LMS Algorithms for Tracking Slow Markov Chains With Applications to Hidden Markov Estimation and Adaptive Multiuser Detection

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***Abstract*—This paper analyzes the tracking properties of the least mean squares (LMS) algorithm when the underlying parameter evolves according to a finite-state Markov chain with infrequent jumps. First, using perturbed Liapunov function methods, mean-square error estimates are obtained for the tracking error. Then using recent results on two-time-scale Markov chains, mean ordinary differential equation and diffusion approximation results are obtained. It is shown that a sequence of the centered tracking errors converges to an ordinary differential equation. Moreover, a suitably scaled sequence of the tracking errors converges weakly to a diffusion process. It is also shown that iterate averaging of the tracking algorithm results in optimal asymptotic convergence rate in an appropriate sense. Two application examples, analysis of the performance of an adaptive multiuser detection algorithm in a direct-sequence code-division multiple-access (DS/CDMA) system, and tracking analysis of the state of a hidden Markov model (HMM) with infrequent jumps, are presented.**

***Index Terms*—Adaptive filtering, admission/access control, direct-sequence code-division multiple-access (DS/CDMA) adaptive multiuser detection, hidden Markov model (HMM), jump Markov parameter, mean square error bound, weak convergence.**

I. INTRODUCTION

IN this work, we consider a class of adaptive least mean squares (LMS) algorithms for the purpose of tracking a time-varying parameter process with infrequent jump changes. The time-varying parameter is modeled as a discrete-time Markov chain whose transition matrix is "almost identity." The dynamics of the parameter process display piecewise-constant behavior with infrequent jumps from one state to another. In what follows, we often call it a slowly varying Markov chain or a slow Markov chain. Our interest lies in figuring out the bounds for tracking errors. To accomplish our goals, we carry out the analysis in several steps. We derive mean-square error bounds, treat an interpolated sequence of the centered estimation errors, proceed with the consideration of a suitably scaled sequence of estimation errors, obtain its diffusion limit, and examine further the error bounds via asymptotic normality.

Most analyses of LMS algorithms with time-varying parameters (see [1], [10], [11], [16], [17], [20], [34]) assume that the parameter varies continuously but slowly over time with a small amount of changes, e.g., a slow random walk. In contrast, we deal with the case when the parameter is constant over long periods of time and then jump changes by possibly a large amount (i.e., a Markov chain with transition probability matrix close to the identity). Unlike the existing results on the analysis of LMS algorithms, we explicitly consider this Markovian time-varying parameter in the analysis. Noting the special feature of the transition matrix, we identify it as a Markov chain with two time scales. Using recent results on two-time-scale Markov chains [38], [39], we examine the asymptotic properties of the tracking algorithm. Using the asymptotic normality resulting from the limit diffusion process, we provide further error bounds on the probability of deviations.

This paper is motivated by several practical applications, for example, fault diagnosis and change detection [1], where the LMS algorithm is used to track a parameter that undergoes infrequent jump changes. Such problems also appear in emerging applications of wireless communication and estimation of hidden Markov models. Specifically, we consider the following two examples.

Example 1: (Effect of Admission/Access Control on Adaptive Multiuser Detector:) Activity detection in direct-sequence code-division multiple-access (DS/CDMA) networks has been studied in detail in [22] and [23]. In these papers, the task of detecting changes in the user population due to the entrance of new users or departure of existing users, is formulated as a sequential detection/isolation problem.

Here, we consider the performance analysis of an adaptive linear multiuser detector in a cellular DS/CDMA wireless network with changing user activity due to an admission or access controller at the base station. Understanding the interaction of an admission control/access control algorithm with the physical layer interference suppression (multiuser detection) is of increasing importance in cross-layer optimization of wireless networks, [33]. An admission controller [33] typically regulates the admission of new users to the network to maintain an acceptable quality of service (e.g., signal-to-interference ratio) and blocking probability (i.e., probability of new user being rejected). An access (scheduling) controller determines on a multiframe-by-multiframe basis (slower time scale than the bit

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interval) which group of users to transmit. Such access control is typically required when the network admits more users than its capacity in order to reduce call blocking and handoff blocking probabilities. The adaptive linear multiuser receiver must dynamically adapt the weight vector of the receiver according to an LMS-type algorithm [12], [26], [35] to track the optimal linear minimum mean-square error (LMMSE). However, the coefficients of this LMMSE receiver jump change at times corresponding to the arrival and departure of users (admission controller) or active user group (access control). Since the number of possible combinations of active users is finite, the coefficients of the LMMSE multiuser detector evolves according to a finite-state process. The dynamics of the evolution of this finite-state process depends on the admission or access control policies. In particular, for Markovian admission or access control policies (e.g., Markov decision process based admission controller [33], [29], or seed exchange (SEEDEX) medium-access protocol [14]), the coefficients of the LMMSE receiver evolves according to a finite-state Markov chain. Given that the adaptive linear multiuser detector uses the LMS algorithm to adaptively track the coefficients of the LMMSE receiver (time-varying parameter), it is of interest to study how the LMS tracks a parameter that evolves according to a Markov chain. In contrast, most papers in adaptive multiuser detection [12], [27] assume a constant optimal LMMSE receiver (weight vector) and examine how the constant step size LMS algorithm (or recursive least squares (RLS) with fixed forgetting factor) hovers about this constant-weight vector.

