots. The set of time slots is denoted by $\mathcal{T} = \{1, \ldots, T\}$. Each PHEV is equipped with a smart charger that allows the battery to be charged in time slot t with controllable charging rate denoted by $r_i(t)$. As shown in Fig. 1, the utility company and the smart chargers are connected through a two-way communication link to receive and transmit data. The following assumptions are made in modelling the charging problem of PHEVs.

A1. PHEV $i \in \mathcal{N}$ arrives at $t_{i,a} \in \mathcal{T}$ and departs at $t_{i,d} \in \mathcal{T}$. Hence, the charging interval for PHEV i is $[t_{i,a}, t_{i,d}]$.

A2. The charging rate for PHEV i is within the minimum and maximum limits $r_i^{\min}$ and $r_i^{\max}$ in time slot t, respectively:

$$r_i^{\min} \leq r_i(t) \leq r_i^{\max}, \qquad i \in \mathcal{N}, t \in \mathcal{T}. \tag{1}$$

A3. The charging load of PHEVs is considered to be the only *controllable load* in the system. The remaining uncontrollable load for household i in time slot t is considered as the *base load* denoted by $l_{b,i}(t)$. Let $L_{base}(t)$ denote the aggregate base load at time t. We have

$$L_{base}(t) = \sum_{i \in \mathcal{H}} l_{b,i}(t), \qquad t \in \mathcal{T}. \tag{2}$$

A4. In power grids, there is always a minimum amount of uncontrollable load demand (such as refrigerator) in the system. Thus, the aggregate base load is greater than the minimum limit $L_{base}^{\min}(t)$ in time slot t. We have

$$0 < L_{base}^{\min}(t) \leq L_{base}(t), \qquad t \in \mathcal{T}. \tag{3}$$

The total load $L_{tot}(t)$ in time slot t is

$$L_{tot}(t) = L_{base}(t) + \sum_{i \in \mathcal{N}} r_i(t), \qquad t \in \mathcal{T}. \tag{4}$$

A5. PHEV i needs to fulfill the predetermined charging demand L_i^D for its battery upon completion of charging. Thus,

$$L_i^D = \sum_{t=t_{i,a}}^{t_{i,d}} r_i(t), \qquad i \in \mathcal{N}. \tag{5}$$

A. Supplier and Customers Side Models

1) Supplier Side: We consider that the utility company has a time-dependent generation cost function $c\big(L_{tot}(t)\big)$ when it supplies power $L_{tot}(t)$ in time slot t. We use the class of increasing quadratic cost functions with the form of $c(L_{tot}(t)) = \lambda_2 L_{tot}(t)^2 + \lambda_1 L_{tot}(t) + \lambda_0$ to model the generation cost of the utility company, where λ_2, λ_1, and λ_0 are the coefficients [9]. The utility company supplies the customers with amount of $x(t)$ in time slot t by solving the following optimization problem:

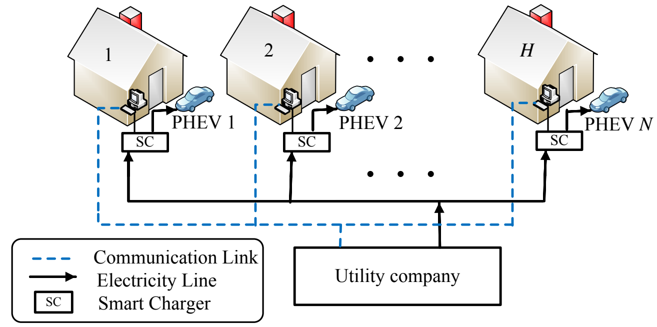

Fig. 1. Diagram of smart grid composed of a utility company, H households, N PHEVs with smart charger, electricity lines, and communication links.

$$\begin{aligned} &\underset{x(t)}{\text{minimize}} \quad c\,(x(t)) \\ &\text{subject to} \quad x(t) \geq L_{tot}(t). \end{aligned} \tag{6}$$

Let $p_e(t)$ denote the Lagrange multiplier associated with the inequality constraint $x \geq L_{tot}(t)$. From the Karush-Kuhn-Tucker (KKT) conditions, we have

$$c'\,(x(t)) - p_e(t) = 0, \tag{7a}$$

$$p_e(t)\,(x(t) - L_{tot}(t)) = 0, \tag{7b}$$

$$p_e(t) \geq 0. \tag{7c}$$

The cost function $c(.)$ is an increasing function. Hence, the solution to problem (6) is $x(t) = L_{tot}(t)$. From (7b) and (7c), we have $p_e(t) > 0$. From (7a), we obtain $c'\,(x(t)) = p_e(t)$. The Lagrange multiplier $p_e(t)$ is the shadow electricity price The utility company sets the price $p_e(t)$ equal to its marginal cost $c'\,(L_{tot}(t))$.

