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REGIME SWITCHING STOCHASTIC APPROXIMATION ALGORITHMS WITH APPLICATION TO ADAPTIVE DISCRETE STOCHASTIC OPTIMIZATION*

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Abstract. This work is devoted to a class of stochastic approximation problems with regime switching modulated by a discrete-time Markov chain. Our motivation stems from using stochastic recursive algorithms for tracking Markovian parameters such as those in spreading code optimization in CDMA (code division multiple access) wireless communication. The algorithm uses constant step size to update the increments of a sequence of occupation measures. It is proved that least squares estimates of the tracking errors can be developed. Assume that the adaptation rate is of the same order of magnitude as that of the time-varying parameter, which is more difficult to deal with than that of slower parameter variations. Due to the time-varying characteristics and Markovian jumps, the usual stochastic approximation (SA) techniques cannot be carried over in the analysis. By a combined use of the SA method and two-time-scale Markov chains, asymptotic properties of the algorithm are obtained, which are distinct from the usual SA results. In this paper, it is shown for the first time that, under simple conditions, a continuous-time interpolation of the iterates converges weakly not to an ODE, as is widely known in the literature, but to a system of ODEs with regime switching, and that a suitably scaled sequence of the tracking errors converges not to a diffusion but to a system of switching diffusion. As an application of these results, the performance of an adaptive discrete stochastic optimization algorithm is analyzed.

Key words. stochastic approximation, Markovian parameter, time-varying parameter, regime switching model, tracking, regime switching ODEs, switching diffusion

AMS subject classifications. 60J10, 60J27, 62L20, 93C50, 93E10

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1. Introduction.

In this paper, we consider a class of stochastic approximation (SA) algorithms for tracking the invariant distribution of a conditional Markov chain (conditioned on another Markov chain whose transition probability matrix is "near" identity). Here and henceforth, we refer to such a Markov chain with infrequent jumps as a slow Markov chain, for simplicity. It is well known that if the parameter changes too drastically, there is no chance one can track the time-varying properties using an SA algorithm. Such a phenomenon is known as tracking capability; see [4] for related discussions. Our objectives include evaluating the tracking capability of the SA algorithm in terms of mean squares tracking error, characterizing the dynamic behavior of the iterates, revealing the structure of a scaled sequence of tracking errors, and obtaining the asymptotic covariance of the associated limit process.

Motivation. While there are several papers that analyze tracking properties of SA algorithms when the underlying parameter varies according to a slow random walk [4, 19], fewer papers consider the case when the underlying parameter evolves according to a slow Markov chain. Yet such slow Markov chain models arise in several

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applications. The main motivation for our work stems from applications in discrete stochastic optimization. Such problems appeared in [21] and were subsequently considered in [2, 3, 10] among others; we refer the reader to [20] for a recent survey of several methods for discrete stochastic optimization including selection and multiple comparison methods, multi-armed bandits, the stochastic ruler, nested partition methods, and discrete stochastic optimization algorithms based on simulated annealing [1, 2, 3, 9].

The discrete stochastic optimization algorithms in [2, 3] can be thought of as random search procedures, in which there is a feasible set $\mathcal{S}$ that contains the minima together with other potential search candidates. One devises a strategy so that the optimal parameter (minimum) is estimated with minimal effort. An important variation of this is to devise and analyze the performance of an *adaptive* discrete stochastic optimization algorithm when the underlying parameter (minimum) is slowly time-varying. Such tracking problems lie at the heart of applications of SA algorithms. In such cases, because the parameter set is finite, it is often reasonable to assume that the underlying parameter (termed "hypermodel" in [4]) evolves according to a slow finite state Markov chain. As will be shown in section 6, the general tracking analysis presented in this paper for a slow Markov chain parameter readily applies to analyzing the tracking performance of such adaptive discrete stochastic optimization algorithms. To the best of our knowledge, this is the first time a tracking analysis has been presented for a discrete stochastic optimization algorithm.

Applications. Discrete stochastic optimization problems arise in emerging applications such as adaptive coding in wireless CDMA (code division multiple access) communication networks. In our recent work [11], we considered optimizing the spreading code of the CDMA system at the transmitter. This was formulated as a discrete stochastic optimization problem (since the spreading codes are finite-length and finite-state sequences), and the random-search–based discrete stochastic optimization algorithm of [2] was used to compute the optimal spreading code. In addition to the random-search–type algorithms, we also designed adaptive SA algorithms with both fixed step size and adaptive step sizes to track slowly time-varying optimal spreading codes caused by fading characteristics of the wireless channel. The numerical results in [11, 12] have shown remarkable improvement compared with that of several heuristic algorithms. Section 6 explicitly derives performance bounds in terms of error probabilities for the adaptive discrete stochastic optimization algorithm.

Outline. This paper considers an algorithm with constant step size and updates that are essentially of the form of occupation measures. We are interested in the analysis of tracking errors. First, using perturbed Lyapunov function methods [16], we derive mean squares–type error bounds. The argument is mainly based on stability analysis. Naturally, one then asks whether an associated limit ODE (ordinary differential equation) can be derived via ODE methods as in the usual analysis of SA and stochastic optimization–type algorithms. The standard ODE method cannot be carried over due to the fact that the system is now time-varying, and the adaptation rate is the same as that of the parameter variation. By a combined use of the updated treatment on SA [16] and two-time-scale Markov chains [22, 23], we demonstrate that a limit system can still be obtained. However, very different from the usual stochastic approximation methods in the existing literature, the limit system is no longer a single ODE, but a system of ODEs modulated by a continuous-time Markov chain. Thus, the limit is not deterministic but stochastic. Such systems are referred to as ODEs with regime switching. Based on the system of switching ODEs obtained, we further examine a sequence of suitably normalized errors aiming at understanding the rate of variation (rate of convergence) of the scaled sequence of tracking errors. It is well known that for an SA algorithm, if the true parameter is a fixed constant, then a suitably scaled sequence of estimation errors has a Gaussian diffusion limit. In contrast, somewhat remarkably, the scaled tracking error sequence generated by the SA algorithm in this paper does not have a diffusion limit. Instead, the limit is a system of diffusions with regime switching. In the limit system, the diffusion coefficient depends on the modulating Markov chain, which reveals the distinctive time-varying nature of the underlying system and provides new insight on Markov modulated SA problems.

Context. The main weak convergence results in this paper in sections 4 and 5 assume that the dynamics of the true parameter (modeled as a slow Markov chain with transition probability matrix $I + \varepsilon Q$) evolves on the same time scale as the adaptive SA algorithm with step size μ, i.e., $\varepsilon = O(\mu)$. We note that the case $\varepsilon = O(\mu)$ addressed in this paper is much more difficult to handle than $\varepsilon = o(\mu)$ (e.g., $\varepsilon = O(\mu^2)$), which is widely used in the analysis of tracking algorithms [4]. The meaning of $\varepsilon = o(\mu)$ is that the true parameter evolves much more slowly than the adaptation speed of the stochastic optimization algorithm and is more restrictive than $\varepsilon = O(\mu)$. Furthermore, with $\varepsilon = o(\mu)$ one obtains a standard ODE and linear diffusion limit, whereas with $\varepsilon = O(\mu)$ we show for the first time in this paper that one obtains a randomly switching system of ODEs and switching diffusion limit. Finally, in several applications arising in wireless telecommunication network optimization, e.g., signature code optimization in spread spectrum systems over fading channels [11, 12], the optimal signature sequence (true parameter) changes as quickly as the adaptation of the algorithm, i.e., $\varepsilon = O(\mu)$.

The rest of the paper is organized as follows. Section 2 contains the formulation of the problem. Section 3 is devoted to obtaining mean squares error bounds. In section 4, we obtain a weak convergence result of an interpolated sequence of the iterates. Section 5 further examines a suitably scaled tracking error sequence of the iterates and derives a switching diffusion limit. Section 6 presents an example of an adaptive discrete stochastic optimization algorithm, which is motivated by [11], where such algorithms have been used to perform adaptive spreading code optimization in wireless CDMA systems. The analysis of section 3 and section 5 is used to derive bounds on the error probability of this adaptive discrete stochastic optimization algorithm.

Before proceeding, a bit of notation is in order. Throughout the paper, z' denotes the transpose of $z \in \mathbb{R}^{\ell\times r}$ for some $\ell, r \geq 1$; unless otherwise noted, all vectors are column vectors; $|z|$ denotes the norm of z; K denotes a generic positive constant whose values may vary for different usage (the conventions $K + K = K$ and $KK = K$ will be used without notice).

2. Formulation of the problem.

We will use the following conditions throughout the paper. Condition (M) characterizes the time-varying underlying parameter as a Markov chain with infrequent transitions, while condition (S) characterizes the observed signal.

(M) Let $\{\theta_n\}$ be a discrete-time Markov chain with finite state space

$$\mathcal{M} = \{\overline{\theta}_1, \dots, \overline{\theta}_{m_0}\} \tag{2.1}$$

and transition probability matrix

$$P^\varepsilon = I + \varepsilon Q, \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter, I is an $m_0 \times m_0$ identity matrix, and $Q = (q_{ij}) \in \mathbb{R}^{m_0 \times m_0}$ is a generator of a continuous-time Markov chain (i.e., Q satisfies $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m_0} q_{ij} = 0$ for each $i = 1, \ldots, m_0$). For simplicity, suppose that the initial distribution $P(\theta_0 = \overline{\theta}_i) = p_{0,i}$ is independent of ε for each $i = 1, \ldots, m_0$, where $p_{0,i} \geq 0$ and $\sum_{i=1}^{m_0} p_{0,i} = 1$. Q is irreducible.

(S) Let $\{X_n\}$ be an S-state conditional Markov chain (conditioned on the parameter process). The state space of $\{X_n\}$ is $\mathcal{S} = \{e_1, \ldots, e_S\}$, where e_i for $i = 1, \ldots, S$ denotes the ith standard unit vectors, with the ith component being 1 and the rest of the components being 0. For each $\theta \in \mathcal{M}$, $A(\theta) = (a_{ij}(\theta)) \in \mathbb{R}^{S \times S}$, the transition probability matrix of X_n is defined by

$$a_{ij}(\theta) = P(X_{n+1} = e_j | X_n = e_i, \theta_n = \theta) = P(X_1 = e_j | X_0 = e_i, \theta_0 = \theta),$$

where $i, j \in \{1, \ldots, S\}$. For $\theta \in \mathcal{M}$, $A(\theta)$ is irreducible and aperiodic.

Remark 2.1. Note that the underlying Markov chain $\{\theta_n\}$ is in fact ε-dependent. We suppress the ε-dependence for notational simplicity. The small parameter ε in (2.2) ensures that the entries of the transition probability matrix are nonnegative, since $p^{\varepsilon}_{ij} = \delta_{ij} + \varepsilon q_{ij} \geq 0$ for $\varepsilon > 0$ small enough, where δ_{ij} denotes the Kronecker δ satisfying $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The use of the generator Q makes the row sum of the matrix P be one. The main idea is that, although the true parameter is time-varying, it is piecewise constant. Moreover, due to the dominating identity matrix in (2.2), $\{\theta_n\}$ varies slowly in time. The time-varying parameter takes a constant value $\overline{\theta}_i$ for a random duration and jumps to another state $\overline{\theta}_j$ with $j \neq i$ at a random time.

The assumptions on irreducibility and aperiodicity of $A(\theta)$ imply that for each $\theta \in \mathcal{M}$ there exists a unique stationary distribution $\pi(\theta) \in \mathbb{R}^{S \times 1}$ satisfying

$$\pi'(\theta) = \pi'(\theta) A(\theta) \quad \text{and} \quad \pi'(\theta) \mathbb{1}_S = 1,$$

where $\mathbb{1}_\ell \in \mathbb{R}^{\ell \times 1}$ with all entries being equal to 1. We aim to use an SA algorithm to track the time-varying distribution $\pi(\theta_n)$ that depends on the underlying Markov chain θ_n.

2.1. Adaptive algorithm. We use the following adaptive algorithm of least mean squares (LMS) type with constant step size in order to construct a sequence of estimates $\{\widehat{\pi}_n\}$ of the time-varying distribution $\pi(\theta_n)$,

$$\widehat{\pi}_{n+1} = \widehat{\pi}_n + \mu(X_{n+1} - \widehat{\pi}_n), \tag{2.3}$$

where μ denotes the step size. Define $\widetilde{\pi}_n = \widehat{\pi}_n - \mathbf{E}\pi(\theta_n)$. Then (2.3) can be rewritten as

$$\widetilde{\pi}_{n+1} = \widetilde{\pi}_n - \mu \widetilde{\pi}_n + \mu(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1})). \tag{2.4}$$

Note that $\widehat{\pi}_n$, $\pi(\theta_n)$, and hence $\widetilde{\pi}_n$ are column vectors (i.e., they take values in $\mathbb{R}^{S \times 1}$).

The underlying parameter θ_n is called a *hypermodel* in [4]. Note that while the dynamics of the hypermodel θ_n is used in our analysis, it does not explicitly enter the implementation of the LMS algorithm (2.3).

To accomplish our goal, we derive a mean squares error bound, proceed with the examination of an interpolated sequence of the iterates, and derive a limit result for a scaled sequence. These three steps are realized in the following three sections.