Example 2: (State Estimation of a Hidden Markov Model with Infrequent Jumps:) We analyze the performance of the LMS algorithm for tracking the state of a *slowly* varying hidden Markov model (HMM) [5], where the underlying Markov chain's transition probability is of the form $I + \varepsilon Q$ with ε being a small parameter and m being the total number of states. The optimal HMM state filter (which yields the conditional mean estimate of the state) requires $O(m^2)$ computations at each time instant and hence intractable for very large m. For sufficiently small ε, it might be expected that the LMS would do a reasonable job tracking the underlying Markov chain since the states change infrequently. As described in Section V-B, the LMS algorithm requires $O(1)$ computational complexity for an m-state HMM (i.e., the complexity is independent of m). Recently, an $O(m)$ complexity asymptotic (steady-state) HMM state filter was proposed in [9]; see also [31]. It is therefore of interest to analyze the performance of an LMS algorithm (in terms of error probabilities) for tracking a time-varying HMM with infrequent jump changes. Such an analysis is presented in Section V-B. From a practical point of view, using an $O(1)$ complexity LMS algorithm for tracking a slow HMM is useful in military sensor network applications—such as unattended ground sensor networks [4], where conserving battery life of individual processing sensors is important.

Main Results: Assuming that the true parameter θ_n evolves according to a slow finite-state Markov chain with transition probability matrix $I+\varepsilon Q$ where $\varepsilon > 0$ is a small parameter, and that the LMS algorithm operates with a step size μ, we summarize the main results of the paper as follows.

i) In Section III, a mean-square stability analysis of the LMS algorithm is performed which shows that the mean-square tracking error is of the order $O(\mu + \varepsilon/\mu)\exp(\mu + \varepsilon/\mu)$. Unlike the usual analysis of tracking algorithms where the error depends on the step size of the algorithm, the dynamics of the error propagate with respect to the adaptation rate (the step size of the tracking algorithm μ) and the variation rate (magnitude of the perturbation ε) of the Markov chain.

ii) Based on the mean-square error bounds, a mean ordinary differential equation (ODE) limit is obtained in Section IV. This ODE captures the evolution of the tracking error of the LMS algorithm as a centered dynamic system. Such a characterization, in turn, provides us with information of the trajectories. It turns out that the ODE is identical to the case when the parameter θ_n is a constant for all time n.

iii) Building upon the mean-square estimates and the limit ODE, the diffusion approximation of Section IV-C (Theorem 18) shows that if the LMS algorithm with step size $\mu = O(\sqrt{\varepsilon})$ yields parameter estimates $\widehat{\theta}_n$, then the estimation error $\widehat{\theta}_n - \theta_n$ is asymptotically normal with zero mean and covariance identical to the case when the parameter θ_n is a constant for all time n. The asymptotic normality translates into approximate normality for finite n under certain conditions on the moments of the regression and noise vectors (Section V-A).

iv) Section IV-D shows that iterate averaging reduces the asymptotic covariance of the estimate of the LMS algorithm. Iterate averaging was originally proposed by Polyak [24] and Ruppert [30] independently (see also [25]) for accelerating the convergence of stochastic approximation algorithms. It is well known [17] that for a constant true parameter and decreasing step size, iterate averaging results in asymptotically optimal convergence rate (the same asymptotic convergence rate as the recursive least squares), which use matrix step sizes, with an order of magnitude lower computational complexity than RLS.

In the tracking case for a random-walk time-varying parameter, it has recently been shown in [19] that the fixed step size LMS algorithm with an iterate averaging has similar properties to a recursive least squares algorithm with forgetting factor. Obviously, for a time-varying parameter, recursive least squares is not the optimal tracking algorithm. However, in our slow Markov chain case, the parameter remains constant over long duration. Hence, heuristically one would expect that iterate averaging yields real benefits. Section IV-D shows that if $\varepsilon = O(\mu^2)$, and the averaging window width is $O(1/\mu)$ (where μ denotes the step size of the LMS algorithm), then iterate averaging results in an asymptotically optimal tracking algorithm. To our knowledge, this is the first example of a case where iterate averaging results in a constant step size LMS algorithm with optimal tracking properties.

v) The convergence analysis of this paper deals with the case when the parameter is a slow Markov chain which has a nonzero (but small) probability of jumping to any other state at any time instant. In Section IV-E, we briefly consider the much simpler case where the parameters can randomly jump change to another state but only after spending at least $O(1/\varepsilon)$ time in the current state. For this case, the above ODE and weak convergence to diffusion results also hold. The model and resulting analysis are relevant to periodic access control (scheduling) policies in wireless networks.

vi) In light of i)–v), Section V-B presents expressions for probability of error of the quantized estimate of the LMS algorithm when tracking the state of a slow HMM. It is shown that iterate averaging results in a lower error probability. Section V-C shows that if the number of active users in a wireless CDMA network evolves as a slowly varying Markov chain, then the adaptive decision directed linear multiuser detector, the adaptive blind multiuser detector, and the adaptive precombining multiuser detector are LMS algorithms tracking a Markov chain with infrequent jumps. More importantly, it establishes that for sufficiently slow change in users (i.e., $\varepsilon = O(\mu^2)$) iterate averaging does indeed result in optimal convergence properties of the adaptive multiuser detector. This gives theoretical justification to results in [13] where it was shown that the blind adaptive multiuser detector using the so-called blind LMS algorithm with iterate averaging has superior convergence properties to the standard blind LMS algorithm.

The rest of the paper is organized as follows. Section II begins with the formulation of the problem. Section III is devoted to obtaining error bounds in terms of mean-square errors. Section IV derives a mean ODE limit, and develops diffusion approximation via weak convergence methods. Section V considers two practical applications—state estimation of a HMM with slow dynamics and adaptive multiuser detection in a DS/CDMA wireless network. Section VI concludes the paper with further remarks. Throughout the paper, for $z \in \mathbb{R}^{\ell\times r}$, we use z' to denote its transpose, and $|z|$ denote its norm. We also use K to denote a generic positive constant. The convention $K+K=K$ and $KK=K$ will be used without notice. For any two functions g_1 and g_2, $g_1 = O(g_2)$ and $g_1 = o(g_2)$ denote $|g_1/g_2| \le K$ and $|g_1/g_2| \to 0$, respectively.