2) Customer Side: We assume that all PHEV owners are price anticipating. That is, they know that the electricity price function. Besides, each customer aims to locally and selfishly choose its PHEV charging profile to maximize its payoff. Consequently, game theory provides a framework to analyze and develop a proper charging scheduling mechanism to determine the charging times and rates.

In the charging game, the smart chargers of the PHEVs are the players. The charging profiles $\mathbf{r}_i = (r_i(1), \ldots, r_i(T))$, $i \in \mathcal{N}$ are the strategies. The payoff function of customer i is

$$\begin{aligned} u_i(\mathbf{r}_i, \mathbf{r}_{-i}) = \sum_{t \in \mathcal{T}} \Big(U_i\big(r_i(t) + l_{b,i}(t)\big) \\ - \big(r_i(t) + l_{b,i}(t)\big) c'\Big(L_{base}(t) + \sum_{j \in \mathcal{N}} r_j(t)\Big)\Big), \end{aligned} \tag{8}$$

where $\mathbf{r}_{-i}$ is the vector of the charging profiles of all PHEVs except PHEV i and is defined as $\mathbf{r}_{-i} = (\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_{i+1}, \ldots, \mathbf{r}_N)$. In (8), $U_i(.)$ is the utility of customer i to show its satisfaction from consuming electricity. The utility is assumed to be an increasing concave function. We use the class of proportional fairness utility functions to model the customers' utility. This class of utility functions has the form of [10]:

$$U_i\big(r_i(t) + l_{b,i}(t)\big) = \omega_i \log\big(r_i(t) + l_{b,i}(t)\big),$$

where ω_i, $i \in \mathcal{N}$ are scaling coefficients to measure the utility function in monetary units. We assume that ω_i, $i \in \mathcal{N}$, are known *a priori* by the customers.

We denote the charging profile for PHEV i in the Nash equilibrium by $\mathbf{r}_i^\star$. In equilibrium, the optimal charging profile $\mathbf{r}_i^\star$ is the solution of the following optimization problem when other PHEVs charging profiles are unchanged.

$$\begin{aligned} &\underset{\mathbf{r}_i}{\text{maximize}} \sum_{t\in\mathcal{T}} \Bigg(U_i\big(r_i(t) + l_{b,i}(t)\big) \\ &\qquad - \big(r_i(t) + l_{b,i}(t)\big) c'\Big(L_{base}(t) + \sum_{j\in\mathcal{N}} r_j(t)\Big)\Bigg) \\ &\text{subject to constraints (1)}-\text{(5)}. \end{aligned} \tag{9}$$

Let $\mathbf{r} = \big(r_1(1), \ldots, r_1(T), \ldots, r_N(1), \ldots, r_N(T)\big)$ denote the charging profile vector of all PHEVs. Consider the following definition of the ordinal potential game.

Definition 1 A game is called an ordinal potential game if it admits an ordinal potential function $P(\mathbf{r})$. That is, for strategy profiles $\mathbf{r} = (\mathbf{r}_i, \mathbf{r}_{-i})$ and $\hat{\mathbf{r}} = (\hat{\mathbf{r}}_i, \mathbf{r}_{-i})$, we have

$$\begin{aligned} &u_i(\mathbf{r}_i, \mathbf{r}_{-i}) - u_i(\hat{\mathbf{r}}_i, \mathbf{r}_{-i}) \geq 0 \\ \Rightarrow\ &P(\mathbf{r}_i, \mathbf{r}_{-i}) - P(\hat{\mathbf{r}}_i, \mathbf{r}_{-i}) \geq 0. \end{aligned} \tag{10}$$

In Appendix A, we show that game (9) is an ordinal potential game with the following potential function

$$\begin{aligned} P(\mathbf{r}) = &\sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} \Big(U_i\big(r_i(t) + l_{b,i}(t)\big) - \log_m\big(r_i(t) + l_{b,i}(t)\big)\Big) \\ &- \sum_{t\in\mathcal{T}} \log_m \bigg(c'\Big(L_{base}(t) + \sum_{j\in\mathcal{N}} r_j(t)\Big)\bigg), \end{aligned} \tag{11}$$

where m is a positive integer. The charging profile $\mathbf{r}^\star$ for all PHEVs is the solution of the following optimization problem

$$\begin{aligned} &\underset{\mathbf{r}}{\text{maximize}}\ P(\mathbf{r}) \\ &\text{subject to constraints (1)}-\text{(5)}. \end{aligned} \tag{12}$$