3. Mean square error. This section establishes a mean square estimate for $\mathbf{E}|\widetilde{\pi}_n|^2 = \mathbf{E}|\widehat{\pi}_n - \mathbf{E}\pi(\theta_n)|^2$. Analyzing SA algorithms often requires the use of Lyapunov-type functions for proving stability; see [7, 16]. In what follows, we obtain the desired estimate via a stability argument using the perturbed Lyapunov function method [16]. Use $\mathbf{E}_n$ to denote the conditional expectation with respect to $\mathcal{F}_n$, the σ-algebra generated by $\{X_k, \theta_k : k \le n\}$.

THEOREM 3.1. *Assume* (M) *and* (S). *In addition, suppose that* $\varepsilon^2 \ll \mu$. *Then for sufficiently large* n,

$$\mathbf{E}|\widetilde{\pi}_n|^2 = O\left(\mu + \varepsilon + \frac{\varepsilon^2}{\mu}\right). \tag{3.1}$$

Proof. Define $V(x) = (x'x)/2$. Direct calculations lead to

$$\begin{aligned}\mathbf{E}_n V(\widetilde{\pi}_{n+1}) - V(\widetilde{\pi}_n) &= \mathbf{E}_n\{\widetilde{\pi}_n'[-\mu\widetilde{\pi}_n + \mu(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})]]\} \\ &\quad + \mathbf{E}_n|-\mu\widetilde{\pi}_n + \mu(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})]|^2.\end{aligned} \tag{3.2}$$

In view of the Markovian assumption and the structure of the transition probability matrix given by (2.2),

$$\begin{aligned}\mathbf{E}_n[\pi(\theta_n) - \pi(\theta_{n+1})] &= \mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})|\theta_n] \\ &= \sum_{i=1}^{m_0} \mathbf{E}[\pi(\overline{\theta}_i) - \pi(\theta_{n+1})|\theta_n = \overline{\theta}_i] I_{\{\theta_n = \overline{\theta}_i\}} \\ &= \sum_{i=1}^{m_0}\left[\pi(\overline{\theta}_i) - \sum_{j=1}^{m_0}\pi(\overline{\theta}_j)p_{ij}^{\varepsilon}\right] I_{\{\theta_n=\overline{\theta}_i\}} \\ &= -\varepsilon\sum_{i=1}^{m_0}\sum_{j=1}^{m_0}\pi(\overline{\theta}_j)q_{ij}I_{\{\theta_n=\overline{\theta}_i\}} \\ &= O(\varepsilon),\end{aligned} \tag{3.3}$$

and likewise, detailed computation also shows that

$$\mathbf{E}_n|\pi(\theta_n) - \pi(\theta_{n+1})|^2 = O(\varepsilon). \tag{3.4}$$

Owing to (2.2), the transition probability matrix P^{ε} is independent of time n. As a result, the k-step transition probability depends only on the time lags and can be denoted by $(P^{\varepsilon})^k$. By an elementary inequality, we have $|\widetilde{\pi}_n| = |\widetilde{\pi}_n|\cdot 1 \le (|\widetilde{\pi}_n|^2+1)/2$. Thus,

$$O(\varepsilon)|\widetilde{\pi}_n| \le O(\varepsilon)(V(\widetilde{\pi}_n) + 1).$$

Noting that the sequence of signals $\{X_n\}$ is bounded, the boundedness of $\{\widehat{\pi}_n\}$, and $O(\varepsilon\mu) = O(\mu^2 + \varepsilon^2)$ via the elementary inequality $ab \le (a^2 + b^2)/2$ for any real numbers a and b, the estimate (3.4) yields

$$\begin{aligned}&\mathbf{E}_n|-\mu\widetilde{\pi}_n + \mu(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})]|^2 \\ &\le K\mathbf{E}_n\Big[\mu^2|\widetilde{\pi}_n|^2 + \mu^2|X_{n+1} - \mathbf{E}\pi(\theta_n)|^2 + \mu^2|\widetilde{\pi}_n'\mathbf{E}(X_{n+1} - \mathbf{E}\pi(\theta_n))| \\ &\qquad + \mu|\widetilde{\pi}_n'\mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1}))| + \mu|(X_{n+1} - \mathbf{E}\pi(\theta_n))'\mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1}))|\Big] \\ &\quad + |\mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1}))|^2 \\ &= O(\mu^2 + \varepsilon^2)(V(\widetilde{\pi}_n) + 1) + |\mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1}))|^2\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\mathbf{E}_n\{\widetilde{\pi}_n'[-\mu\widetilde{\pi}_n+\mu(X_{n+1}-\mathbf{E}\pi(\theta_n))+\mathbf{E}(\pi(\theta_n)-\pi(\theta_{n+1}))]\}\\&\quad=-2\mu V(\widetilde{\pi}_n)+\mu\mathbf{E}_n\widetilde{\pi}_n'(X_{n+1}-\mathbf{E}\pi(\theta_n))+\mathbf{E}_n\widetilde{\pi}_n'\mathbf{E}(\pi(\theta_n)-\pi(\theta_{n+1})).\end{aligned}\tag{3.6}$$

Using (3.5) and (3.6) in (3.2) together with (3.3), we obtain

$$\begin{aligned}&\mathbf{E}_nV(\widetilde{\pi}_{n+1})-V(\widetilde{\pi}_n)\\&\quad=-2\mu V(\widetilde{\pi}_n)+\mu\mathbf{E}_n\widetilde{\pi}_n'(X_{n+1}-\mathbf{E}\pi(\theta_n))+\mathbf{E}_n\widetilde{\pi}_n'\mathbf{E}(\pi(\theta_n)-\pi(\theta_{n+1}))\\&\qquad+O(\mu^2+\varepsilon^2)(V(\widetilde{\pi}_n)+1).\end{aligned}\tag{3.7}$$

To obtain the desired estimate, we need to "average out" the second to the fourth terms on the right-hand side of (3.7). To do so, for any $0<T<\infty$, we define the following perturbations:

$$\begin{aligned}V_1^\varepsilon(\widetilde{\pi},n)&=\mu\sum_{j=n}^{T/\varepsilon}\widetilde{\pi}'\mathbf{E}_n(X_{j+1}-\mathbf{E}\pi(\theta_j)),\\V_2^\varepsilon(\widetilde{\pi},n)&=\sum_{j=n}^{T/\varepsilon}\widetilde{\pi}'\mathbf{E}(\pi(\theta_j)-\pi(\theta_{j+1})).\end{aligned}\tag{3.8}$$

In the above and hereafter, T/ε is understood to be $\lfloor T/\varepsilon\rfloor$, i.e., the integer part of T/ε.

Throughout the rest of the paper, we often need to use the notion of fixed-θ processes. For example, by $X_j(\theta)$ for $n\le j\le O(1/\varepsilon)$, we mean a process in which $\theta_j=\theta$ is fixed for all j with $n\le j\le O(1/\varepsilon)$.

For $V_1^\varepsilon(\widetilde{\pi},n)$ defined in (3.8),

$$\begin{aligned}\left|\sum_{j=n}^{T/\varepsilon}\mathbf{E}_n[X_{j+1}-\pi(\theta_j)]\right|&\le\left|\sum_{j=n}^{T/\varepsilon}\mathbf{E}_n[X_{j+1}-\mathbf{E}X_{j+1}]\right|\\&\quad+\left|\sum_{j=n}^{T/\varepsilon}[\mathbf{E}X_{j+1}-\mathbf{E}\pi(\theta_j)]\right|.\end{aligned}\tag{3.9}$$

Using the ϕ-mixing property of $\{X_j\}$ (see [5, p. 166]),

$$\left|\sum_{j=n}^{T/\varepsilon}\mathbf{E}_n[X_{j+1}-\mathbf{E}X_{j+1}]\right|\le K<\infty\qquad\text{uniformly in } n.\tag{3.10}$$

We can also show

$$\left|\sum_{j=n}^{T/\varepsilon}[\mathbf{E}X_{j+1}-\mathbf{E}\pi(\theta_j)]\right|<\infty.\tag{3.11}$$

Thus, using (3.9)–(3.11), for each $\widetilde{\pi}$,

$$|V_1^\varepsilon(\widetilde{\pi},n)|\le O(\mu)(V(\widetilde{\pi})+1).\tag{3.12}$$

By virtue of the definition of $V_2^\varepsilon(\cdot)$ and (2.2), it follows that there exists an N_ε for all $n\ge N_\varepsilon$ such that

$$\begin{aligned}|V_2^\varepsilon(\widetilde{\pi},n)|&=\left|\sum_{j=n}^{T/\varepsilon}\widetilde{\pi}'[\mathbf{E}(\pi(\theta_j)-\pi(\theta_{j+1}))]\right|\\&=\left|\widetilde{\pi}'\mathbf{E}[\pi(\theta_n)-\pi(\theta_{T/\varepsilon})]\right|\\&\le|\widetilde{\pi}|O(\varepsilon)\\&\le O(\varepsilon)(V(\widetilde{\pi})+1).\end{aligned}\tag{3.13}$$

We next show that they result in the desired cancellation in the error estimate. Note that

$$\begin{aligned} &\mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n) \\ &\quad = \mathbf{E}_n[V_1^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n+1)] + \mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_n, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n). \end{aligned} \tag{3.14}$$

It can be seen that

$$\mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_n, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n) = -\mu \mathbf{E}_n \widetilde{\pi}_n'(X_{n+1} - \mathbf{E}\pi(\theta_n)) \tag{3.15}$$

and

(3.16)

$$\begin{aligned} &\mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - \mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_n, n+1) \\ &= \mu \sum_{j=n+1}^{T/\varepsilon} \mathbf{E}_n \widetilde{\pi}_{n+1}' \mathbf{E}_{n+1}(X_{j+1} - \mathbf{E}\pi(\theta_j)) - \mu \sum_{j=n+1}^{T/\varepsilon} \mathbf{E}_n \widetilde{\pi}_n' \mathbf{E}_{n+1}(X_{j+1} - \mathbf{E}\pi(\theta_j)) \\ &= \mu \sum_{j=n+1}^{T/\varepsilon} \mathbf{E}_n(\widetilde{\pi}_{n+1} - \widetilde{\pi}_n)' \mathbf{E}_{n+1}(X_{j+1} - \mathbf{E}\pi(\theta_j)) \\ &= \mu \sum_{j=n+1}^{T/\varepsilon} \mathbf{E}_n[-\mu\widetilde{\pi}_n + \mu(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1}))]' \mathbf{E}_{n+1}[X_{j+1} - \mathbf{E}\pi(\theta_j)] \\ &= O(\mu^2)(V(\widetilde{\pi}_n) + 1) + O(\mu\varepsilon) = O(\mu^2)(V(\widetilde{\pi}_n) + 1) + O(\varepsilon^2). \end{aligned}$$

In the above, we have used $O(\mu\varepsilon) = O(\mu^2 + \varepsilon^2)$, (2.4), and (3.2) to obtain

$$\begin{aligned} |\mathbf{E}_n[\widetilde{\pi}_{n+1} - \widetilde{\pi}_n]| &\le \mu \mathbf{E}_n|\widetilde{\pi}_n| + \mu \mathbf{E}_n|X_{n+1} - \mathbf{E}\pi(\theta_n)| + O(\varepsilon) \\ &= O(\mu)(V(\widetilde{\pi}_n) + 1) + O(\varepsilon). \end{aligned} \tag{3.17}$$

Thus

$$\begin{aligned} &\mathbf{E}_n V_1^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n) \\ &\quad = -\mu \mathbf{E}_n \widetilde{\pi}_n'(X_{n+1} - \mathbf{E}\pi(\theta_n)) + O(\mu^2)(V(\widetilde{\pi}_n) + 1) + O(\varepsilon^2). \end{aligned} \tag{3.18}$$

Analogous estimates yield that

$$\begin{aligned} &\mathbf{E}_n V_2^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - \mathbf{E}_n V_2^\varepsilon(\widetilde{\pi}_n, n+1) \\ &\quad = \sum_{j=n+1}^{T/\varepsilon} \mathbf{E}_n(\widetilde{\pi}_{n+1} - \widetilde{\pi}_n)' \mathbf{E}(\pi(\theta_j) - \pi(\theta_{j+1})) \\ &\quad = O(\mu\varepsilon)(V(\widetilde{\pi}_n) + 1) + O(\varepsilon^2) = O(\varepsilon^2 + \mu^2)(V(\widetilde{\pi}_n) + 1), \end{aligned} \tag{3.19}$$

and that

$$\mathbf{E}_n V_2^\varepsilon(\widetilde{\pi}_n, n+1) - V_2^\varepsilon(\widetilde{\pi}_n, n) = -\widetilde{\pi}_n' \mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1})). \tag{3.20}$$

Thus,

$$\begin{aligned} &\mathbf{E}_n V_2^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_2^\varepsilon(\widetilde{\pi}_n, n) \\ &= -\widetilde{\pi}_n' \mathbf{E}(\pi(\theta_n) - \pi(\theta_{n+1})) + O(\mu^2 + \varepsilon^2)(V(\widetilde{\pi}_n) + 1). \end{aligned} \tag{3.21}$$

Redefine V_1^ε and V_2^ε with T/ε replaced by ∞. Estimates (3.9)–(3.21) still hold.