II. Formulation

Let $\{y_n\}$ be a sequence of real-valued signals representing the observations obtained at time n, and $\{\theta_n\}$ be the time-varying true parameter, an $\mathbb{R}^r$-valued random process. Suppose that

$$y_n = \varphi_n'\theta_n + e_n, \qquad n = 0,1,\ldots \tag{1}$$

where $\varphi_n \in \mathbb{R}^r$ is the regression vector and $e_n \in \mathbb{R}$ is a sequence of zero-mean random vectors. Note that (1) is a variant of the usual linear regression model, in which, a time-varying stochastic process θ_n is in place of a fixed parameter. Throughout the paper, we assume that θ_n is a discrete-time Markov chain. The precise conditions on the parameter process is given as follows.

Assumptions on the Markov Chain θ_n:

(A1) Suppose that there is a small parameter $\varepsilon > 0$ and that θ_n is a discrete-time homogeneous Markov chain, whose state space and transition probability matrix are given by

$$\mathcal{M} = \{\overline{\theta}_1, \ldots, \overline{\theta}_m\} \tag{2}$$

and

$$P^\varepsilon = I + \varepsilon Q \tag{3}$$

respectively, where I is an $\mathbb{R}^{m\times m}$ identity matrix and $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$ is a generator of a continuous-time Markov chain (i.e., Q satisfies $q_{ij} \ge 0$ for $i \ne j$ and $\sum_{j=1}^m q_{ij} = 0$ for each $i = 1,\ldots,m$). For simplicity, assume the initial distribution $P(\theta_0 = \overline{\theta}_i) = p_{0,i}$ to be independent of ε for each $i = 1,\ldots,m$, where $p_{0,i} \ge 0$ and $\sum_{i=1}^m p_{0,i} = 1$.

Remark 3: Note that the small parameter $\varepsilon > 0$ in (A1) ensures that the identity matrix I dominates. In fact, $q_{ij} \ge 0$ for $i \ne j$ thus, the small parameter $\varepsilon > 0$ ensures the entries of the transition matrix to be positive since $p_{ij}^\varepsilon = \delta_{ij} + \varepsilon q_{ij} \ge 0$ for $\varepsilon > 0$ small enough, where $\delta_{ij} = 1$ if $i = j$ and is 0 otherwise. The use of the generator Q makes the row sum of the matrix P be one since

$$\sum_{j=1}^m p_{ij}^\varepsilon = 1 + \varepsilon \sum_{j=1}^m q_{ij} = 1.$$

The essence is that although the true parameter is time varying, it is piecewise constant. In addition, the process does not change too frequently due to the dominating identity matrix in the transition matrix (1). It remains as a constant most of the time and jumps into another state at random instance.

Adaptive Algorithm: The adaptive algorithm is of LMS adaptive filtering type with a constant step size. To track the parameter $\{\theta_n\}$, we construct a sequence of estimates $\{\widehat{\theta}_n\}$ according to

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \mu\varphi_n\left(y_n - \varphi_n'\widehat{\theta}_n\right), \qquad n = 0,1,\ldots \tag{4}$$

where $\mu > 0$ is a small constant step size for the algorithm. By using (1) with $\widetilde{\theta}_n = \widehat{\theta}_n - \theta_n$, we obtain

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n + (\theta_n - \theta_{n+1}). \tag{5}$$

Our task to follow is to figure out the bounds on the deviation $\widetilde{\theta}_n = \widehat{\theta}_n - \theta_n$. This goal is accomplished by the following four steps.

1) Obtain a mean-square error bounds for $\boldsymbol{E}|\widehat{\theta}_n - \theta_n|^2$.
2) Obtain a limit ODE of centered process.
3) Obtain a weak convergence result of a suitably scaled sequence.
4) Obtain probabilistic bounds on $P(|\widehat{\theta}_n - \theta_n| > \alpha)$ for $\alpha > 0$, and hence probability of error bounds based on the result obtained in part 2).

Remark 4: The parameter θ_n is called a *hypermodel* in [1]. Note that while the dynamics of the hypermodel θ_n are used in our analysis, it does not enter the implementation of the LMS algorithm (4) explicitly.

Assumptions on the Signals: Let $\mathcal{F}_n$ be the σ-algebra generated by $\{(\varphi_j, e_j), j < n, \theta_j, j \le n\}$, and denote the conditional expectation with respect to $\mathcal{F}_n$ by $\boldsymbol{E}_n$. We will use the following conditions on the signals.

(A2) The signal $\{\varphi_n, e_n\}$ is independent of $\{\theta_n\}$. Either $\{\varphi_n, e_n\}$ is a sequence of bounded signals such that there is a symmetric and positive-definite matrix $B \in \mathbb{R}^{r\times r}$ such that $\boldsymbol{E}\varphi_n\varphi_n' = B$

$$\left|\sum_{j=n}^{\infty} \boldsymbol{E}_n\left[\varphi_j\varphi_j' - B\right]\right| \le K \tag{6}$$

and in addition

$$\left|\sum_{j=n}^{\infty} \boldsymbol{E}_n \varphi_n e_j\right| \le K \tag{7}$$

or $\{\varphi_n, e_n\}$ is a sequence of martingale difference signals satisfying $\sup_n E|\varphi_n|^{4+\Delta} < \infty$ and $\sup_n E|\varphi_n e_n|^{2+\Delta} < \infty$ for some $\Delta > 0$.