In potential game (12), the payoff function of all players is mapped onto a potential function $P(\mathbf{r})$. The set of pure Nash equilibria can be determined by locating the local optima of the potential function [11]. Problem (12) implies that the set of Nash equilibria of game (9) corresponds to the set of maximum points of the potential function (11) on the feasible set defined by constraints (1)-(5). In Appendix B, we show that when m is sufficiently large, the function $P(\mathbf{r})$ is concave. Consequently, the optimal point of the potential function (11) exists and is unique. The optimal point of the potential function corresponds to the pure strategy Nash equilibrium. Therefore, the Nash equilibrium of charging game (9) exists and is unique.

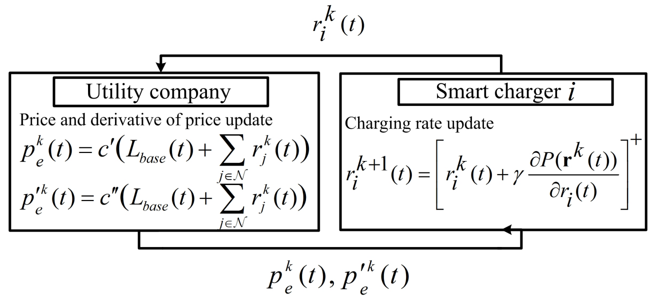

Fig. 2. The interactions between the utility company and the smart chargers.

B. Distributed Algorithm

We propose a distributed algorithm to determine the optimal charging rate of each PHEV. Let k denote the iteration number. Let $r_i^k(t)$, $i \in \mathcal{N}$, denote the charging rate of PHEV i in iteration k in time slot t. $\mathbf{r}^k(t) = (r_1^k(t), \ldots, r_N^k(t))$ is the vector of charging rate of all PHEVs in iteration k in time slot t. $p_e^k(t)$ and $p_e'^k(t)$ denote the electricity price and the derivative of the electricity price function in iteration k in time slot t, respectively. The interaction between each smart charger and the utility company is shown in Fig. 2. In each iteration, the utility company updates the electricity price $p_e^k(t)$. The utility company also updates $p_e'^k(t)$ by calculating the second derivative of the cost function $c(.)$. The distributed algorithm of smart charger i is summarized in Algorithm 1. In Line 1, smart charger i randomly initializes the charging rate in time t. The loop in Lines 2 to 6 describes the interaction between the smart charger and the utility company. Within this loop, smart charger i communicates the charging rate to the utility company in Line 3. It receives the updated value of $p_e^k(t)$ and $p_e'^k(t)$ in Line 4. It then updates the charging rate as follows:

$$r_i^{k+1}(t) = \left[r_i^k(t) + \gamma \frac{\partial P(\mathbf{r}^k(t))}{\partial r_i(t)} \right]^+, \tag{13}$$

where γ is the step size and $[.]^+$ is the projection onto the feasible set defined by constraints (1)-(5). In (13), smart charger i knows it is an ordinal potential game with potential function (11). Considering (11), smart charger i does not need the charging rate of the other chargers since $\frac{\partial P(\mathbf{r}^k(t))}{\partial r_i(t)}$ only depends on its charging rate, $p_e^k(t)$ and $p_e'^k(t)$.

Algorithm 1 Executed by smart charger $i \in \mathcal{N}$.

1: Initialization: $k = 0$, $\theta = 10^{-6}$.
2: **Repeat**
3: Send the charging rate $r_i^k(t)$ to the utility company.
4: Receive the updated $p_e^k(t)$ and $p_e'^k(t)$ from the utility company.
5: Update the charging rate $r_i^k(t)$ according to (13).
6: $k := k + 1$.
7: **Until** $||r_i^k(t) - r_i^{k-1}(t)|| < \theta$.