Define

$$W(\widetilde{\pi}, n) = V(\widetilde{\pi}) + V_1^\varepsilon(\widetilde{\pi}, n) + V_2^\varepsilon(\widetilde{\pi}, n).$$

Then, using the above estimates, we have

$$\begin{aligned} &\mathbf{E}_n W(\widetilde{\pi}_{n+1}, n+1) - W(\widetilde{\pi}_n, n) \\ &\quad = \mathbf{E}_n V(\widetilde{\pi}_{n+1}) - V(\widetilde{\pi}_n) + \mathbf{E}_n [V_1^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\pi}_n, n)] \\ &\qquad + \mathbf{E}_n [V_2^\varepsilon(\widetilde{\pi}_{n+1}, n+1) - V_2^\varepsilon(\widetilde{\pi}_n, n)] \\ &\quad = -2\mu V(\widetilde{\pi}_n) + O(\mu^2 + \varepsilon^2)(V(\widetilde{\pi}_n) + 1). \end{aligned} \tag{3.22}$$

This, together with (3.12) and (3.13) and T/ε replaced by ∞, implies

$$\begin{aligned} &\mathbf{E}_n W(\widetilde{\pi}_{n+1}, n+1) - W(\widetilde{\pi}_n, n) \\ &\quad \le -2\mu W(\widetilde{\pi}_n, n) + O(\mu^2 + \varepsilon^2)(W(\widetilde{\pi}_n, n) + 1). \end{aligned} \tag{3.23}$$

Choose μ and ε small enough so that there is a $\lambda > 0$ satisfying

$$-2\mu + O(\varepsilon^2) + O(\mu^2) \le -\lambda\mu.$$

Then, we get

$$\mathbf{E}_n W(\widetilde{\pi}_{n+1}, n+1) \le (1 - \lambda\mu) W(\widetilde{\pi}_n, n) + O(\mu^2 + \varepsilon^2). \tag{3.24}$$

Taking the expectation and iterating on the resulting inequality yields

$$\begin{aligned} \mathbf{E} W(\widetilde{\pi}_{n+1}, n+1) &\le (1 - \lambda\mu)^{n - N_\varepsilon} \mathbf{E} W(\widetilde{\pi}_0, 0) + \sum_{j=N_\varepsilon}^{n} (1 - \lambda\mu)^{j - N_\varepsilon} O(\mu^2 + \varepsilon^2) \\ &\le (1 - \lambda\mu)^{n - N_\varepsilon} \mathbf{E} W(\widetilde{\pi}_0, 0) + O\left(\mu + \frac{\varepsilon^2}{\mu}\right). \end{aligned} \tag{3.25}$$

By taking n large enough, we can make $(1 - \lambda\mu)^{n - N_\varepsilon} = O(\mu)$. Then

$$\mathbf{E} W(\widetilde{\pi}_{n+1}, n+1) \le O\left(\mu + \frac{\varepsilon^2}{\mu}\right). \tag{3.26}$$

Finally, applying (3.12) and (3.13) again, replacing $W(\widetilde{\pi}, n)$ by $V(\widetilde{\pi})$ adds another $O(\varepsilon)$ term. Thus we obtain

$$\mathbf{E} V(\widetilde{\pi}_{n+1}) \le O\left(\mu + \varepsilon + \frac{\varepsilon^2}{\mu}\right). \tag{3.27}$$

This concludes the proof. □

Remark 3.2. In view of Theorem 3.1, in order that our adaptive algorithm can track the time-varying parameter, the ratio ε/μ must not be large. Given the order-of-magnitude estimate $O(\mu + \varepsilon + \varepsilon^2/\mu)$, to balance the two terms μ and ε^2/μ, we need to choose $\varepsilon = O(\mu)$. Therefore, we obtain the following result.

COROLLARY 3.3. *Under the conditions of Theorem* 3.1, *if* $\varepsilon = O(\mu)$, *then for sufficiently large* n, $\mathbf{E}|\widetilde{\pi}_n|^2 = O(\mu)$.

4. Limit system of regime switching ODEs. Our objective in this section is to derive a limit system for an interpolated sequence of the iterates. Different from the usual approach of stochastic approximation [4], where $\varepsilon = o(\mu)$, here and henceforth, we take $\varepsilon = O(\mu)$. For notational simplicity, however, we use $\varepsilon = \mu$. For $0 < T < \infty$, we construct a sequence of piecewise constant interpolation of the stochastic approximation iterates $\widehat{\pi}_n$ as

$$\widehat{\pi}^\mu(t) = \widehat{\pi}_n, \qquad t \in [\mu n, \mu n + \mu). \tag{4.1}$$

The process $\widehat{\pi}^\mu(\cdot)$ so defined is in $D([0,T];\mathbb{R}^S)$, which is the space of functions defined on $[0,T]$ taking values in $\mathbb{R}^S$ that are right continuous, have left limits, and are endowed with the Skorohod topology. We use weak convergence methods to carry out the analysis. The application of weak convergence ideas usually requires proof of tightness and the characterization of the limit processes. Different from the usual approach of stochastic approximation, the limit is not a deterministic ODE but rather a system of ODEs modulated by a continuous-time Markov chain.

LEMMA 4.1. *Under conditions* (M) *and* (S), $\{\pi^\mu(\cdot)\}$ *is tight in* $D([0,T];\mathbb{R}^S)$.

Proof. By using the tightness criteria [14, p. 47], it suffices to verify that for any $\delta > 0$ and $0 < s \le \delta$,

$$\lim_{\delta\to 0}\limsup_{\mu\to 0} \mathbf{E}|\widehat{\pi}^\mu(t+s) - \widehat{\pi}^\mu(t)|^2 = 0. \tag{4.2}$$

To begin, note that

$$\begin{aligned}\widehat{\pi}^\mu(t+s) - \widehat{\pi}^\mu(t) &= \widehat{\pi}_{(t+s)/\mu} - \widehat{\pi}_{t/\mu}\\ &= \mu \sum_{k=t/\mu}^{(t+s)/\mu - 1} (X_{k+1} - \widehat{\pi}_k).\end{aligned} \tag{4.3}$$

Note also that both the iterates and the observations are bounded uniformly. Then the boundedness of $\{X_k\}$ and $\{\widehat{\pi}_k\}$ implies that

$$\begin{aligned}&\mathbf{E}|\widehat{\pi}^\mu(t+s) - \widehat{\pi}^\mu(t)|^2\\ &\quad= \mathbf{E}\left[\mu \sum_{k=t/\mu}^{(t+s)/\mu-1} (X_{k+1} - \widehat{\pi}_k)'\right]\left[\mu \sum_{j=t/\mu}^{(t+s)/\mu-1} (X_{j+1} - \widehat{\pi}_j)\right]\\ &\quad= \mu^2 \sum_{k=t/\mu}^{(t+s)/\mu-1} \sum_{j=t/\mu}^{(t+s)/\mu-1} \mathbf{E}(X_{k+1} - \widehat{\pi}_k)'(X_{j+1} - \widehat{\pi}_j)\\ &\quad\le K\mu^2 \left(\frac{t+s}{\mu} - \frac{t}{\mu}\right)^2\\ &\quad= K((t+s)-t)^2 = O(s^2).\end{aligned} \tag{4.4}$$

Taking $\limsup_{\mu\to 0}$ and then $\lim_{\delta\to 0}$ in (4.4), equation (4.2) is verified, and so the desired tightness follows. □

4.1. Limit of the modulating Markov chain. Consider the Markov chain θ_n. Regarding the probability vector and the n-step transition probability matrix, we have the following approximation results.

LEMMA 4.2. *Suppose that α_n^η is a Markov chain with a finite state space $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l$ and transition probability matrix*

$$P^\eta = \mathrm{diag}(P^1, \dots, P^l) + \eta Q, \tag{4.5}$$

where for each i, P^i is a transition probability matrix that is irreducible and aperiodic, and Q is a generator of a continuous-time Markov chain. For simplicity, denote $\mathcal{M} = \{1, \dots, m_0\}$, $p_n^\eta = (P(\alpha_n^\eta = 1), \dots, P(\alpha_n^\eta = m_0))$ with $p_0^\eta = p_0$, and the stationary distribution of P^i by ν^i (a row vector) for $i = 1, \dots, l$. Then for some $k_0 > 0$,

$$p_n^\eta = \mathrm{diag}(\nu^1, \dots, \nu^l) z(t) + O\left(\eta + \exp\left(\frac{-k_0 t}{\eta}\right)\right), \tag{4.6}$$

where $z(t) \in \mathbb{R}^{1\times l}$ (with $t = \eta n$) satisfies

$$\frac{dz(t)}{dt} = z(t)\overline{Q}, \;\; z(0) = p_0 \, \mathrm{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l}),$$

with

$$\overline{Q} = \mathrm{diag}(\nu^1, \dots, \nu^l) Q \, \mathrm{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l}). \tag{4.7}$$

In addition, for $n \le O(1/\eta)$, the n-step transition probability matrix satisfies (with $t = \eta n$),

$$(P^\eta)^n = \Xi(t) + O\left(\eta + \exp\left(\frac{-k_0 t}{\eta}\right)\right), \tag{4.8}$$

where

$$\begin{aligned} &\Xi(t) = \mathrm{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l}) \Theta(t) \, \mathrm{diag}(\nu^1, \dots, \nu^l), \\ &\frac{d\Theta(t)}{dt} = \Theta(t)\overline{Q}, \;\; \Theta(0) = I. \end{aligned} \tag{4.9}$$

Proof. The proof is that of Theorems 3.5 and 4.3 of [23]. ◻

LEMMA 4.3. *Suppose that α_n^η is the Markov chain given in Lemma 4.2. Define an aggregated process $\overline{\alpha}_n^\eta = i$ if $\alpha_n^\eta \in \mathcal{M}_i$, and define an interpolated process $\overline{\alpha}^\eta(\cdot)$ by $\overline{\alpha}^\eta(t) = \overline{\alpha}_n^\eta$ if $t \in [n\eta, n\eta + \eta)$. Then $\overline{\alpha}^\eta(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, which is a continuous-time Markov chain generated by $\overline{Q}$ given in* (4.7).

Proof. The proof of this result can be found in [24].

With the above two lemmas, we can now derive a result that will be used in the subsequent analysis. The proof is essentially an application of the above lemmas.

PROPOSITION 4.4. *Assume* (M). *Choose $\varepsilon = \mu$ and consider the Markov chain θ_n. Then the following assertions hold:*

- *Denote $p_n^\mu = (P(\theta_n = \overline{\theta}_1), \dots, P(\theta_n = \overline{\theta}_{m_0}))$. Then*

$$\begin{aligned} &p_n^\mu = z(t) + O\left(\mu + \exp\left(\frac{-k_0 t}{\mu}\right)\right), \qquad z(t) \in \mathbb{R}^{1\times m_0}, \\ &\frac{dz(t)}{dt} = z(t)Q, \qquad z(0) = p_0, \\ &(P^\mu)^n = Z(t) + O\left(\mu + \exp\left(\frac{-k_0 t}{\mu}\right)\right), \\ &\frac{dZ(t)}{dt} = Z(t)Q, \qquad Z(0) = I. \end{aligned} \tag{4.10}$$

- *Define the continuous-time interpolation of θ_n^μ by $\theta^\mu(t) = \theta_n$ if $t \in [n\mu, n\mu + \mu)$. Then $\theta^\mu(\cdot)$ converges weakly to $\theta(\cdot)$, which is a continuous-time Markov chain generated by Q.*

Proof. Observe that the identity matrix in (2.2) can be written as

$$I = \text{diag}(1, \dots, 1) \in \mathbb{R}^{m_0 \times m_0}.$$

Each of the 1's can be thought of as a 1×1 "transition matrix." Note that under the conditions for the Markov chain θ_n, the $\text{diag}(\nu^1, \dots, \nu^l)$ defined in (4.7) becomes $I \in \mathbb{R}^{m_0 \times m_0}$, and $\text{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l})$ in (4.7) is also I. Moreover, the $\overline{Q}$ defined in (4.7) is now simply Q. Straightforward applications of Lemmas 4.2 and 4.3 then yield the desired results. □

4.2. Characterization of the limit. Consider the pair $(\widehat{\pi}^\mu(\cdot), \theta^\mu(\cdot))$. Then $\{\widehat{\pi}^\mu(\cdot), \theta^\mu(\cdot)\}$ is tight in $D([0, T]; \mathbb{R}^S \times \mathcal{M})$ for $T > 0$ by virtue of Proposition 4.4 and Lemma 4.1 together with the Cramér–Wold device [5, p. 48]. By virtue of Prohorov's theorem, we can extract convergent subsequences. Do that, and still index the subsequence by μ for notational simplicity. Denote the limit by $\widehat{\pi}(\cdot)$. By virtue of the Skorohod representation, $\widehat{\pi}^\mu(\cdot)$ converges to $\widehat{\pi}(\cdot)$ w.p.1, and the convergence is uniform on any compact set. We proceed to characterize the limit $\widehat{\pi}(\cdot)$. The result is stated in the following theorem.

THEOREM 4.5. *Under conditions* (M) *and* (S), *$(\widehat{\pi}^\mu(\cdot), \theta^\mu(\cdot))$ converges weakly to $(\widehat{\pi}(\cdot), \theta(\cdot))$, which is a solution of the following switching ODE:*

$$\frac{d}{dt}\widehat{\pi}(t) = \pi(\theta(t)) - \widehat{\pi}(t), \qquad \widehat{\pi}(0) = \widehat{\pi}_0. \tag{4.11}$$

Remark 4.6. The above switching ODE displays a very different behavior than the trajectories of systems derived from the classical ODE approach for SA. It involves a random element since $\theta(t)$ is a continuous-time Markov chain with generator Q. Because of the regime switching, the system is qualitatively different from the existing literature on SA methods. To analyze SA algorithms, the ODE methods (see [15, 16] and [17]) are now standard and widely used in various applications. The rationale is that the discrete iterations are compared with the continuous dynamics given by a limit ODE. The ODE is then used to analyze the asymptotic properties of the recursive algorithms. Dealing with tracking algorithms having time-varying features, sometimes, one may obtain a nonautonomous differential equation [16, section 8.2.6], but the systems are still purely deterministic. Unlike those mentioned above, the limit dynamic system in Theorem 4.5 is only piecewise deterministic due to the underlying Markov chain. In lieu of one ODE, we have a number of ODEs modulated by a continuous-time Markov chain. At any given instance, the Markov chain dictates which regime the system belongs to, and the corresponding system then follows one of the ODEs until the modulating Markov chain jumps into a new location, which explains the time-varying and regime switching nature of the systems under consideration.