Remark 5: Although $\{y_n\}$ depends on θ_n, we have assumed $\{\varphi_n, e_n\}$ to be independent of θ_n. The signal models we are dealing with include a large class of practical applications. Inequalities (6) and (7) are modeled after mixing processes and are in the almost sure (a.s.) sense with the constant K independent of ω, the sample point. (Note, however, we use the same kind of notation as in, for example, the mixing inequalities [2, p. 166, eq. (20.4)] and [15, p. 82, eqs. (6.6) and (6.7)].) This allows us to work with correlated signals whose remote past and distant future are asymptotically independent. To obtain the desired result, the distribution of the signal need not be known. The boundedness is a mild restriction; for example, one may consider truncated Gaussian processes, etc. Moreover, dealing with recursive procedures in practice, in lieu of (4), one often uses a projection or truncation algorithm. For instance, one may use

$$\widehat{\theta}_{n+1} = \pi_H\left[\widehat{\theta}_n + \mu\varphi_n\left(y_n - \varphi_n'\widehat{\theta}_n\right)\right] \tag{8}$$

where π_H is a projection operator and H is a bounded set. When the iterates are outside H, it will be projected back to the constrained set H. Extensive discussions for such projection algorithms can be found in [17]. On the other hand, for the possibly unbounded signals, we can treat martingale difference sequences. With some modification, such an approach can also be used to treat moving average type signals.

In the subsequent development, we will concentrate mainly on the processes satisfying (6) and (7). The proof for the unbounded martingale difference sequence is simpler (for example, in the mean square estimate to follow, no perturbed Liapunov functions are needed). For brevity, we will omit the verbatim proof for such processes.

III. Mean-Square Error Bounds

This section establishes a mean-square error estimate for $\boldsymbol{E}|\widehat{\theta}_n - \theta_n|^2$. We obtain the desired estimate via a stability argument using perturbed Liapunov function methods. It is important to note that the mean-square error analysis below holds for small positive but fixed μ and ε. Indeed, let $\lambda_{\min} > 0$ denote the smallest eigenvalue of the symmetric positive definite matrix B defined in (A2). Then, in the following theorem it is sufficient to pick out μ and ε small enough so that

$$\lambda_{\min}\mu > O(\mu^2) + O(\varepsilon\mu); \tag{9}$$

see (25) in the proof below. The phrase "for sufficiently large n" in what follows means that there is an $n_0 = n_0(\varepsilon, \mu)$ such that (10) holds for $n \ge n_0$. In fact, (10) holds uniformly for $n \ge n_0$.

Theorem 6: Under conditions (A1) and (A2), for sufficiently large n, as $\varepsilon \to 0$ and $\mu \to 0$

$$\boldsymbol{E}|\widetilde{\theta}_n|^2 = \boldsymbol{E}|\widehat{\theta}_n - \theta_n|^2 = O\left(\mu + \varepsilon/\mu\right)\exp(\mu + \varepsilon/\mu). \tag{10}$$

Proof: It suffices to prove (10) for the Euclidian norm $|x| = (x'x)^{1/2}$. Define $V(x) = (x'x)/2$. Direct calculation leads to

$$\begin{aligned}\boldsymbol{E}_n V(\widetilde{\theta}_{n+1}) - V(\widetilde{\theta}_n) &= \boldsymbol{E}_n\left\{\widetilde{\theta}_n'\left[-\mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n + (\theta_n - \theta_{n+1})\right]\right\}\\ &\quad + \boldsymbol{E}_n\left|-\mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n + (\theta_n - \theta_{n+1})\right|^2. \end{aligned}\tag{11}$$

In view of the Markovian assumption, the independence of the Markov chain with the signals $\{(\varphi_n, e_n)\}$, and the structure of the transition probability matrix given by (3)

$$\begin{aligned}\boldsymbol{E}_n(\theta_n - \theta_{n+1}) &= \sum_{i=1}^{m}\boldsymbol{E}(\overline{\theta}_i - \theta_{n+1}|\theta_n = \overline{\theta}_i)I_{\{\theta_n=\overline{\theta}_i\}}\\ &= \sum_{i=1}^{m}\left[\overline{\theta}_i - \sum_{j=1}^{m}\overline{\theta}_j p_{ij}^{\varepsilon}\right]I_{\{\theta_n=\overline{\theta}_i\}}\\ &= -\varepsilon\sum_{i=1}^{m}\sum_{j=1}^{m}\overline{\theta}_j q_{ij} I_{\{\theta_n=\overline{\theta}_i\}}\\ &= O(\varepsilon). \end{aligned}\tag{12}$$

Using an elementary inequality $ab \le (a^2+b^2)/2$ for two real numbers a and b, we have

$$|\widetilde{\theta}_n| = \left|\widetilde{\theta}_n\right|\cdot 1 \le \left(|\widetilde{\theta}_n|^2 + 1\right)/2 \quad \text{so}$$
$$O(\varepsilon)|\widetilde{\theta}_n| \le O(\varepsilon)\left(V(\widetilde{\theta}_n) + 1\right).$$

By virtue of the boundedness of the signal $\{(\varphi_n, e_n)\}$

$$\boldsymbol{E}_n\left|-\mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n + (\theta_n - \theta_{n+1})\right|^2 = O(\mu^2 + \mu\varepsilon + \varepsilon)(V(\widetilde{\theta}_n) + 1).$$