C. Lyapunov Stability

In this subsection, we study the convergence of the proposed distributed algorithm. Algorithm 1 can be modeled as a dynamic system, in which smart charger i updates the charging rate under the following differential equation [12].

$$\frac{\partial r_i(t,\xi)}{\partial \xi} = \gamma \left[\frac{\partial P(\mathbf{r}(t,\xi))}{\partial r_i(t,\xi)}\right]^+, \qquad i \in \mathcal{N}, t \in \mathcal{T}, \qquad (14)$$

where ξ is the evolution parameter, and $r_i(t,\xi)$ is the evolutionary function for PHEV i [12]. $r_i(t,\xi)$ shows the variation of the charging rate of PHEV i in time slot t. The iterative equation in (13) and the dynamic differential equation (14) describe the discrete and the continuous variation of the charging rate $r_i(t)$ in time slot t, respectively. The evolution parameter ξ in (14) plays the same role as the iteration number k in (13). ξ varies from 0 to $+\infty$ continuously but the iteration number k only takes on positive integer numbers. Similarly, $\mathbf{r}(t,\xi) = \big(r_1(t,\xi), \dots, r_N(t,\xi)\big)$ is the vector of charging rates in time slot t when ξ varies from 0 to $+\infty$.

The rest point of the dynamic system (14) in time slot t, (i.e., the point in which $\frac{\partial r_i(t,\xi)}{\partial \xi} = 0$, $i \in \mathcal{N}$), corresponds to the unique Nash equilibrium of the charging game. Let $\mathbf{r}^*(t)$ denote the charging profile of the PHEVs in time slot t in the Nash equilibrium. If the system always converges to the rest point for any solution with any initial condition, then $\mathbf{r}^*(t)$ is global asymptotically stable. We can use the Lyapunov's direct method to determine the stability by using Lyapunov functions. In Appendix C, it is shown that by modeling the problem as a potential game, we can define a proper Lyapunov function for the dynamic system (14), and the candidate Lyapunov function satisfies the necessary conditions of the Lyapunov's direct method. Hence, it is guaranteed that the equilibrium is globally asymptotically stable and Algorithm 1 converges to the unique Nash equilibrium from any initial condition [13].

III. Performance Evaluation

In this section, we evaluate the performance of our proposed algorithm and compare it with the algorithm given in [8] to show the efficiency of our potential game model. We consider a neighborhood with a utility company, and $H = 1000$ homes, $N = 800$ of which are equipped with PHEVs. The Chevrolet Volt PHEV with 16 kWh Lithium-ion battery is used for modeling the vehicles [14]. The day is divided to $T = 24$ one-hour time slots. The generation cost function for the utility company is modeled as the quadratic function $c(L_{tot}(t)) = \lambda_2 L_{tot}(t)^2$, where $\lambda_2 = 2 \times 10^{-3}$ when $L_{tot}(t)$ is measured in kilowatts. These parameters are pre-determined coefficients and are known by the utility company. The utility company sends the electricity price signal to the customers and does not need to reveal the generation cost function data. The electricity price is equal to the marginal cost. Thus, we have $p_e(L_{tot}(t)) = 4 \times 10^{-3} \times L_{tot}(t)$. The scaling coefficient in the utility functions $U_i(.)$, $i \in \mathcal{N}$ are set to $\omega_i = 30$, $i \in \mathcal{N}$. Besides, $m = 10$ is considered for the potential function (11). The generation capacity is assumed to be sufficiently large to supply both the base load and charging load in all time slots. To model the customers' base load profiles, we use sample daily residential load profiles with off-peak of about 1 kW and high peak of 5 kW [15]. The aggregate base load and the total load profiles are shown in Fig. 3.

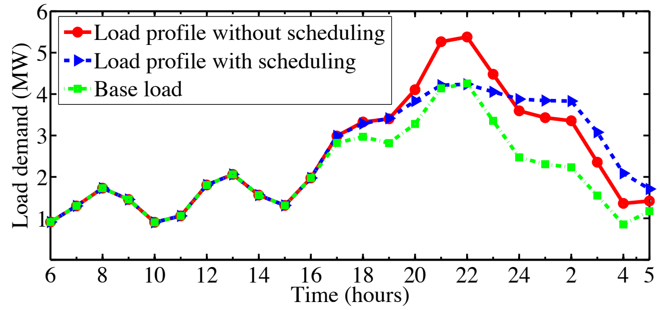

Fig. 3. Base load and total load profiles with and without load scheduling.

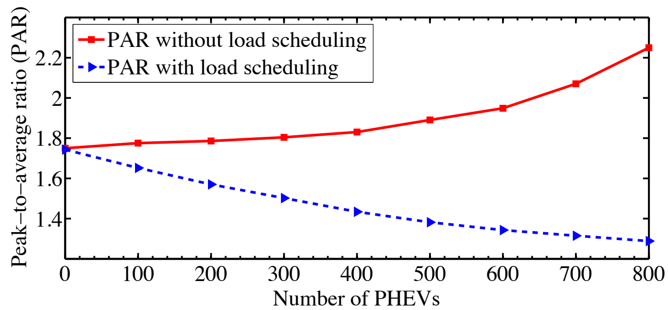

Fig. 4. Peak-to-average ratio (PAR) with and without load scheduling.