Proof. To obtain the desired limit, we prove that the limit $(\widehat{\pi}(\cdot), \theta(\cdot))$ is the solution of the martingale problem with operator L_1 given by

$$L_1 f(x, \overline{\theta}_i) = \nabla f'(x, \overline{\theta}_i)(\pi(\overline{\theta}_i) - x) + Qf(x, \cdot)(\overline{\theta}_i) \quad \text{for each } \overline{\theta}_i \in \mathcal{M}, \tag{4.12}$$

where

$$Qf(x, \cdot)(\overline{\theta}_i) = \sum_{j \in \mathcal{M}} q_{ij} f(x, \overline{\theta}_j) = \sum_{j \neq i} q_{ij}[f(x, \overline{\theta}_j) - f(x, \overline{\theta}_i)] \quad \text{for each } \overline{\theta}_i \in \mathcal{M},$$

and for each $\overline{\theta}_i \in \mathcal{M}$, $f(\cdot,\overline{\theta}_i)$ is twice continuously differentiable with compact support. In the above, $\nabla f(x,\overline{\theta}_i)$ denotes the gradient of $f(x,\overline{\theta}_i)$ with respect to x. Using an argument as in [22, Lemma 7.18], it can be shown that the martingale problem associated with the operator L_1 has a unique solution. Thus, it remains to show that the limit $(\widehat{\pi}(\cdot),\theta(\cdot))$ is the solution of the martingale problem. To this end, we need only show that for any positive integer ℓ_0, any $t>0$, $s>0$, and $0<t_j\le t$, and any bounded and continuous function $h_j(\cdot,\overline{\theta}_i)$ for each $\overline{\theta}_i \in \mathcal{M}$ with $j\le \ell_0$,

$$\begin{aligned}
&\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}(t_j),\theta(t_j)) \\
&\quad\times\left[f(\widehat{\pi}(t+s),\theta(t+s))-f(\widehat{\pi}(t),\theta(t))-\int_t^{t+s} L_1 f(\widehat{\pi}(u),\theta(u))du\right]=0.
\end{aligned}\tag{4.13}$$

To verify (4.13), we work with the processes indexed by μ and prove that the above equation holds as $\mu\to 0$.

First by the weak convergence of $(\widehat{\pi}^\mu(\cdot),\theta^\mu(\cdot))$ to $(\widehat{\pi}(\cdot),\theta(\cdot))$ and the Skorohod representation,

$$\begin{aligned}
&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[f(\widehat{\pi}^\mu(t+s),\theta^\mu(t+s))-f(\widehat{\pi}^\mu(t),\theta^\mu(t))\right] \\
&\quad=\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}(t_j),\theta(t_j))\left[f(\widehat{\pi}(t+s),\theta(t+s))-f(\widehat{\pi}(t),\theta(t))\right].
\end{aligned}\tag{4.14}$$

On the other hand, choose a sequence n_μ such that $n_\mu\to\infty$ as $\mu\to 0$, but $\mu n_\mu\to 0$. Divide $[t,\, t+s]$ into intervals of width $\delta_\mu=\mu n_\mu$. We have

$$\begin{aligned}
&\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[f(\widehat{\pi}^\mu(t+s),\theta^\mu(t+s))-f(\widehat{\pi}^\mu(t),\theta^\mu(t))\right] \\
&\quad=\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\left[f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu+n_\mu})-f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu})\right]\right. \\
&\qquad\left.+\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\left[f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu})-f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu})\right]\right].
\end{aligned}\tag{4.15}$$

By virtue of the smoothness and boundedness of $f(\cdot,\theta)$, it can be seen that

$$\begin{aligned}
&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\left[f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu+n_\mu})-f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu})\right]\right] \\
&\quad=\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\left[f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu+n_\mu})-f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu})\right]\right].
\end{aligned}\tag{4.16}$$

Thus we need only work with the latter term. Moreover, letting $\mu \to 0$ and $l\delta_\mu = \mu l n_\mu \to u$ and using nested expectation, we can insert $\mathbf{E}_k$ and obtain

(4.17)
$$\begin{aligned}
&\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} [f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu+n_\mu}) - f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu})]\right] \\
&\quad = \mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \Bigg[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \sum_{j=1}^{m_0}\sum_{i=1}^{m_0}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} [f(\widehat{\pi}_{ln_\mu},\overline{\theta}_i) \\
&\qquad\qquad \times P(\theta_{k+1}=\overline{\theta}_i \big| \theta_k=\overline{\theta}_j) - f(\widehat{\pi}_{ln_\mu},\overline{\theta}_j)] I_{\{\theta_k=\overline{\theta}_j)\}}\Bigg] \\
&\quad = \mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \left[\frac{\delta_\mu}{n_\mu}\sum_{j=1}^{m_0}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} Qf(\widehat{\pi}_{ln_\mu},\cdot)(\theta_k) I_{\{\theta_k=\overline{\theta}_j\}}\right]\right] \\
&\quad \to \mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}(t_j),\theta(t_j)) \left[\int_t^{t+s} Qf(\widehat{\pi}(u),\theta(u))du\right] \quad \text{as } \mu \to 0.
\end{aligned}$$

Since $\widehat{\pi}^\mu_{ln_\mu}$ and θ_{ln_μ} are $\mathcal{F}_{ln_\mu}$-measurable, by virtue of the continuity and boundedness of $\nabla f(\cdot,\theta)$,

$$\begin{aligned}
&\mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} [f(\widehat{\pi}_{ln_\mu+n_\mu},\theta_{ln_\mu}) - f(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu})] \\
&\quad = \mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \left[\mu\nabla f'(\widehat{\pi}_{ln_\mu},\theta_{ln_\mu}) \sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu}(X_{k+1}-\widehat{\pi}_k)\right] \\
&\qquad + o(1),
\end{aligned}$$

where $o(1) \to 0$ as $\mu \to 0$. Next, consider the term

$$\lim_{\mu\to 0} \mathbf{E}\prod_{j=1}^{\ell_0} h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j)) \left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \delta_\mu \left[\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu} X_{k+1}\right]\right]. \tag{4.18}$$

Consider a fixed-θ process $X_k(\theta)$, which is a process with θ_k fixed at $\theta_k = \theta$ for $ln_\mu \le k \le O(1/\mu)$. Close scrutiny of the inner summation shows that

$$\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu} X_{k+1} \text{ can be approximated by } \frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu} X_{k+1}(\theta) \tag{4.19}$$

with an approximation error going to 0, since, $E_{ln_\mu}[X_{k+1} - X_{k+1}(\theta)] = O(\varepsilon) = O(\mu)$

by use of the transition matrix (2.2). Thus we have

$$\begin{aligned}&\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\mathbf{E}_{ln_\mu}X_{k+1}\\&=\sum_{j=1}^{m_0}\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\mathbf{E}\left(X_{k+1}(\overline{\theta}_j)I_{\{\theta_{ln_\mu}=\overline{\theta}_j\}}\big|\theta_{ln_\mu}=\overline{\theta}_j\right)+o(1)\\&=\sum_{j=1}^{m_0}\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\sum_{j_1=1}^{\mathcal{S}}e_{j_1}[A(\overline{\theta}_j)]^{k+1-ln_\mu}I_{\{\theta_{ln_\mu}=\overline{\theta}_j\}}+o(1),\end{aligned}$$

where $o(1)\to 0$ in probability as $\mu\to 0$. Henceforth, we write $\mathbb{1}$ in lieu of $\mathbb{1}_S$. Note that for each $j=1,\ldots,S$, as $n_\mu\to\infty$ (recall that $\delta_\mu=\mu n_\mu$),

$$\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}[A(\overline{\theta}_j)]^{k+1-ln_\mu}\to\mathbb{1}\pi'(\overline{\theta}_j).$$

Note that $I_{\{\theta_{ln_\mu}=\overline{\theta}_j\}}$ can be written as $I_{\{\theta^\mu(l\delta_\mu)=\overline{\theta}_j\}}$. As $\mu\to 0$ and $l\delta_\mu\to u$, by the weak convergence of $\theta^\mu(\cdot)$ to $\theta(\cdot)$ and the Skorohod representation, $I_{\{\theta^\mu(\mu ln_\mu)=\overline{\theta}_j\}}\to I_{\{\theta(u)=\overline{\theta}_j\}}$ w.p.1. Consequently, since $\mathbb{1}\pi'(\overline{\theta}_j)$ has identical rows,

$$\begin{aligned}\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\mathbf{E}_{ln_\mu}X_{k+1}&\to\sum_{j=1}^{m_0}\pi(\overline{\theta}_j)I_{\{\theta(u)=\overline{\theta}_j\}}\\&=\pi(\theta(u)).\end{aligned}\tag{4.20}$$

That is, the limit does not depend on the value of initial state, a salient feature of Markov chains. As a result,

$$\begin{aligned}&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0}h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\mathbf{E}_{ln_\mu}X_{k+1}\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0}h_j(\widehat{\pi}(t_j),\theta(t_j))\left[\sum_{j=1}^{m_0}\int_t^{t+s}\pi(\overline{\theta}_j)I_{\{\theta(u)=\overline{\theta}_j\}}du\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0}h_j(\widehat{\pi}(t_j),\theta(t_j))\left[\int_t^{t+s}\pi(\theta(u))du\right].\end{aligned}\tag{4.21}$$

Likewise, it can be shown that, as $\mu\to 0$,

$$\begin{aligned}&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0}h_j(\widehat{\pi}^\mu(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\delta_\mu\frac{1}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\widehat{\pi}_k\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0}h_j(\widehat{\pi}(t_j),\theta(t_j))\left[\int_t^{t+s}\widehat{\pi}(u)du\right].\end{aligned}\tag{4.22}$$

Combining (4.14), (4.17), (4.21), and (4.22), the desired result follows. □

5. Switching diffusion limit. By Theorem 3.1, $\{\frac{\hat{\pi}_n - \mathbf{E}\pi(\theta_n)}{\sqrt{\mu}}\}$ is tight for $n \geq n_0$, for some positive integer n_0. In an effort to evaluate the rate of variation of the tracking error sequence, we define a scaled sequence of the tracking errors $\{v_n\}$ and its continuous-time interpolation $v^\mu(\cdot)$ by

$$v_n = \frac{\widehat{\pi}_n - \mathbf{E}\{\pi(\theta_n)\}}{\sqrt{\mu}}, \quad n \geq n_0, \quad v^\mu(t) = v_n \ \text{ for } \ t \in [n\mu, n\mu + \mu). \tag{5.1}$$

We will derive a limit process for $v^\mu(\cdot)$ as $\mu \to 0$. Similarly to the rate of convergence study when θ is a fixed parameter (see [16, Chapter 10]), the scaling factor $\sqrt{\mu}$, together with the asymptotic covariance of the limit process, gives us a "rate of convergence" result.

Note that from Proposition 4.4

$$\mathbf{E}\{\pi(\theta_n)\} = \bar{\pi}(\mu n) + O(\mu + \exp(-k_0 n)), \quad \text{where } \bar{\pi}(\mu n) \stackrel{\text{def}}{=} \sum_{i=1}^{S} z^i(\mu n)\pi(\overline{\theta}_i), \tag{5.2}$$

where $z^i(t)$ is the ith component of $z(t)$ given in Proposition 4.4. By (M), $\{\theta_n\}$ is a Markov chain with stationary (time-invariant) transition probabilities, so in view of (2.3),

$$v_{n+1} = v_n - \mu v_n + \sqrt{\mu}(X_{n+1} - \mathbf{E}\{\pi(\theta_n)\}) + \frac{\mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})]}{\sqrt{\mu}}. \tag{5.3}$$

Our task in what follows is to figure out the asymptotic properties of $v^\mu(\cdot)$. We aim to show that the limit is a switching diffusion using a martingale problem formulation.

5.1. Truncation and tightness. Owing to the definition (5.1), $\{v_n\}$ is not a priori bounded. A convenient way to circumvent this difficulty is to use a truncation device [16]. Let $N > 0$ be a fixed but otherwise arbitrary real number, $S_N(z) = \{z \in \mathbb{R}^S : |z| \leq N\}$ be the sphere with radius N, and $\tau^N(z)$ be a smooth function satisfying

$$\tau^N(z) = \begin{cases} 1 & \text{if } |z| \leq N, \\ 0 & \text{if } |z| \geq N+1. \end{cases}$$

Note that $\tau^N(z)$ is "smoothly" connected between the sphere S_N and S_{N+1}. Now define

$$v^N_{n+1} = v^N_n - \mu v^N_n \tau^N(v^N_n) + \sqrt{\mu}(X_{n+1} - \mathbf{E}\pi(\theta_n)) + \frac{\mathbf{E}[\pi(\theta_n) - \pi(\theta_{n+1})]}{\sqrt{\mu}}, \tag{5.4}$$

and define $v^{\mu,N}(\cdot)$ to be the continuous-time interpolation of v^N_n. It then follows that

$$\lim_{k_0 \to \infty} \limsup_{\mu \to 0} P\left(\sup_{0 \leq t \leq T} |v^{\mu,N}(t)| \geq k_0 \right) = 0 \ \text{ for each } \ T < \infty$$

and that $v^{\mu,N}(\cdot)$ is a process that is equal to $v^\mu(\cdot)$ up until the first exit from S_N, and hence an N-truncation process of $v^\mu(\cdot)$ [16, p. 284]. To proceed, we work with $\{v^{\mu,N}(\cdot)\}$ and derive its tightness and weak convergence first. Finally, we let $N \to \infty$ to conclude the proof.