Using the above two inequalities in (11) together with (12) yields

$$\begin{aligned}\boldsymbol{E}_n V(\widetilde{\theta}_{n+1}) - V(\widetilde{\theta}_n) &= \boldsymbol{E}_n\left\{\widetilde{\theta}_n'\left[-\mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n\right]\right\}\\ &\quad + O(\varepsilon)(V(\widetilde{\theta}_n) + 1)\\ &\quad + O(\mu^2 + \mu\varepsilon)(V(\widetilde{\theta}_n) + 1). \end{aligned}\tag{13}$$

To obtain the desired estimate, we need to "average out" the terms inside the curly bracket {} in (13). Roughly speaking, if φ_n, e_n were independent and also independent of $\widetilde{\theta}_n$, then if $\boldsymbol{E}_n\{\varphi_n\varphi_n'\} = B$, the expectation of the term in the curly brackets would be $-\mu\widetilde{\theta}_n B\theta_n$. The idea that follows is to extend this rigorously to correlated noise (of mixing type) $\{\varphi_n, e_n\}$ that satisfies (A2) by using perturbed Liapunov functions.

To do so, define two perturbations of the Liapunov function by

$$V_1^\varepsilon(\widetilde{\theta}, n) = -\mu \sum_{j=n}^{\infty} \boldsymbol{E}_n \widetilde{\theta}' \left(\varphi_j\varphi_j' - B\right)\widetilde{\theta}$$
$$V_2^\varepsilon(\widetilde{\theta}, n) = \mu \sum_{j=n}^{\infty} \widetilde{\theta}' \boldsymbol{E}_n \varphi_j e_j.$$

For each $\widetilde{\theta}$, by virtue of (A2), it is easily verified that

$$\mu \left| \sum_{j=n}^{\infty} \left[\boldsymbol{E}_n \varphi_j\varphi_j' - B\right]\right| |\widetilde{\theta}|^2 \le O(\mu)(V(\widetilde{\theta})+1)$$

so

$$|V_1^\varepsilon(\widetilde{\theta}, n)| \le O(\mu)(V(\widetilde{\theta})+1). \tag{14}$$

Similarly, for each $\widetilde{\theta}$

$$|V_2^\varepsilon(\widetilde{\theta}, n)| \le O(\mu)(V(\widetilde{\theta})+1). \tag{15}$$

Note that

$$\begin{aligned}\boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\theta}_n, n) &= \boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_{n+1}, n+1) - \boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_n, n+1)\\ &\quad + \boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_n, n+1) - V_1^\varepsilon(\widetilde{\theta}_n, n). \end{aligned}\tag{16}$$

It follows that

$$\boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_n, n+1) - V_1^\varepsilon(\widetilde{\theta}_n, n) = \mu \boldsymbol{E}_n \widetilde{\theta}_n' \left(\varphi_n\varphi_n' - B\right)\widetilde{\theta}_n \tag{17}$$

by virtue of (A2). In addition

$$\begin{aligned}\boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_{n+1}, n+1) &- \boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_n, n+1)\\ &= -\mu \sum_{j=n+1}^{\infty} \boldsymbol{E}_n (\widetilde{\theta}_{n+1} - \widetilde{\theta}_n)' \left[\boldsymbol{E}_{n+1}\varphi_j\varphi_j' - B\right]\widetilde{\theta}_{n+1}\\ &\quad - \mu \sum_{j=n+1}^{\infty} \boldsymbol{E}_n \widetilde{\theta}_n' \left[\boldsymbol{E}_{n+1}\varphi_j\varphi_j' - B\right](\widetilde{\theta}_{n+1} - \widetilde{\theta}_n). \end{aligned}\tag{18}$$

Using (5), similar estimates as that of (12) yield

$$\begin{aligned}\boldsymbol{E}_n|\widetilde{\theta}_{n+1} - \widetilde{\theta}_n| &\le \mu \boldsymbol{E}_n\left|\varphi_n\varphi_n'\right||\widetilde{\theta}_n| + \mu\boldsymbol{E}_n|\varphi_n e_n| + O(\varepsilon)\\ &= O(\mu)(V(\widetilde{\theta}_n)+1) + O(\varepsilon). \end{aligned}\tag{19}$$

Moreover,

$$\begin{aligned}&\left|\mu \sum_{j=n+1}^{\infty} \boldsymbol{E}_n \widetilde{\theta}_n' \left[\boldsymbol{E}_{n+1}\varphi_j\varphi_j' - B\right](\widetilde{\theta}_{n+1} - \widetilde{\theta}_n)\right|\\ &\quad \le K\mu \boldsymbol{E}_n \sum_{j=n+1}^{\infty} |\boldsymbol{E}_{n+1}\varphi_j\varphi_j' - B||\widetilde{\theta}_n||\widetilde{\theta}_{n+1} - \widetilde{\theta}_n|\\ &\quad \le K\mu|\widetilde{\theta}_n|\boldsymbol{E}_n|\widetilde{\theta}_{n+1} - \widetilde{\theta}_n|\\ &\quad \le O(\mu^2 + \mu\varepsilon)(V(\widetilde{\theta}_n)+1). \end{aligned}\tag{20}$$

Likewise

$$\begin{aligned}&\left|-\mu \sum_{j=n+1}^{\infty} \boldsymbol{E}_n (\widetilde{\theta}_{n+1} - \widetilde{\theta}_n)' \left[\boldsymbol{E}_{n+1}\varphi_j\varphi_j' - B\right]\widetilde{\theta}_{n+1}\right|\\ &\qquad \le O(\mu^2 + \mu\varepsilon)(V(\widetilde{\theta}_n)+1). \end{aligned}\tag{21}$$