To model the charging rate of the smart chargers, AC level 1 charging method is used. AC Level 1 is a 120-volt AC plug, and a full charge can take between 5 and 10 hours, depending on the battery capacity of the vehicle. The charging rate in AC level 1 is less than 5 kW [14]. The charging demand L_i^D, arrival time $t_{i,a}$ and departure time $t_{i,d}$ of the PHEVs are chosen randomly from intervals $[10\,\text{kWh}, 16\,\text{kWh}]$, $[16, 24]$ (at night) and $[4, 6]$ (in the morning), respectively. We have used AC level 1 charging method. Hence, the initial value of the charging rate for PHEV $i \in \mathcal{N}$ can take on any value in $[0, 5\,\text{kW}]$. For simplicity, we assume that the initial value of the charging rate for PHEV $i \in \mathcal{N}$ is equal to the average demand over the charging period. Hence, we have $r_i^0(t) = \frac{L_i^D}{t_{i,d} - t_{i,a}}$.

Fig. 3 shows the total load profiles with and without load scheduling. Without load scheduling, the peak load is 5400 kW. By applying the proposed algorithm, the charging load is shifted to off-peak time slots and the peak load decreases to 4200 kW. In Fig. 4, the peak-to-average ratio (PAR) with and without load scheduling for different number of PHEVs is shown. Without load scheduling, the charging demand of PHEVs further increases the existing peak load, and the PAR is high. Besides, the PAR increases when the number of PHEVs increases. However, with load scheduling, peak load is reduced and the charging load is shifted to off-peak periods. Hence, the average load increases and the peak load decreases, thus the PAR decreases even for high number of PHEVs.

By participating in the proposed charging game, the daily payoff of the customers increases. Fig. 5 presents the payoff of the PHEV owners 1 to 30 as an example. This figure shows that the payoff for the vehicle owners is increased when they charge their PHEV strategically. That is, the outcome with

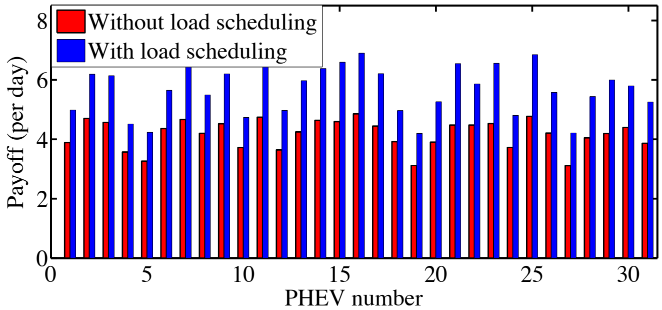

Fig. 5. The payoff from PHEV 1 to PHEV 30 with and without scheduling.

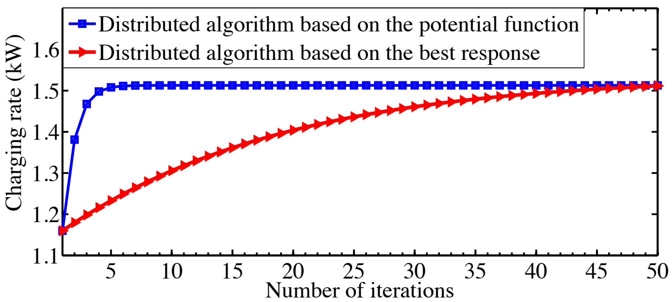

Fig. 6. The convergence of charging rates.

implementing the scheduling motivates the customers to utilize smart chargers to manage the charging load of their PHEVs.

In real-world scenarios, the charging algorithm is deployed for a large number of PHEVs. Thus, the running time of the algorithm is an important factor to evaluate its efficiency. We compare our proposed algorithm with the algorithm based on the best response of the customers given in [9]. Fig. 6 shows the convergence of the charging rates for PHEV 1 in time slot 18 using both algorithms. The step size is 0.05. The algorithm based on the potential function has a computational complexity of $O(N)$ since all payoff functions are mapped onto one potential function. It converges in 10 iterations in about 2 minutes when $N = 800$. However, the algorithm based on the best response has an exponential complexity and converges in 50 iterations in 19 minutes. The running time of the algorithms for different number of PHEVs are given in Table I. It confirms that the proposed algorithm can be implemented in scenarios with high PHEV penetration level.