LEMMA 5.1. *Under conditions* (M) *and* (S), $\{v^{\mu,N}(\cdot)\}$ *is tight in* $D(S[0,T];\mathbb{R}^S)$, *and the process* $\{v^{\mu,N}(\cdot),\theta^\mu(\cdot)\}$ *is tight in* $D([0,T];\mathbb{R}^S\times\mathcal{M})$.

Proof. In fact, only the first assertion needs to be verified. In view of (5.4), for any $\delta>0$ and $t,s\geq 0$ with $s\leq\delta$,

(5.5)
$$v^{\mu,N}(t+s)-v^{\mu,N}(t)=-\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}v_k^N\tau^N(v_k^N)+\sqrt{\mu}\sum_{k=t/\mu}^{(t+s)/\mu-1}(X_{k+1}-\mathbf{E}\pi(\theta_k))$$
$$+\frac{1}{\sqrt{\mu}}\sum_{k=t/\mu}^{(t+s)/\mu-1}\mathbf{E}(\pi(\theta_k)-\pi(\theta_{k+1})).$$

Owing to the N-truncation used,

$$\left|\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}v_k^N\tau^N(v_k^N)\right|\leq Ks,$$

and as a result,

$$\lim_{\delta\to 0}\limsup_{\mu\to 0}\mathbf{E}\left|\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}v_k^N\tau^N(v_k^N)\right|^2=0. \tag{5.6}$$

Next, by virtue of (M), the irreducibility of the conditional Markov chain $\{X_n\}$ implies that it is ϕ-mixing with exponential mixing rate [5, p. 167], $\mathbf{E}\pi(\theta_k)-\mathbf{E}X_{k+1}\to 0$ exponentially fast, and consequently

$$\mathbf{E}\left|\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}(X_{k+1}-\mathbf{E}\pi(\theta_k))\right|^2$$
$$=\mathbf{E}\left|\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}[(X_{k+1}-\mathbf{E}X_{k+1})-(\mathbf{E}\pi(\theta_k)-\mathbf{E}X_{k+1})]\right|^2=O(s).$$

This yields that

$$\lim_{\delta\to 0}\limsup_{\mu\to 0}\mathbf{E}\left|\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}(X_{k+1}-\mathbf{E}\pi(\theta_k))\right|^2=0. \tag{5.7}$$

In addition,

$$\frac{1}{\sqrt{\mu}}\sum_{k=t/\mu}^{(t+s)/\mu-1}\mathbf{E}(\pi(\theta_k)-\pi(\theta_{k+1}))=\frac{1}{\sqrt{\mu}}[\mathbf{E}\pi(\theta_{t/\mu})-\mathbf{E}\pi(\theta_{(t+s)/\mu})]=O(\sqrt{\mu}). \tag{5.8}$$

Combining (5.5)–(5.8), we have

$$\lim_{\delta\to 0}\limsup_{\mu\to 0}\mathbf{E}\left|v^{\mu,N}(t+s)-v^{\mu,N}(t)\right|^2=0,$$

and hence the criterion [14, p. 47] implies that $\{v^{\mu,N}(\cdot)\}$ is tight. ∎

5.2. Representation of covariance. The main results to follow, Lemma 5.4 and Corollary 5.5 for the diffusion limit in section 5.3, require representation of the covariance of the conditional Markov chain $\{X_k\}$. This is again worked out via the use of fixed-θ process $X_k(\theta)$ similar in spirit to (4.19). For any integer $m \geq 0$, for $m \leq k \leq O(1/\mu)$, with θ_k fixed at θ, $X_{k+1}(\theta)$ is a finite-state Markov chain with 1-step irreducible transition matrix $A(\theta)$ and stationary distribution $\pi(\theta)$. Thus [5, p. 167] implies that $\{X_{k+1}(\theta) - \mathbf{E}X_{k+1}(\theta)\}$ is a ϕ-mixing sequence with zero mean and exponential mixing rate, and hence it is strongly ergodic. Similarly to (4.19), $X_{k+1} - \mathbf{E}X_{k+1}$ can be approximated by a fixed θ process $X_{k+1}(\theta) - \mathbf{E}X_{k+1}(\theta)$. Taking $n = n_\mu \leq O(1/\mu)$ as $\mu \to 0$, $n \to \infty$, and

(5.9)
$$\lim_{\mu\to 0} \frac{1}{n} \sum_{k_1=m}^{n+m-1} \sum_{k=m}^{n+m-1} (X_{k+1}(\theta) - \mathbf{E}X_{k+1}(\theta))(X_{k_1+1}(\theta) - \mathbf{E}X_{k_1+1}(\theta))' = \Sigma(\theta) \quad \text{w.p.1},$$

where $\Sigma(\theta)$ is an $S \times S$ deterministic matrix and

(5.10)
$$\lim_{\mu\to 0} \frac{1}{n} \sum_{k_1=m}^{n+m-1} \sum_{k=m}^{n+m-1} \mathbf{E}\left\{(X_{k+1}(\theta) - \mathbf{E}X_{k+1}(\theta))(X_{k_1+1}(\theta) - \mathbf{E}X_{k_1+1}(\theta))'\right\} = \Sigma(\theta).$$

Note that (5.9) is a consequence of ϕ-mixing and strong ergodicity, and (5.10) follows from (5.9) by means of the dominated convergence theorem. Clearly, $\Sigma(\theta)$ is symmetric and nonnegative definite. The following lemma gives an explicit formula for $\Sigma(\theta)$ in terms of $\pi(\theta)$ and $A(\theta)$ and is useful for computational purposes.

LEMMA 5.2. *The covariance matrix* $\Sigma(\theta)$ *in* (5.10) *can be explicitly computed as*

$$\Sigma(\theta) = Z'(\theta)D(\theta) + D(\theta)Z(\theta) - D(\theta) - \pi(\theta)\pi'(\theta), \tag{5.11}$$

where $D(\theta) = \operatorname{diag}(\pi_1(\theta), \ldots, \pi_{m_0}(\theta))$ *and* $Z(\theta)$ *is given by*

$$Z(\theta) = (I - A(\theta) + \mathbb{1}\pi'(\theta))^{-1}.$$

Remark 5.3. The $Z(\theta)$ is termed the "fundamental" matrix [6, p. 226]. As shown in the aforementioned reference, because $A(\theta)$ is irreducible, $Z(\theta)$ is nonsingular.

Proof. Note that $\Sigma(\theta) = \lim_{\mu\to 0} \Sigma^\mu(\theta)$, where $\Sigma^\mu(\theta)$ can be expressed in terms of $\pi(\theta)$ as

$$\begin{aligned} \Sigma^\mu(\theta) &= \mathbf{E}\xi_0(\theta)\xi_0'(\theta) + \sum_{k=-\lfloor 1/\mu\rfloor}^{-1} \mathbf{E}\xi_k(\theta)\xi_0'(\theta) + \sum_{k=1}^{\lfloor 1/\mu\rfloor} \mathbf{E}\xi_k(\theta)\xi_0'(\theta), \\ \xi_k(\theta) &\stackrel{\text{def}}{=} X_k(\theta) - \pi(\theta), \end{aligned} \tag{5.12}$$

and $\{X_k(\theta)\}$ is a fixed-θ Markov chain with $\theta_{-\lfloor 1/\mu\rfloor} = \theta$ and $\theta_k = \theta$ for all integer $k \leq O(1/\mu)$. Consider the terms in the above equation. For $0 < k \leq O(1/\mu)$,

$$\mathbf{E}\xi_k(\theta)\xi_0'(\theta) = \mathbf{E}X_k(\theta)X_0'(\theta) - \pi(\theta)\pi'(\theta) = (A^k(\theta))'\mathbf{E}\{X_0(\theta)X_0'(\theta)\} - \pi(\theta)\pi'(\theta).$$

Since $\{X_k(\theta)\}$ is geometrically ergodic and starts at $k = -\lfloor 1/\mu\rfloor$, $X_0(\theta)$ has distribution $\pi(\theta)$, so $\mathbf{E}\{X_0(\theta)X_0'(\theta)\} = D(\theta)$. Then using the fact that $\pi(\theta) = D(\theta)\mathbb{1}$, it follows that $\mathbf{E}\xi_k(\theta)\xi_0'(\theta) = (A^k(\theta) - \mathbb{1}\pi'(\theta))'D(\theta)$. Thus it is easily checked that

$$\lim_{\mu\to 0}\sum_{k=1}^{\lfloor 1/\mu\rfloor}\mathbf{E}\xi_k(\theta)\xi_0'(\theta) = \lim_{\mu\to 0}\sum_{k=1}^{\lfloor 1/\mu\rfloor}\left(A^k(\theta) - \mathbb{1}\pi'(\theta)\right)' D(\theta) = (Z(\theta)-I)'D(\theta); \tag{5.13}$$

see also [6, p. 226], where it was shown that $\lim_{\mu\to 0}\sum_{k=1}^{\lfloor 1/\mu\rfloor}(A^k(\theta)(\theta) - \mathbb{1}\pi'(\theta)) = Z(\theta) - I$. Similarly,

$$\begin{aligned}&\lim_{\mu\to 0}\sum_{k=-\lfloor 1/\mu\rfloor}^{-1}\mathbf{E}\xi_k(\theta)\xi_0'(\theta) = D(\theta)(Z(\theta)-I),\\ &\mathbf{E}\xi_0(\theta)\xi_0'(\theta) = D(\theta) - \pi(\theta)\pi'(\theta).\end{aligned} \tag{5.14}$$

The expression (5.12) and the limits in (5.13) and (5.14) yield (5.11). □

5.3. Weak limit via a martingale problem solution. To obtain the desired weak convergence result, we work with the pair $(v^{\mu,N}(\cdot),\theta^\mu(\cdot))$. By virtue of the tightness and Prohorov's theorem, we can extract a weakly convergent subsequence (still denoted by $(v^{\mu,N}(\cdot),\theta^\mu(\cdot))$ for simplicity) with limit $(v^N(\cdot),\theta(\cdot))$. We will show that the limit is a switching diffusion.

To proceed with the diffusion approximation, similarly as in the proof of Theorem 4.5, we will use the martingale problem formulation to derive the desired result. For $v\in\mathbb{R}^S$, $\theta\in\mathcal{M}$, and any twice continuously differentiable function $f(\cdot,\theta)$ with compact support, consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}f(v,\theta) = -\nabla f'(v,\theta)v + \frac{1}{2}\mathrm{tr}[\nabla^2 f(v,\theta)\Sigma(\theta)] + Qf(v,\cdot)(\theta), \tag{5.15}$$

where $\Sigma(\theta)$ is given by (5.10) and $\nabla^2 f(v,\theta)$ denotes $(\partial^2/\partial v_i\partial v_j)f(v,\theta)$, the mixed second-order partial derivatives. For any positive integer ℓ_0, any $t>0$, $s>0$, any $0<t_j\le t$ with $j\le \ell_0$, and any bounded and continuous function $h_j(\cdot,\theta)$ for each $\theta\in\mathcal{M}$, we aim to derive an equation similar to (4.13) with the operator L_1 replaced by $\mathcal{L}$. As in the proof of Theorem 4.5, we work with the sequence indexed by μ. Choose n_μ such that $n_\mu\to\infty$ but $\delta_\mu = \mu n_\mu \to 0$. The tightness of $\{v^{\mu,N}(\cdot),\theta^\mu(\cdot)\}$ and the Skorohod representation yield that (4.14)–(4.16) hold with $\widehat{\pi}^\mu(\cdot)$ and $\widehat{\pi}(\cdot)$ replaced by $v^{\mu,N}(\cdot)$ and $v^N(\cdot)$, respectively.

LEMMA 5.4. *Assume the conditions of Lemma* 5.1 *and that* $(v^{\mu,N}(0),\theta^\mu(0))$ *converges weakly to* $(v^N(0),\theta(0))$. *Then* $(v^{\mu,N}(\cdot),\theta^\mu(\cdot))$ *converges weakly to* $(v^N(\cdot),\theta(\cdot))$, *which is a solution of the martingale problem with operator* $\mathcal{L}^N$ *given by*

$$\mathcal{L}^N f(v,\theta) = -\nabla f'(v^N,\theta)v^N\tau^N(v^N) + \frac{1}{2}\mathrm{tr}[\nabla^2 f(v^N,\theta)\Sigma(\theta)] + Qf(v^N,\cdot)(\theta), \tag{5.16}$$

or equivalently $v^N(\cdot)$ *satisfies*

$$dv^N(t) = -v^N(t)\tau^N(v^N(t)) + \Sigma^{1/2}(\theta(t))dw, \tag{5.17}$$

where $w(\cdot)$ *is a standard* S*-dimensional Brownian motion and* $\Sigma(\theta)$ *is given by* (5.10).