Thus, we arrive at

$$\begin{aligned}&\boldsymbol{E}_n V_1^\varepsilon(\widetilde{\theta}_{n+1}, n+1) - V_1^\varepsilon(\widetilde{\theta}_n, n)\\ &\quad = \mu \boldsymbol{E}_n \widetilde{\theta}_n' \left(\varphi_n\varphi_n' - B\right)\widetilde{\theta}_n + O(\mu^2 + \mu\varepsilon)(V(\widetilde{\theta}_n)+1). \end{aligned}\tag{22}$$

An analogues estimate for $V_2^\varepsilon(\widetilde{\theta}_n, n)$ yields that

$$\begin{aligned}\boldsymbol{E}_n V_2^\varepsilon(\widetilde{\theta}_{n+1}, n+1) &- V_2^\varepsilon(\widetilde{\theta}_n, n)\\ &= O(\mu^2 + \mu\varepsilon)(V(\widetilde{\theta}_n)+1). \end{aligned}\tag{23}$$

Define

$$W(\widetilde{\theta}, n) = V(\widetilde{\theta}) + V_1^\varepsilon(\widetilde{\theta}, n) + V_2^\varepsilon(\widetilde{\theta}, n).$$

Then, using (11), (22), and (23), we obtain

$$\begin{aligned}\boldsymbol{E}_n W(\widetilde{\theta}_{n+1}, n+1) &- W(\widetilde{\theta}_n, n)\\ &\le -\mu\widetilde{\theta}_n' B\widetilde{\theta}_n + O(\mu^2 + \mu\varepsilon + \varepsilon)(V(\widetilde{\theta}_n)+1). \end{aligned}\tag{24}$$

Since B is positive definite, there is a $\lambda > 0$ such that $\widetilde{\theta}' B\widetilde{\theta} \ge \lambda V(\widetilde{\theta})$. For example, from the Rayleigh–Ritz theorem, $\lambda = \lambda_{\min}$ satisfies $\widetilde{\theta}' B\widetilde{\theta} \ge \lambda V(\widetilde{\theta})$, where $\lambda_{\min} > 0$ is the smallest eigenvalue of B. This together with (14) and (15) implies

$$\begin{aligned}\boldsymbol{E}_n W(\widetilde{\theta}_{n+1}, n+1) &- W(\widetilde{\theta}_n, n)\\ &\le -\lambda\mu W(\widetilde{\theta}_n, n) + O(\mu^2 + \mu\varepsilon + \varepsilon)(W(\widetilde{\theta}_n, n)+1). \end{aligned}$$

Choose μ and ε small enough so that there is a $\lambda_0 > 0$ satisfying $\lambda_0 \le \lambda$ and

$$-\lambda\mu + O(\mu^2) + O(\mu\varepsilon) \le -\lambda_0\mu. \tag{25}$$

Note that this is equivalent to (9). Then for such a small fixed μ and ε, we obtain (using $(O(\mu^2 + \varepsilon\mu + \varepsilon) \le O(\mu^2 + \varepsilon))$

$$\begin{aligned}\boldsymbol{E}_n W(\widetilde{\theta}_{n+1}, n+1) &\le (1 - \lambda_0\mu)W(\widetilde{\theta}_n, n)\\ &\quad + O(\varepsilon)W(\widetilde{\theta}_n, n) + O(\mu^2 + \varepsilon). \end{aligned}$$

Taking expectation above and iterating on the resulting inequality yields

$$\begin{aligned}\boldsymbol{E}W(\widetilde{\theta}_{n+1}, n+1) &\le (1 - \lambda_0\mu)^n \boldsymbol{E}W(\widetilde{\theta}_0, 0)\\ &\quad + \varepsilon\sum_{j=0}^{n}(1 - \lambda_0\mu)^{n-j}\boldsymbol{E}W(\widetilde{\theta}_j, j) + O(\mu + \varepsilon/\mu). \end{aligned}$$

By taking n large enough, we can make $(1 - \lambda_0\mu)^n \le O(\mu)$. Thus, an application of the Gronwall's inequality leads to

$$\begin{aligned}\boldsymbol{E}W(\widetilde{\theta}_{n+1}, n+1) &\le O(\mu + \varepsilon/\mu)\exp\left(\varepsilon\sum_{j=0}^{n}(1 - \lambda_0\mu)^{n-j}\right)\\ &\le O(\mu + \varepsilon/\mu)\exp(\varepsilon/\mu). \end{aligned}$$

Finally, applying (14) and (15) again, we also obtain

$$\boldsymbol{E}V(\widetilde{\theta}_{n+1}) \le O(\mu + \varepsilon/\mu)\exp(\varepsilon/\mu).$$

Thus, the desired result follows. □

In view of Theorem 6, it is clear that in order that the adaptive algorithm can track the time-varying parameter, due to the presence of the term $\exp(\varepsilon/\mu)$, we need to have at least $\varepsilon/\mu = O(1)$. Thus, the ratio ε/μ must not be large. A glance of the order of magnitude estimate $O(\mu+\varepsilon/\mu)$, to balance the two terms μ and ε/μ, we need to choose $\varepsilon = O(\mu^2)$. Therefore, we arrive at the following corollary.

Corollary 7: Under the conditions of Theorem 6, if $\varepsilon = O(\mu^2)$, then for sufficiently large n, $\boldsymbol{E}V(\widetilde{\theta}_n) = O(\mu)$.

IV. Mean ODE and Diffusion Approximation

This section is devoted to getting the mean dynamics of the tracking error and the diffusion approximation limit under suitable scales. The Markov chain that characterizes the evolution of the parameter θ_n we are working with in fact is ε-dependent. That is, θ_n should have been written as θ_n^ε. Nevertheless, for notational simplicity, we have suppressed the ε-dependence up to this point. Due to the form of the transition matrix given by (3), the underlying Markov chain is one belonging to the category of two-time-scale Markov chains. For some of the recent work on this subject, we refer the reader to [38], [39] and the references therein.