IV. Conclusion

In this paper, we have proposed a scheduling algorithm to reduce the peak load caused by the charging loads of the PHEVs. The charging problem is formulated as a potential game to maximize the customers' payoff. Furthermore, the electricity price is obtained by solving the revenue maximization problem of the utility company. We have shown that the Nash equilibrium exists and is unique. We have modeled the game as a dynamical system, and we showed that Nash equilibrium is also asymptotically stable. We have proposed a distributed algorithm the potential function and compared it with an algorithm based on the PHEVs' best response. We have shown that the potential function based algorithm

TABLE I
Running Time for the Best Response and Potential Function Based Algorithms

Number of PHEVs	Running time (in sec) of best response based algorithm	Running time (in sec) of potential function based algorithm
20	3.04	2.93
100	22.11	15.28
200	83.49	32.4
300	185.56	49.01
500	510.123	79.65
800	1145.2	121.36

is polynomial time and converges from any arbitrary initial condition to the Nash equilibrium. Simulation results show that the proposed algorithm increases the customers' payoff and reduces the peak load. For future work, we will consider the uncertainty in the charging demand of the PHEVs, which is dependent on the daily trip of the vehicles.

Appendix

A. Obtaining the Potential Function

We substitute (8) into the left-hand side of inequality (10). After some algebraic manipulation, we can show that the left-hand side of inequality (10) is equivalent to

$$\sum_{t\in\mathcal{T}}\Big(U_i\big(r_i(t)+l_{b,i}(t)\big)-U_i\big(\hat{r}_i(t)+l_{b,i}(t)\big)\Big) \geq \sum_{t\in\mathcal{T}}\big(K_1(t)-K_2(t)\big), \quad (15)$$

where

$$K_1(t)=\Big(r_i(t)+l_{b,i}(t)\Big)c'\Big(r_i(t)+L_{base}(t)+\sum_{j\in\mathcal{N}\setminus\{i\}} r_j(t)\Big),$$

$$K_2(t)=\Big(\hat{r}_i(t)+l_{b,i}(t)\Big)c'\Big(\hat{r}_i(t)+L_{base}(t)+\sum_{j\in\mathcal{N}\setminus\{i\}} r_j(t)\Big).$$

Consider the following two lemmas.

Lemma 1. *Consider $f(x)=\log_m(x)$ and $\varepsilon>0$ for which $x>\varepsilon$. Then, we have $f'(x)<1$ for $m>e^{\frac{1}{\varepsilon}}$ and all feasible x.*

Proof. For function $f(x)=\log_m(x)$, we have $f'(x)=\frac{1}{x\ln m}$. For $x>\varepsilon$, if $m>e^{\frac{1}{\varepsilon}}$, we have $\frac{1}{x\ln m}<\frac{1}{\varepsilon\frac{1}{\varepsilon}}=1$. Hence, we have $f'(x)<1$ for $m>e^{\frac{1}{\varepsilon}}$. ■

Lemma 2. *Consider an increasing, continuous, and differentiable function $f(x)$ with $0<f'(x)<1$. Then, for all $a>b$ in the domain of $f(x)$, the following inequality holds*

$$f(a)-f(b)<a-b. \quad (16)$$

Proof. From mean value theorem, there exists α in $[a,b]$ such that $f'(\alpha)=\frac{f(a)-f(b)}{a-b}$. We have $0<f'(\alpha)<1$. Hence, $0<\frac{f(a)-f(b)}{a-b}<1$, and inequality (16) holds. ■

Consider the logarithmic function $f(x)=\log_m(x)$ with $x=K_1(t)$ and $x=K_2(t)$. There exist small enough $\varepsilon_1,\varepsilon_2>0$, where $K_1(t)>\varepsilon_1$ and $K_2(t)>\varepsilon_2$. Lemma 1 guarantees that for $m > \max(e^{\frac{1}{\varepsilon_1}}, e^{\frac{1}{\varepsilon_2}})$, we have $f'(x) < 1$. According to Lemma 2, for each time slot t, we have

$$\log_m\big(K_1(t)\big) - \log_m\big(K_2(t)\big) < K_1(t) - K_2(t). \tag{17}$$

Considering (15) and (17), we have

$$\begin{aligned}&\sum_{t\in\mathcal{T}}\Big(\log_m\big(K_1(t)\big) - \log_m\big(K_2(t)\big)\Big)\\ &<\sum_{t\in\mathcal{T}}\Big(U_i\big(r_i(t)+l_{b,i}(t)\big) - U_i\big(\hat{r}_i(t)+l_{b,i}(t)\big)\Big).\end{aligned} \tag{18}$$