Proof. In view of (5.8), the term $\sum_{k=t/\mu}^{(t+s)/\mu-1}[\mathbf{E}\pi(\theta_k) - \mathbf{E}\pi(\theta_{k+1})]/\sqrt{\mu} = O(\sqrt{\mu})$

can be ignored in the characterization of the limit process. Moreover,

$$\begin{aligned}&\sqrt{\mu}\sum_{k=t/\mu}^{(t+s)/\mu-1}[X_{k+1}-\mathbf{E}\pi(\theta_k)]\\&=\sqrt{\mu}\sum_{k=t/\mu}^{(t+s)/\mu-1}(X_{k+1}-\mathbf{E}X_{k+1})+\sqrt{\mu}\sum_{k=t/\mu}^{(t+s)/\mu-1}(\mathbf{E}X_{k+1}-\mathbf{E}\pi(\theta_k)).\end{aligned}$$

Since $\mathbf{E}X_{k+1}-\mathbf{E}\pi(\theta_k)\to 0$ exponentially fast owing to the elementary properties of a Markov chain, the last term above is $o(1)$ that goes to 0 as $\mu\to 0$. Thus,

$$v^{\mu,N}(t+s)-v^{\mu,N}(t)=-\mu\sum_{k=t/\mu}^{(t+s)/\mu-1}v_k^N\tau^N(v_k^N)+\sqrt{\mu}\sum_{k=t/\mu}^{(t+s)/\mu-1}(X_{k+1}-\mathbf{E}X_{k+1})+o(1). \tag{5.18}$$

Similarly to the argument in the proof of Theorem 4.5,

$$\begin{aligned}&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}[f(v_{ln_\mu}^N,\theta_{ln_\mu+n_\mu})-f(v_{ln_\mu}^N,\theta_{ln_\mu})]\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^N(t_j),\theta(t_j))\left[\int_t^{t+s}Qf(v^N(u),\theta(u))du\right].\end{aligned} \tag{5.19}$$

In addition,

$$\begin{aligned}&\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[-\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\frac{\delta_\mu}{n_\mu}\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}\nabla f'(v_{ln_\mu}^N,\theta_{ln_\mu})v_k^N\tau^N(v_k^N)\right]\\&=\lim_{\mu\to 0}\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[-\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\delta_\mu\nabla f'(v_{ln_\mu}^N,\theta_{ln_\mu})v_{ln_\mu}^N\tau^N(v_{ln_\mu}^N)\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^N(t_j),\theta(t_j))\left[-\int_t^{t+s}\nabla f'(v^N(u),\theta(u))v^N(u)\tau^N(v^N(u))du\right].\end{aligned} \tag{5.20}$$

Next we note that

$$\begin{aligned}&\left|\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[\sqrt{\mu}\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\nabla f'(v_{ln_\mu}^N,\theta_{ln_\mu})\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}[X_{k+1}-\mathbf{E}X_{k+1}]\right]\right|\\&\le\left|\mathbf{E}\prod_{j=1}^{\ell_0}h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[\sqrt{\mu}\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}|\nabla f'(v_{ln_\mu}^N,\theta_{ln_\mu})|\right.\right.\\&\qquad\left.\left.\times\sum_{k=ln_\mu}^{ln_\mu+n_\mu-1}|\mathbf{E}_{ln_\mu}[X_{k+1}-\mathbf{E}X_{k+1}]|\right]\right|\\&\to 0\ \text{ as }\ \mu\to 0\end{aligned} \tag{5.21}$$

owing to the mixing property.

Finally, define

$$g_{ln_\mu} g'_{ln_\mu} = \frac{1}{n_\mu} \sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \sum_{k_1=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu} [X_{k+1} - \mathbf{E}X_{k+1}][X_{k_1+1} - \mathbf{E}X_{k_1+1}]'.$$

It follows that

$$\begin{aligned}
&\mathbf{E} \prod_{j=1}^{\ell_0} h_j(v^{\mu,N}(t_j), \theta^\mu(t_j)) \left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \mathrm{tr}[\nabla^2 f(v^N_{ln_\mu}, \theta_{ln_\mu})(v^N_{ln_\mu+n_\mu} - v^N_{ln_\mu}) \right.\\
&\qquad\qquad \left. \times (v^N_{ln_\mu+n_\mu} - v^N_{ln_\mu})'] \right]\\
&= \mathbf{E} \prod_{j=1}^{\ell_0} h_j(v^{\mu,N}(t_j), \theta^\mu(t_j)) \left[\sum_{j=1}^{m_0} \sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \mathrm{tr}[\nabla^2 f(v^N_{ln_\mu}, \theta_{ln_\mu})(v^N_{ln_\mu+n_\mu} - v^N_{ln_\mu}) \right.\\
&\qquad\qquad \left. \times (v^N_{ln_\mu+n_\mu} - v^N_{ln_\mu})'] I_{\{\theta_{ln_\mu} = \overline{\theta}_j\}} \right]\\
&= \mathbf{E} \prod_{j=1}^{\ell_0} h_j(v^{\mu,N}(t_j), \theta^\mu(t_j)) \left[\sum_{j=1}^{m_0} \sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1} \delta_\mu \mathrm{tr}[\nabla^2 f(v^N_{ln_\mu}, \theta_{ln_\mu}) \mathbf{E}_{ln_\mu} g_{ln_\mu} g'_{ln_\mu}] \right.\\
&\qquad\qquad \left. \times I_{\{\theta_{ln_\mu} = \overline{\theta}_j\}} \right] + \rho_\mu,
\end{aligned}$$

where $\rho_\mu \to 0$ as $\mu \to 0$. Since it is conditioned on $\theta_{ln_\mu} = \overline{\theta}_j$, $X_{k+1} - \mathbf{E}X_{k+1}$ can be approximated by a fixed-$\overline{\theta}_j$ process $X_{k+1}(\overline{\theta}_j) - \mathbf{E}X_{k+1}(\overline{\theta}_j)$, and since $X_{k+1}(\overline{\theta}_j) - \mathbf{E}X_{k+1}(\overline{\theta}_j)$ is a finite-state Markov chain with irreducible transition matrix $A(\overline{\theta}_j)$, it is ϕ-mixing, and the argument in (5.10) implies that for each $\overline{\theta}_j \in \mathcal{M}$ with $j = 1, \ldots, m_0$,

(5.22)
$$\begin{aligned}
&\frac{1}{n_\mu} \sum_{k=ln_\mu}^{ln_\mu+n_\mu-1} \sum_{k_1=ln_\mu}^{ln_\mu+n_\mu-1} \mathbf{E}_{ln_\mu} (X_{k+1}(\overline{\theta}_j) - \mathbf{E}X_{k+1}(\overline{\theta}_j))(X_{k_1+1}(\overline{\theta}_j) - \mathbf{E}X_{k_1+1}(\overline{\theta}_j))'\\
&\to \Sigma(\overline{\theta}_j) \quad \text{w.p.1 as} \quad \mu \to 0,
\end{aligned}$$

where $\Sigma(\theta)$ is defined in (5.10). By virtue of Lemma 4.3, $\theta^\mu(\cdot)$ converges weakly to $\theta(\cdot)$. As a result, by Skorohod representation, sending $\mu \to 0$ and $l\delta_\mu \to u$ leads to $\theta^\mu(\mu l n_\mu)$ converging to $\theta(u)$ w.p.1. In addition, $I_{\{\theta^\mu(l\delta_\mu) = \overline{\theta}_j\}} \to I_{\{\theta(u) = \overline{\theta}_j\}}$ w.p.1. It follows that

(5.23)
$$\begin{aligned}&\mathbf{E}\prod_{j=1}^{\ell_0} h_j(v^{\mu,N}(t_j),\theta^\mu(t_j))\left[\sum_{ln_\mu=t/\mu}^{(t+s)/\mu-1}\operatorname{tr}\big[\nabla^2 f(v^N_{ln_\mu},\theta_{ln_\mu})(v^N_{ln_\mu+n_\mu}-v^N_{ln_\mu})\right.\\&\qquad\left.\times(v^N_{ln_\mu+n_\mu}-v^N_{ln_\mu})'\big]\right]\\&\to\mathbf{E}\prod_{j=1}^{\ell_0} h_j(v^N(t_j),\theta(t_j))\left[\int_t^{t+s}\sum_{j=1}^{m_0}\operatorname{tr}\big[\nabla^2 f(v^N(u),\overline{\theta}_j)\Sigma(\overline{\theta}_j)\big]I_{\{\theta(u)=\overline{\theta}_j\}}du\right]\\&=\mathbf{E}\prod_{j=1}^{\ell_0} h_j(v^N(t_j),\theta(t_j))\left[\int_t^{t+s}\operatorname{tr}\big[\nabla^2 f(v^N(u),\theta(u))\Sigma(\theta(u))\big]du\right].\end{aligned}$$

In view of (5.19)–(5.23), the desired result follows. □

COROLLARY 5.5. *Under the conditions of Lemma* 5.4, *the untruncated process* $(v^\mu(\cdot),\theta^\mu(\cdot))$ *converges weakly to* $(v(\cdot),\theta(\cdot))$ *satisfying the switching diffusion equation*

$$dv(t)=-v(t)dt+\Sigma^{1/2}(\theta(t))dw, \tag{5.24}$$

where $w(\cdot)$ *is a standard Brownian motion and* $\Sigma(\theta)$ *is given by* (5.10).

Proof. The uniqueness of the associated martingale problem can be proved similarly to that of [22, Lemma 7.18]. The rest of the proof follows from a similar argument as in [16, Step 4, p. 285]. □

Combining Lemma 5.1, Lemma 5.4, and Corollary 5.5, we have proved the following result.

THEOREM 5.6. *Assume conditions* (M) *and* (S) *and that* $(v^\mu(0),\theta^\mu(0))$ *converges weakly to* $(v(0),\theta(0))$. *Then* $(v^\mu(\cdot),\theta^\mu(\cdot))$ *converges weakly to* $(v(\cdot),\theta(\cdot))$, *which is the solution of the martingale problem with operator defined by* (5.15), *or equivalently, it is the solution of the system of diffusions with regime switching* (5.24).

Remark 5.7. The reason for obtaining a result such as Theorem 5.6 stems from the motivation for figuring out rates of convergence. If θ were a fixed parameter, we would obtain a diffusion limit as those in [16, Chapter 10]. As a consequence, the sequence v_n will be approximately normal. Now, our motivation is still for getting the rate of convergence. However, Theorem 5.6 reveals that v_n is an asymptotically Gaussian mixture. The mixture results from the time-varying parameter.

Remark 5.8. *Occupation measure for hidden Markov model.* The development thus far concerns recursive estimation of the occupation measure $\pi(\theta_n)$, given exact measurements of the conditional Markov sequence $\{X_n\}$. The above results can be extended to the hidden Markov model (HMM) case where the process $\{X_n\}$ is observed in noise as $\{Y_n\}$, where

$$Y_n=X_n+\zeta_n. \tag{5.25}$$

Assume that $\{\zeta_n\}$ satisfies the standard noise assumptions of an HMM [8, 13], i.e., it is a mutually independent and identically distributed (i.i.d.) noise process independent of X_n and θ_n. Then, given $\{Y_n\}$, to recursively estimate $\pi(\theta_n)$, the following modified version of the LMS algorithm (2.3) can be used. Replace X_{n+1} in algorithm (2.3) by Y_{n+1}. The mean square error analysis, switching ODE, and switching diffusion results of the previous sections carry over to this HMM case. More precisely, the following theorem holds.

THEOREM 5.9. *Consider the LMS algorithm* (2.3)*, where* X_{n+1} *is replaced by the HMM observation* Y_{n+1} *defined in* (5.25)*. Assume that the conditions of Theorem* 5.6 *hold, that* $\{\zeta_n\}$ *is a sequence of i.i.d. random variables with zero mean and* $E|\zeta_1|^2 < \infty$*, and that* $\{\zeta_n\}$ *is independent of* $\{X_n\}$ *and* $\{\theta_n\}$*. Then the conclusions of Theorems* 3.1*,* 4.5*, and* 5.6 *continue to hold.*

6. Application—Adaptive discrete stochastic optimization. In this section we apply the results developed in sections 3–5 to analyzing the tracking performance of an adaptive version of a discrete stochastic optimization algorithm proposed by Andradóttir [2]. Throughout this section we assume that the $\mathcal{M}$ in (2.1) is $\mathcal{M} = \mathcal{S} = \{e_1, \ldots, e_S\}$, where e_i denotes the standard unit vector. In what follows, $\mathcal{M}$ denotes the set of candidate values from which the time-varying global minimizer is chosen at each time instant (according to a slow Markov chain). $\mathcal{S}$ is the set of candidate solutions for the discrete optimization. Because we assume $\mathcal{M} = \mathcal{S}$, we do not use the notation $\mathcal{S}$ in this section. Note that the assumption that $\mathcal{M} = \mathcal{S}$ is made purely for notational convenience. Indeed, the set $\mathcal{M}$ of possible values from which the time-varying optimum is drawn can be any subset of $\mathcal{S}$.

6.1. Static discrete stochastic optimization. Consider the following discrete stochastic optimization problem:

$$\min_{\overline{\theta}\in\mathcal{M}} \mathbf{E}\{c_n(\overline{\theta})\}, \tag{6.1}$$

where for each fixed $\overline{\theta} \in \mathcal{M}$, $\{c_n(\overline{\theta})\}$ is a sequence of i.i.d. random variables with finite variance. Let $\mathcal{K} \subset \mathcal{M}$ denote the set of global minimizers for (6.1). The problem is static in the sense that the set $\mathcal{K}$ of global minima does not evolve with time.