Here and in the following sections, we assume $\varepsilon = O(\mu^2)$ (see Corollary 7), i.e., the adaptation speed of the LMS algorithm (4) is faster than the Markov chain dynamics. Recall that the mean-square error analysis in Section III deals with the mean-square behavior of the random variable $\widetilde{\theta}_n = \widehat{\theta}_n - \theta_n$ as $n \to \infty$, for small but fixed μ and ε. In contrast, the mean ODE and diffusion approximation analysis of this section deal with how the entire discrete-time trajectory (stochastic process) $\{\widetilde{\theta}_n : n = 0, 1, 2 \ldots, \}$ converges (weakly) to a the limiting continuous-time process (on a suitable function space) as $\mu \to 0$ on a time scale $O(1/\mu)$. In Section IV-B, we show that this limiting continuous-time process satisfies an ODE. The next step is to determine the asymptotic error distribution—or, equivalently, the rate of convergence. In Section IV-C, we show that the limiting continuous-time process for the discrete-time process $\{\widetilde{\theta}_n/\sqrt{\mu}\}$ satisfies a linear diffusion. Since the underlying true parameter θ_n evolves according to a Markov chain (unlike standard stochastic approximation proofs, where the parameter is assumed constant), the proofs of the ODE and diffusion limit in Section IV-C are nonstandard and require use of the so-called "martingale problem" formulation.

To proceed, we first provide some preliminary results on two-time-scale Markov chains relevant to our problem. Then, in the second part, we will use these auxiliary results to derive the limit diffusion process.

A. Properties of θ_n

Define a probability vector by

$$p_n^\varepsilon = (P(\theta_n = \overline{\theta}_1), \ldots, P(\theta_n = \overline{\theta}_m)) \in \mathbb{R}^{1\times m}.$$

Note that $p_0^\varepsilon = (p_{0,1}, \ldots, p_{0,m})$, which is given in (A1). Note also that $(P^\varepsilon)^n$ is the n-step transition probability matrix with P^ε given by (3) since the Markov chain is time homogeneous.

Recall from (A1) that Q is the generator of a m-state continuous Markov chain. Then the $1 \times m$-dimensional state probability vector $\pi(\tau) = (\pi_1(\tau), \ldots, \pi_m(\tau))$ of this continuous Markov chain at any continuous time $\tau \in \mathbb{R}^+$ satisfies the Chapman–Kolmogorov equation

$$\frac{d\pi(\tau)}{d\tau} = \pi(\tau)Q, \qquad \pi(0) = p_0, \tag{26}$$

where p_0 is the initial probability defined in (A1).

Regarding the probability vector p_n^ε and the $(n - n_0)$-step transition matrix, we have the following approximation results.

Lemma 8: Assume (A1). Then the following assertions hold.

a) For some $\kappa_0 > 0$

$$p_n^\varepsilon = \pi(\varepsilon n) + O(\varepsilon + \exp(-\kappa_0 n)), \quad 0 \le n \le O(1/\varepsilon) \tag{27}$$

where $\pi(\varepsilon n)$ is defined in (26) with $\tau = \varepsilon n$. In addition

$$(P^\varepsilon)^{n-n_0} = \Xi(\varepsilon n_0, \varepsilon n) + O(\varepsilon + \exp(-\kappa_0(n - n_0))) \tag{28}$$

where with $\tau_0 = \varepsilon n_0$ and $\tau = \varepsilon n$, $\Xi(\tau_0, \tau)$ satisfies

$$\begin{cases} \frac{d\Xi(\tau_0,\tau)}{d\tau} = \Xi(\tau_0, \tau)Q \\ \Xi(\tau_0, \tau_0) = I. \end{cases} \tag{29}$$

b) Define the continuous-time interpolation $\theta^\varepsilon(t)$ of θ_n as

$$\theta^\varepsilon(t) = \theta_n, \quad \text{if } t \in [n\varepsilon, n\varepsilon + \varepsilon). \tag{30}$$

Then $\theta^\varepsilon(\cdot)$ converges weakly to $\theta(\cdot)$, which is a continuous-time Markov chain generated by Q with state space $\mathcal{M}$.

Proof: The proof for a) is essentially in that of Theorems 3.5 and 4.3 of [39], whereas the proof of b) can be found in [40]. □

Remark 9: Let us first explain the idea a little bit. In view of [39], we can consider the transition matrix (3) as one that includes all recurrent states. The dominating part of the matrix P^ε can be thought of as $I = \mathrm{diag}(1, 1, \ldots, 1) \in \mathbb{R}^{m\times m}$. Each of the scalar 1 can be viewed as an irreducible transition matrix. Thus, the result of the aforementioned paper can be applied. The statement of Lemma 8 indicates that the probability vector p_n^ε and the transition matrix $(P^\varepsilon)^{n-n_0}$ can be approximated by $\pi(\varepsilon n)$ and $\Xi(\varepsilon n_0, \varepsilon n)$, respectively. The errors turn out to be of the order $O(\varepsilon + \exp(-\kappa_0 n))$ for $n \le O(1/\varepsilon)$.

In fact, a full asymptotic series was constructed in [39]. However, for our purpose, the leading term is sufficient. The error term $O(\varepsilon + \exp(-\kappa_0 n))$ comes from the initial layer corrections. Note that in [39], the running time is k instead of n, and the notation for $\pi(\varepsilon n)$ in that paper is $\theta(\varepsilon n)$. In addition, here we have used $\exp(-\kappa_0 n)$ in lieu of λ^n for a $0 < \lambda < 1$ as in [39], which are equivalent.