Substituting $K_1(t)$ and $K_2(t)$ into (18) shows that for $m > \max(e^{\frac{1}{\varepsilon_1}}, e^{\frac{1}{\varepsilon_2}})$, (11) is the potential function of (9). ■

B. Convexity of the Potential Function

We show that when m is sufficiently large, function (11) is strictly concave, i.e., its Jacobian matrix $J = \nabla^2 P(\mathbf{r})$ is negative definite. The diagonal and non-diagonal entries are

$$\begin{aligned}J_{ii} = \frac{\partial^2 P(\mathbf{r})}{\partial r_i^2(t)} = &\sum_{t\in\mathcal{T}} U_i''\big(r_i(t)+l_{b,i}(t)\big) + \frac{1}{\big(r_i(t)+l_{b,i}(t)\big)\ln m}\\ &- \frac{c'(L_{tot}(t))c'''(L_{tot}(t)) - c''(L_{tot}(t))^2}{c'(L_{tot}(t))^2 \ln m},\end{aligned} \tag{19}$$

$$J_{ij} = \frac{\partial^2 P(\mathbf{r})}{\partial r_i(t)\partial r_j(t)} = \sum_{t\in\mathcal{T}} \frac{c''(L_{tot}(t))^2 - c'(L_{tot}(t))c'''(L_{tot}(t))}{c'(L_{tot}(t))^2 \ln m} \tag{20}$$

The sum of all non-diagonal elements in row i is

$$\sum_{j\in\mathcal{N}\backslash\{i\}} J_{ij} = \sum_{t\in\mathcal{T}} \frac{(N-1)\big(c''(L_{tot}(t))^2 - c'(L_{tot}(t))c'''(L_{tot}(t))\big)}{c'(L_{tot}(t))^2 \ln m}. \tag{21}$$

From assumption A4, we have $r_i(t)+l_{b,i}(t) > 0$. The utility function for customer i is concave. Hence, there exists $\varepsilon_i > 0$ where $-U_i''\big(r_i(t)+l_{b,i}(t)\big) > \varepsilon_i$. The diagonal element J_{ii} is less than the sum of non-diagonal elements in its row iff

$$\begin{aligned}&-U_i''\big(r_i(t)+l_{b,i}(t)\big) > \varepsilon_i\\ &> \sum_{t\in\mathcal{T}}\Bigg(\frac{(N-2)\big(c'(L_{tot}(t))c'''(L_{tot}(t)) - c''(L_{tot}(t))^2\big)}{c'(L_{tot}(t))^2 \ln m}\\ &\qquad - \frac{1}{\big(r_i(t)+l_{b,i}(t)\big)\ln m}\Bigg).\end{aligned} \tag{22}$$

The charging rate $r_i(t)$ is bounded. Hence, the right-hand side of (22) is bounded. Function $\ln m$ is a strictly increasing function. For a sufficiently large m, the above inequality holds for all rows $i \in \mathcal{N}$. Thus, J_{ii} is less than the sum of all non-diagonal elements in its row. Hence, matrix J is negative definite, and the potential function is strictly concave [16]. ■

C. Lyapunov Function and Nash Equilibrium Stability

In this part, we introduce a candidate Lyapunov function to show the stability of dynamical system (14). Consider the following Lyapunov function $V(\mathbf{r}(t)) = P(\mathbf{r}^*(t)) - P(\mathbf{r}(t))$, where $\mathbf{r}^*(t)$ is the optimum point of the potential function P in time slot t. Thus, $V(\mathbf{r}(t)) \geq 0$. Function $P(\mathbf{r}(t))$ is concave. Hence, $V(\mathbf{r}(t))$ is a convex function. Besides, $V(\mathbf{r}(t)) = 0$ iff $\mathbf{r}(t) = \mathbf{r}^*(t)$. Considering the dynamical model (14), we have

$$\frac{\partial V\big(\mathbf{r}(t,\xi)\big)}{\partial \xi} = -\left(\frac{\partial \mathbf{r}(t,\xi)}{\partial \xi}\right)^T \nabla_{\mathbf{r}} P(\mathbf{r}(t,\xi)). \tag{23}$$