When the expected value $\mathbf{E}\{c_n(\overline{\theta})\}$ can be evaluated analytically, (6.1) may be solved using standard integer programming techniques. A more interesting and important case motivated by applications in operations research [20] and wireless communication networks [11] is when $\mathbf{E}\{c_n(\overline{\theta})\}$ cannot be evaluated analytically and only $c_n(\overline{\theta})$ can be measured via simulation.

If a closed form solution of $\mathbf{E}\{c_n(\overline{\theta})\}$ cannot be obtained, a brute force method [18, Chapter 5.3] of solving the discrete stochastic optimization problem involves an exhaustive enumeration. It proceeds as follows: For each possible $\overline{\theta} \in \mathcal{M}$, compute the empirical average

$$\widehat{c}_N(\overline{\theta}) = \frac{1}{N}\sum_{i=1}^{N} c_i(\overline{\theta})$$

via simulation for large N, and pick out $\widehat{\theta} = \arg\min_{\overline{\theta}\in\mathcal{M}} \widehat{c}_N(\overline{\theta})$.

Since for any fixed $\overline{\theta} \in \mathcal{M}$, $\{c_n(\overline{\theta})\}$ is an i.i.d. sequence of random variables with finite variance, by virtue of Kolmogorov's strong law of large numbers, $\widehat{c}_N(\overline{\theta}) \to \mathbf{E}\{c_1(\overline{\theta})\}$ w.p.1 as $N \to \infty$. This and the finiteness of $\mathcal{M}$ imply that, as $N \to \infty$,

$$\arg\min_{\overline{\theta}\in\mathcal{M}} \widehat{c}_N(\overline{\theta}) \to \arg\min_{\overline{\theta}\in\mathcal{M}} \mathbf{E}\left\{c_1(\overline{\theta})\right\} \text{ w.p.1.} \tag{6.2}$$

In principle, the above brute force simulation method can solve the discrete stochastic optimization problem (6.1) for large N and the estimate is *consistent*, i.e., (6.2) holds. However, the method is highly inefficient since $\widehat{c}_N(\overline{\theta})$ needs to be evaluated for each $\overline{\theta} \in \mathcal{M}$. The evaluations of $\widehat{c}_N(\overline{\theta})$ for $\overline{\theta} \notin \mathcal{K}$ are wasted because they contribute nothing to the estimation of $\widehat{c}_N(\theta)$, $\theta \in \mathcal{K}$.

The idea of discrete stochastic optimization in [3] is to design an algorithm that is both *consistent* and *attracted* to the minimum. That is, the algorithm should spend more time obtaining observations $c_n(\overline{\theta})$ in areas of the state space $\mathcal{M}$ near the minimizer θ, and less so in other areas. Thus in discrete stochastic optimization the aim is to devise an *efficient* [18, Chapter 5.3] adaptive search (sampling plan), which allows us to find the maximizer with as few samples as possible by not making unnecessary observations at nonpromising values of $\overline{\theta}$.

In the papers [2] and [3], Andradóttir has proposed random search–based discrete stochastic optimization algorithms for computing the global minimizer in (6.1). In this subsection a brief outline of the assumptions and algorithm in [2] is given. Sections 6.2 and 6.3 analyze the performance of an adaptive version of the algorithm for tracking a time-varying minimum. In [2], the following stochastic ordering assumption was used.

(O) For each $e_i, e_j \in \mathcal{M}$, there exists some random variable Y^{e_i,e_j} such that for all $e_i \in \mathcal{K}$, $e_j \in \mathcal{K}$, and $e_l \in \mathcal{M}$, $l \neq i, j$,

$$\begin{aligned} &P(Y^{e_j,e_i} > 0) \geq P(Y^{e_i,e_j} > 0), \qquad P(Y^{e_l,e_i} > 0) \geq P(Y^{e_l,e_j} > 0), \\ &P(Y^{e_i,e_l} \leq 0) \geq P(Y^{e_j,e_l} \leq 0). \end{aligned} \tag{6.3}$$

Roughly speaking, this assumption ensures that the algorithm is more likely to jump towards a global minimum than away from it; see [2] for details. Some examples on how to choose Y^{e_i,e_j} are given in [2]. For example, suppose $c_n(\overline{\theta}) = \overline{\theta} + w_n(\overline{\theta})$ in (6.1) for each $\overline{\theta} \in \mathcal{M}$, where $\{w_n(\overline{\theta})\}$ has a symmetric continuous probability density function with zero mean. In this case simply choose $Y^{e_i,e_j} = c_n(e_i) - c_n(e_j)$. It is easily established that such a Y^{e_i,e_j} satisfies assumption (O). In [10] a stochastic comparison algorithm is presented for this example.

The static discrete stochastic optimization algorithm presented in [2] is as follows.

ALGORITHM 1 (static discrete stochastic optimization algorithm).

a. `Step 0:` (Initialization) At time $n = 0$, select starting point $X_0 \in \mathcal{M}$. Set $\widehat{\pi}_0 = X_0$, and select $\widehat{\theta}_0^* = X_0$.
b. `Step 1:` (Random search) At time n, sample $\widetilde{X}_n$ with uniform distribution from $\mathcal{M} - \{X_n\}$.
c. `Step 2:` (Evaluation and acceptance) Generate observation $Y^{X_n,\widetilde{X}_n}$. If $Y^{X_n,\widetilde{X}_n} > 0$, set $X_{n+1} = \widetilde{X}_n$; else, set $X_{n+1} = X_n$.
d. `Step 3:` (LMS algorithm for updating occupation probabilities of X_n) Construct $\widehat{\pi}_{n+1}$ as

$$\widehat{\pi}_{n+1} = \widehat{\pi}_n + \frac{1}{n}(X_{n+1} - \widehat{\pi}_n).$$

e. `Step 4:` (Compute estimate of the solution) $\widehat{\theta}_n^* = e_{i^*}$, where

$$i^* = \arg \max_{i \in \{1,\dots,S\}} \widehat{\pi}_{n+1}^*;$$

set $n \to n+1$ and go to Step 1 ($\widehat{\pi}_{n+1}^i$ denotes the ith component of the S-dimensional vector $\widehat{\pi}_{n+1}$).

The main convergence results in [2] for the above algorithm can be summarized as follows.

THEOREM 6.1. *Under assumption* (O)*, the sequence* $\{X_n\}$ *generated by Algorithm* 1 *is a homogeneous, aperiodic, irreducible Markov chain with state space* $\mathcal{M}$.

Furthermore, for sufficiently large n, $\{X_n\}$ spends more time in $\mathcal{K}$ than other states; i.e., if $\theta = e_i$ is a global minimizer of (6.1), then the stationary distribution $\pi(\theta)$ of $\{X_n\}$ satisfies $\pi^i(\theta) \geq \pi^j(\theta)$, $e_j \in \mathcal{M} - \mathcal{K}$, where $\pi^i(\theta)$ denotes the ith component of $\pi(\theta)$.

The theorem shows that $\widehat{\theta}_n^*$ is attracted to and converges almost surely to an element in $\mathcal{K}$.

6.2. Adaptive discrete stochastic optimization algorithm. Motivated by problems in spreading code optimization of CDMA wireless networks [11], we consider a variant of Algorithm 1 where the optimal solution $\theta \in \mathcal{M}$ of (6.1) is time-varying. Denote this time-varying optimal solution as θ_n. We subsequently refer to θ_n as the *true parameter* or *hypermodel*. Tracking such time-varying parameters is at the very heart of applications of adaptive SA algorithms. We propose the following adaptive algorithm.

Algorithm 2 (adaptive discrete stochastic optimization algorithm).

a. `Steps 0-2:` identical to Algorithm 1.

b. `Step 3:` (Constant step-size) Replace Step 3 of Algorithm 1 with a fixed-step-size algorithm, i.e.,

$$\widehat{\pi}_{n+1} = \widehat{\pi}_n + \mu \left(X_{n+1} - \widehat{\pi}_n \right), \tag{6.4}$$

where the step size μ is a small positive constant.

c. `Step 4:` identical to Algorithm 1.

Note that as long as $0 < \mu < 1$, $\widehat{\pi}_n$ is guaranteed to be a probability vector. Intuitively, the constant step size μ introduces exponential forgetting of the past occupation probabilities and permits tracking of slowly time-varying θ_n. The rest of this section is devoted to obtaining bounds on the error probability of the estimate $\widehat{\theta}_n^*$ generated by Algorithm 2.

6.3. Convergence analysis of adaptive discrete SA algorithm. In adaptive filtering (e.g., LMS), a typical method for analyzing the tracking performance of an adaptive algorithm is to postulate a *hypermodel* for the variation in the true parameter $\{\theta_n\}$. Since $\theta_n \in \mathcal{M}$ and $\mathcal{M}$ is a finite state space, it is reasonable to describe $\{\theta_n\}$ as a slow Markov chain on $\mathcal{M}$ for the subsequent analysis. Henceforth, we assume that (M) holds for $\{\theta_n\}$. Note that the hypermodel assumption is used only for the analysis and does not enter the actual algorithm implementation; see Algorithm 2.

Theorem 6.1 says that for fixed $\theta_n = \theta$ the sequence $\{X_n\}$ generated by Algorithm 2 is a conditional Markov chain (conditioned on θ_n); i.e., assumption (S) of section 2 holds. The update of the occupation probabilities (6.4) is identical to (2.3). Thus the behavior of the sequence $\{\widehat{\pi}_n\}$ generated by Algorithm 2 exactly fits the model of section 2 with $m_0 = S$. In particular, the mean squares analysis of section 3, the limit system of switching ODEs, and switching diffusion limit of section 5 hold.

Owing to the discrete nature of the underlying parameter θ_n, it makes sense to give bounds on the probability of error of the estimates $\widehat{\theta}_n^*$ generated by Step 4 of Algorithm 2. Define the error event E and probability of error $P(E)$ as

$$E = \{\widehat{\theta}_n^* \neq \theta_n\}, \quad P(E) = P(\widehat{\theta}_n^* \neq \theta_n). \tag{6.5}$$

Clearly E depends on n and the step size μ; we suppress the n here for notational simplicity. When we wish to emphasize the n- and μ-dependence, we write it as E_n^μ. Based on the mean square error of Theorem 3.1, the following result holds.

THEOREM 6.2. *Under conditions* (M) *and* (S), *if* $\mu = \varepsilon$, *then there is an* n_1 *such that for all* $n \geq n_1$ *the error probability of the estimate* $\widehat{\theta}_n^*$ *generated by Algorithm* 2 *satisfies*

$$P(E) = P(E_n^\mu) \leq K\mu^{1-2\gamma}, \quad 0 < \gamma < \frac{1}{2}, \tag{6.6}$$

where K *is a positive constant independent of* μ *and* ε.

The above result can be used to check the consistency: As $\mu \to 0$, the probability of error $P(E)$ of the tracking algorithm goes to zero. The constant K can be explicitly determined; however, it is highly conservative.

Proof. The estimate of the maximum generated by the discrete stochastic optimization algorithm at time n is $\widehat{\pi}_n^* = \arg\max_j \widehat{\pi}_n^j$ (where $\widehat{\pi}_n^j$ denotes the jth component of the S-dimensional vector $\widehat{\pi}_n$). Thus the error event E in (6.5) is equivalent to $E = \left\{\arg\max_i \pi^i(\theta_n) \neq \arg\max_j \widehat{\pi}_n^j\right\}$. Then clearly the complement event $\bar{E} = \left\{\arg\max_i \pi^i(\theta_n) = \arg\max_j \widehat{\pi}_n^j\right\}$ satisfies

$$\begin{aligned}\bar{E} &\supseteq \left\{\left|\max_i \pi^i(\theta_n) - \max_j \widehat{\pi}_n^j\right| \leq \min_{i,j} |\pi^i(\theta_n) - \widehat{\pi}_n^j|\right\} \\ &\supseteq \left\{\left|\max_i \pi^i(\theta_n) - \max_j \widehat{\pi}_n^j\right| \leq L\right\},\end{aligned}$$

where

$$L \leq \min_{i,j} |\pi^i(\theta_n) - \widehat{\pi}_n^j| \tag{6.7}$$

is a positive constant. Then the probability of no error is

$$P(\bar{E}) = P\left(\arg\max_i \pi^i(\theta_n) = \arg\max_j \widehat{\pi}_n^j\right) > P\left(\left|\max_i \pi^i(\theta_n) - \max_j \widehat{\pi}_n^j\right| \leq L\right)$$

for any sufficiently small positive number L. Then, using the above equation and Theorem 3.1,

$$\begin{aligned}P(E) &\leq P\left(\left|\max_i \pi^i(\theta_n) - \max_j \widehat{\pi}_n^j\right| > L\right) \\ &\leq P\left(\max_i |\pi^i(\theta_n) - \widehat{\pi}_n^i| > L\right).\end{aligned} \tag{6.8}$$

Applying Chebyshev's inequality to (3.1) yields, for any i,

$$P(|\pi^i(\theta_n) - \widehat{\pi}_n^i| > L) \leq \frac{1}{L^2} K\mu$$

for some constant K. Thus (6.8) yields

$$P\left(\max_i |\pi^i(\theta_n) - \widehat{\pi}_n^i| > L\right) \leq \frac{1}{L^2} K\mu. \tag{6.9}$$

It only remains to pick a sufficiently small L. Choose $L = \mu^\gamma$, where $0 < \gamma < \frac{1}{2}$ is arbitrary. It is clear that, for sufficiently small μ, L satisfies (6.7). Then (6.9) yields $P(E) \leq K\mu^{1-2\gamma}$. □

Using the diffusion approximation Corollary 5.5 and Theorem 5.6, a sharper upper bound for the error probability can be obtained as follows. First, without loss of generality we may order the states $\overline{\theta}_i \in \mathcal{M}$ so that the covariances $\Sigma(\overline{\theta})$ are, in ascending order,

$$\Sigma(\overline{\theta}_1) \le \Sigma(\overline{\theta}_2) \le \cdots \le \Sigma(\overline{\theta}_S), \tag{6.10}$$

where $\Sigma(\overline{\theta}_i) \le \Sigma(\overline{\theta}_j)$ (resp., $\Sigma(\overline{\theta}_i) < \Sigma(\overline{\theta}_j)$) means that $\Sigma(\overline{\theta}_i) - \Sigma(\overline{\theta}_j)$ is nonnegative definite (resp., positive definite). Note that $\Sigma(\overline{\theta}_i)$ is explicitly computable using (5.11). Define

$$e^{ji} \stackrel{\text{def}}{=} e_j - e_i, \quad \sigma^{ji}(\overline{\theta}) \stackrel{\text{def}}{=} \sqrt{e^{ji,\prime}\Sigma(\overline{\theta})e^{ji}}. \tag{6.11}$$

THEOREM 6.3. *Assume that conditions* (M) *and* (S) *hold and that* $\mu = \varepsilon$. *Then for sufficiently large* n *the error probability of the estimate* $\widehat{\theta}_n$ *generated by Algorithm* 2 *satisfies*

$$P(E) = \sum_{i=1}^{S} P(\theta_n = \overline{\theta}_i)P(E|\theta_n = \overline{\theta}_i) = \sum_{i=1}^{S} z^i(\mu n)P(E|\theta_n = \overline{\theta}_i) + O(\mu + \exp(-k_0 n)), \tag{6.12}$$

$$\begin{aligned} P(E|\theta_n = \overline{\theta}_i) \le \sum_{\substack{j=1\\ j\neq i}}^{S} \Bigg[& I(e^{ji,\prime}\overline{\pi}(\mu n) \le 0)\Phi^c\left(\frac{-e^{ji,\prime}\overline{\pi}(\mu n)/\sqrt{\mu}}{\sigma^{ji}(\overline{\theta}_1)/2}\right) \\ & + I(e^{ji,\prime}\overline{\pi}(\mu n) > 0)\Phi^c\left(\frac{-e^{ji,\prime}\overline{\pi}(\mu n)/\sqrt{\mu}}{\sigma^{ji}(\overline{\theta}_S)/2}\right)\Bigg], \end{aligned} \tag{6.13}$$

where $z^i(\cdot), \overline{\pi}(\cdot)$ *are defined in* (5.2), *and* $\sigma^{ji}(\cdot)$ *are defined in* (6.11), *which can be computed using* (5.11) *and* $\Phi^c(\cdot) = 1 - \Phi(\cdot)$, *with* $\Phi(\cdot)$ *being the standard normal distribution function.*

Proof. Clearly $P(E) = \sum_{i=1}^{S} P(\theta_n = \overline{\theta}_i)P(E|\theta_n = \overline{\theta}_i)$. Then (5.2) yields (6.12). Now

$$\begin{aligned} P(E|\theta_n = \overline{\theta}_i) &= P\left(\arg\max_j \widehat{\pi}_n^j \neq e_i|\theta_n = \overline{\theta}_i\right) \\ &= P\left(\bigcup_{\substack{j=1\\ j\neq i}}^{S} \{\widehat{\pi}_n^j - \widehat{\pi}_n^i > 0\} \mid \theta_n = \overline{\theta}_i\right) \\ &\le \sum_{\substack{j=1\\ j\neq i}}^{S} P\left(\widehat{\pi}_n^j - \widehat{\pi}_n^i > 0 \mid \theta_n = \overline{\theta}_i\right) \qquad \text{(union bound)}. \end{aligned}$$

Upper bounds for each of the $S-1$ terms in the above summation will now be constructed.

Using (5.1), with $\overline{\pi}(\mu n)$ defined in (5.2),

$$\widehat{\pi}_n = \mathbf{E}\{\pi(\theta_n)\} + \sqrt{\mu}v_n = \overline{\pi}(\mu n) + \sqrt{\mu}v_n + O(\mu + \exp(-k_0 n)), \tag{6.14}$$

where $v(t)$, the limit of the interpolation of v_n, satisfies the switching diffusion (5.24), and $\Sigma(\overline{\theta}_i)$ are in ascending order as in (6.10).

Define scalar processes β_n^{ji} and $\beta^{ji}(t)$ as $\beta_n^{ji} = e^{ji,\prime} v_n$ and $\beta^{ji}(t) = e^{ji,\prime} v(t)$. Then $\beta^{ji}(t)$ satisfies the real-valued switching diffusion

$$d\beta^{ji}(t) = -\beta^{ji}(t)dt + \sigma_{ji}(\theta(t))db(t),$$

where $\sigma^{ji}(\theta(t))$ is defined in (6.11) and $b(t)$ is a real-valued standard Brownian motion.

Owing to (6.14), $\widehat{\pi}_n^j - \widehat{\pi}_n^i = e^{ji,\prime}\widehat{\pi}_n = e^{ji,\prime}\overline{\pi}(\mu n) + \sqrt{\mu}\beta_n^{ji} + O(\mu + \exp(-k_0 n))$. Since the $O(\mu + \exp(-k_0 n))$ does not contribute to the limit in distribution, we drop it henceforth. We have

$$P\left(\widehat{\pi}_n^j - \widehat{\pi}_n^i > 0 \mid \theta_n = \overline{\theta}_i\right) = P\left(\beta_n^{ji} > \frac{-e^{ji,\prime}\overline{\pi}(\mu n)}{\sqrt{\mu}} \,\middle|\, \theta_n = \overline{\theta}_i\right). \tag{6.15}$$

Since the process β_n^{ji} is a Gaussian mixture and the limiting process $\beta^{ji}(t)$ is a switching diffusion, it is difficult to explicitly compute the right-hand side of (6.15). However, it can be upper-bounded by considering the Gaussian diffusion processes $\underline{\beta}^{ji}(t)$ and $\overline{\beta}^{ji}(t)$, which are defined as follows:

$$\begin{aligned} d\underline{\beta}^{ji}(t) &= -\underline{\beta}^{ji}(t)dt + \sigma_{ji}(\overline{\theta}_1)db(t), \quad \underline{\beta}^{ji}(0) = \beta^{ji}(0), \\ d\overline{\beta}^{ji}(t) &= -\overline{\beta}^{ji}(t)dt + \sigma_{ji}(\overline{\theta}_S)db(t), \quad \overline{\beta}^{ji}(0) = \beta^{ji}(0). \end{aligned}$$

Due to the ordering of the positive definite matrices $\Sigma(\overline{\theta}_i)$ in (6.10), the scalars $\sigma_{ji}(\overline{\theta}_i)$ satisfy

$$\sigma_{ji}(\overline{\theta}_1) \le \sigma_{ji}(\overline{\theta}_2) \le \cdots \le \sigma_{ji}(\overline{\theta}_S). \tag{6.16}$$

To proceed, we claim the following result and postpone the proof until later.

LEMMA 6.4. *For any $a > 0$, $P(\underline{\beta}^{ji}(t) \le a) \ge P(\beta^{ji}(t) \le a | \theta(t) = \overline{\theta}_i) \ge P(\overline{\beta}^{ji}(t) \le a)$. For any $a \le 0$, $P(\underline{\beta}^{ji}(t) \le a) \le P(\beta^{ji}(t) \le a | \theta(t) = \overline{\theta}_i) \le P(\overline{\beta}^{ji}(t) \le a)$.*

Lemma 6.4 implies that

$$P(\beta^{ji}(t) > a | \theta(t) = \overline{\theta}_i) \le I(a > 0) P(\underline{\beta}^{ji}(t) > a) + I(a \le 0) P(\overline{\beta}^{ji}(t) > a). \tag{6.17}$$

Since $\underline{\beta}^{ji}(t)$ and $\overline{\beta}^{ji}(t)$ are real-valued diffusions and are stable, their stationary covariances are easily computed as $\underline{\sigma}^2 = \sigma_{ji}^2(\overline{\theta}_1)/2$ and $\overline{\sigma}^2 = \sigma_{ji}^2(\overline{\theta}_S)/2$, respectively. Thus, asymptotically $\underline{\beta}^{ji}(t)$, $\overline{\beta}^{ji}(t)$ are Gaussian random variables with zero mean and variance $\sigma_{ji}^2(\overline{\theta}_1)/2$ and $\sigma_{ji}^2(\overline{\theta}_S)/2$, respectively. Then (6.17) yields

$$P(\beta^{ji}(t) > a | \theta(t) = \overline{\theta}_i) \le I(a > 0)\Phi^c\left(\frac{a}{\sigma_{ji}(\overline{\theta}_1)/2}\right) + I(a \le 0)\Phi^c\left(\frac{a}{\sigma_{ji}(\overline{\theta}_S)/2}\right).$$

Thus for sufficiently large n and sufficiently small $\mu > 0$,

$$P(\beta_n^{ji} > a | \theta_n = \overline{\theta}_i) \le I(a > 0)\Phi^c\left(\frac{a}{\sigma^{ji}(\overline{\theta}_1)/2}\right) + I(a \le 0)\Phi^c\left(\frac{a}{\sigma^{ji}(\overline{\theta}_S)/2}\right).$$

Using this in (6.15) proves the theorem. $\square$

Proof of Lemma 6.4. Let $t_1 < t_2 < \cdots < t_N \le t$ denote the sequence of jump times of the Markov chain $\{\theta(t)\}$. Let $\mathcal{G}_t$ denote the σ-algebra generated by $\{\theta(s) : s < t, \theta(t)\}$. Then

$$\beta^{ji}(t) = e^{-t}\left[\sigma_{ji}(\theta(0))\int_0^{t_1-} e^{\tau}db(\tau) + \sigma_{ji}(\theta(t_1))\int_{t_1}^{t_2-} e^{\tau}db(\tau) + \cdots + \sigma_{ji}(\theta(t_N))\int_{t_N}^{t} e^{\tau}db(\tau)\right],$$

$$\underline{\beta}^{ji}(t) = e^{-t}\left[\sigma_{ji}(\overline{\theta}_1)\int_0^{t_1-} e^{\tau}db(\tau) + \sigma_{ji}(\overline{\theta}_1)\int_{t_1}^{t_2-} e^{\tau}db(\tau) + \cdots + \sigma_{ji}(\overline{\theta}_1)\int_{t_N}^{t} e^{\tau}db(\tau)\right],$$

where $\underline{\beta}^{ji}(t)$ is a zero mean scalar Gaussian variable. Conditioned on $\mathcal{G}_t$, $\beta^{ji}(t)$ is a zero mean scalar Gaussian random variable. Since $\sigma_{ji}(\overline{\theta}_1) \le \sigma_{ji}(\theta(t))$ for all t by (6.16), clearly $\mathbf{E}\{\underline{\beta}^{ji}(t)\}^2 \le \mathbf{E}\{\beta^{ji}(t)\}^2$. Hence for $x > 0$, $\mathbf{E}\{I(\underline{\beta}^{ji}(t) \le x)\} > \mathbf{E}\{I(\beta^{ji}(t) \le x)|\mathcal{G}_t, \theta(t)\}$. Taking $\mathbf{E}\{\cdot|\theta(t)\}$ on both sides and using the fact that $\underline{\beta}^{ji}(t)$ is independent of $\theta(t)$ yields $P(\underline{\beta}^{ji}(t) \le x) > P(\beta^{ji}(t) \le x|\theta(t))$. The result for $\overline{\beta}^{ji}(t)$ is established similarly. □

Remark 6.5. First, Markov chain Monte Carlo–based simulation methods can be used to evaluate the probability of error of the algorithm. In addition, a Gaussian approximation–based heuristic expression can be obtained for the probability error bounds of Algorithm 2 in lieu of Theorem 6.3. Consider a real-valued switching diffusion process

$$dx = -xdt + \sigma(\theta(t))db,$$

where $\theta(t)$ is the limit Markov chain as in section 5. The negative term $-x$ implies that the system is stable. Thus, by virtue of an argument as in [16, p. 323], the covariance is given by

$$\mathbf{E}x(t)x(0) = \mathbf{E}\left(\int_{-\infty}^{t} \exp(-(t-s))\sigma(\theta(s))db(s)\right)\left(\int_{-\infty}^{0} \exp(-s)\sigma(\theta(s))db(s)\right).$$

Assume in addition that the generator Q of the Markov chain $\theta(t)$ (the one given in condition (M)) is irreducible, which implies (see [22]) that, except for zero, all other eigenvalues are on the left half of the complex plan. As a result, the stationary covariance exists and is given by

$$\widetilde{\sigma}^2 = \mathbf{E}\sum_{l=1}^{S}\int_0^{\infty} \exp(-2s)\sigma^2(\overline{\theta}_l) I_{\{\theta(s)=\overline{\theta}_l\}}ds. \tag{6.18}$$

This covariance may be computed via the Monte Carlo method. Using $\widetilde{\sigma}^2$, an approximation of the probability of error for Algorithm 2 can be computed.

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