Substituting (14) into (23) gives

$$\begin{aligned}\frac{\partial V\big(\mathbf{r}(t,\xi)\big)}{\partial \xi} &= -\gamma \nabla_{\mathbf{r}}^T P(\mathbf{r}(t,\xi)) \nabla_{\mathbf{r}} P(\mathbf{r}(t,\xi))\\ &= -\gamma ||\, [\nabla_{\mathbf{r}} P(\mathbf{r}(t,\xi))]^+ \,||^2 \leq 0.\end{aligned} \tag{24}$$

Thus, $V\big(\mathbf{r}(t,\xi)\big)$ is decreasing function of ξ and $V(\mathbf{r}(t))$ is a valid Lyapunov function [13]. The Lyapunov direct method [13] states that if there exists a decreasing and convex Lyapunov function, then the solutions originating in the compact constraints set will converge to the largest invariant of set $S = \{\mathbf{r}(t,\xi) \in \Omega \mid \frac{\partial V(\mathbf{r}(t,\xi))}{\partial \xi} = 0\}$, where Ω is the feasible set defined by (1)-(5). Set S contains only the unique Nash equilibrium of the game since, from (24), we have $\frac{\partial V\left(\mathbf{r}(t,\xi)\right)}{\partial \xi} = 0$ iff $[\nabla_{\mathbf{r}} P(\mathbf{r}(t,\xi))]^+ = 0$. Thus, $\mathbf{r}(t)$ is equal to the optimal point of $P(\mathbf{r}(t,\xi))$ at time t, which is $\mathbf{r}^*(t)$. Therefore, the dynamical system (14) converges to Nash equilibrium from any arbitrary initial condition. ■

References

[1] G. Li and X. P. Zhang, "Modeling of plug-in hybrid electric vehicle charging demand in probabilistic power flow calculations," *IEEE Trans. on Smart Grid*, vol. 3, no. 1, pp. 492–499, Mar. 2012.

[2] R. Arai, K. Yamamoto, and M. Morikura, "Differential game-theoretic framework for a demand-side energy management system," in *Proc. of IEEE SmartGridComm*, Vancouver, Canada, Oct. 2013.

[3] S. Bahrami and M. Parniani, "Game theoretic based charging strategy for plug-in hybrid electric vehicles," *IEEE Trans. on Smart Grid,*, vol. 5, no. 5, pp. 2368–2375, Sept. 2014.

[4] C. Wu, H. Mohsenian-Rad, and J. Huang, "Vehicle-to-aggregator interaction game," *IEEE Trans. on Smart Grid*, vol. 3, no. 1, pp. 434–442, Mar. 2012.

[5] W. Shi and V. W. S. Wong, "Real-time vehicle-to-grid control algorithm under price uncertainty," in *Proc. of IEEE SmartGridComm.*, Brussels, Belgium, Oct 2011.

[6] R. Wang, Y. Li, P. Wang, and D. Niyato, "Design of a V2G aggregator to optimize PHEV charging and frequency regulation control," in *Proc. of IEEE SmartGridComm*, Vancouver, Canada, Oct. 2013.

[7] R. Couillet, S. M. Perlaza, H. Tembine, and M. Debbah, "Electrical vehicles in the smart grid: A mean field game analysis," *IEEE J. on Selected Areas in Commun.*, vol. 30, no. 6, pp. 1086–1096, Jul. 2012.

[8] Z. Fan, "A distributed demand response algorithm and its application to PHEV charging in smart grids," *IEEE Trans. on Smart Grid*, vol. 3, no. 3, pp. 1280–1291, Sept. 2012.

[9] H. K. Nguyen and J. B. Song, "Optimal charging and discharging for multiple PHEVs with demand side management in vehicle-to-building," *J. of Commun. and Networks*, vol. 14, no. 6, pp. 662–671, Dec. 2012.

[10] R. Faranda, A. Pievatolo, and E. Tironi, "Load shedding: A new proposal," *IEEE Trans. on Power Systems*, vol. 22, no. 4, pp. 2086–2093, Nov. 2007.

[11] D. Monderer and L. S. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 3124–143, May 1996.

[12] H. K. Khalil, *Nonlinear Systems*, 3rd ed. NJ: Prentice Hall, 2001.

[13] A. Bacciotti and L. Rosier, *Lyapunov Function and Stability in Control Theory*, 2nd ed. NY: Springer, 2005.

[14] Chevy Volt website. [Online]. Available: http://www.chevyvolt.net/chevrolet-volt-specs.htm.

[15] National Renewable Energy Labratory (NREL) website. [Online]. Available: http://www.nrel.gov/analysis/data_resources.html

[16] J. Stewart, *Calculus*, 4th ed. CA: Brooks/Cole Pub Co., 1999.