As a corollary of Lemma 8, we obtain an approximation of expected value of θ_n. As a further direct consequence, we can verify that $\theta_n - \boldsymbol{E}\theta_n$ is a mixing sequence with exponential mixing rate. We state the results below for future reference.

Corollary 10: The $\boldsymbol{E}\theta_n$ can be approximated by

$$\boldsymbol{E}\theta_n = \overline{\theta}_*(\varepsilon n) + O(\varepsilon + \exp(-\kappa_0 n)), \quad \text{for } n \le O(1/\varepsilon)$$
$$\overline{\theta}_*(\varepsilon n) \stackrel{\text{def}}{=} \sum_{j=1}^m \overline{\theta}_j \pi_j(\varepsilon n).$$

Proof: Direct computation reveals that for $n \le O(1/\varepsilon)$

$$\begin{aligned}\boldsymbol{E}\theta_n &= \sum_{j=1}^{m} \overline{\theta}_j P(\theta_n = \overline{\theta}_j) \\ &= \sum_{j=1}^{m} \overline{\theta}_j \pi_j(\varepsilon n) + \sum_{j=1}^{m} \overline{\theta}_j O(\varepsilon + \exp(-\kappa_0 n)) \\ &= \sum_{j=1}^{m} \overline{\theta}_j \pi_j(\varepsilon n) + O(\varepsilon + \exp(-\kappa_0 n)). \end{aligned} \tag{31}$$

The proof of the corollary is thus concluded. □

Corollary 11: Define $\xi_n = \theta_n - \boldsymbol{E}\theta_n$. Then for $n \le O(1/\varepsilon)$

$$\begin{aligned} |\boldsymbol{E}_k \xi_{n+k} - \boldsymbol{E}\xi_{n+k}| &\le K \exp(-\kappa_0 n) \\ |\boldsymbol{E}\xi_{n+k}\xi_k - \boldsymbol{E}\xi_{n+k}\boldsymbol{E}\xi_k| &\le K \exp(-\kappa_0 n). \end{aligned} \tag{32}$$

B. Mean ODE

This subsection is devoted to deriving the limit dynamic system resulted in interpolation of centered tracking errors. We examine $\widetilde{\theta}_n = \widehat{\theta}_n - \theta_n$. Define

$$\widetilde{\theta}^\mu(t) = \widetilde{\theta}_n, \qquad \text{for } t \in [n\mu, n\mu + \mu). \tag{33}$$

Then $\widetilde{\theta}^\mu(\cdot) \in D^r[0, \infty)$, the space of $\mathbb{R}^r$-valued functions that are right continuous, have left-hand limits endowed with the Skorohod topology [2], [6]. We will need another condition.

(A3) As $n \to \infty$

$$\frac{1}{n}\sum_{j=n_1}^{n_1+n} \boldsymbol{E}_{n_1}\varphi_j e_j \to 0, \text{ in probability}$$
$$\frac{1}{n}\sum_{j=n_1}^{n_1+n} \boldsymbol{E}_{n_1}\varphi_j \varphi_j' \to B, \text{ in probability.}$$

In the following analysis, we need to deal with a term of the form

$$\sum_{k=t/\mu}^{(t+s)/\mu-1} (\theta_k - \theta_{k+1})/\sqrt{\mu}.$$

Here and henceforth, quantities such as t/μ and $(t+s)/\mu$ are meant to be their integer parts, for notational simplicity. For subsequent use, we present the following lemma.

Lemma 12: As $\mu \to 0$

$$\boldsymbol{E}\left|\sum_{k=t/\mu}^{(t+s)/\mu-1} (\theta_k - \theta_{k+1})\right| = O(\mu); \tag{34}$$

for $0 < s$ satisfying $s\mu < 1$

$$\boldsymbol{E}\left[\left|\sum_{k=t/\mu}^{(t+s)/\mu-1} (\theta_k - \theta_{k+1})\right|^2 |\theta_j : j \le t/\mu\right] = O(s\mu). \tag{35}$$

Proof: To verify the first equality, noting the elementary property

$$P(\theta_k = \overline{\theta}_i, \theta_{k+1} = \overline{\theta}_j) = P(\theta_k = \overline{\theta}_i) P(\theta_{k+1} = \overline{\theta}_j | \theta_k = \overline{\theta}_i)$$

the fact $p_{ij}^\varepsilon = \delta_{ij} + \varepsilon q_{ij}$ given by (3), and $P(\theta_k = \overline{\theta}_i) \le 1$, for all k, we have

$$\begin{aligned}\boldsymbol{E}\left|\sum_{k=t/\mu}^{(t+s)/\mu-1} (\theta_k - \theta_{k+1})\right| &\le \sum_{k=t/\mu}^{(t+s)/\mu-1} \boldsymbol{E}|\theta_k - \theta_{k+1}| \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=t/\mu}^{(t+s)/\mu-1} |\overline{\theta}_i - \overline{\theta}_j| P(\theta_k = \overline{\theta}_i, \theta_{k+1} = \overline{\theta}_j) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=t/\mu}^{(t+s)/\mu-1} |\overline{\theta}_i - \overline{\theta}_j| P(\theta_k = \overline{\theta}_i)(\delta_{ij} + \varepsilon q_{ij}) \\ &= K\varepsilon/\mu = O(\mu). \end{aligned}$$

As for the second inequality, by the Markov property and using telescoping

$$\begin{aligned}&\boldsymbol{E}\left[\left|\sum_{k=t/\mu}^{(t+s)/\mu-1} (\theta_k - \theta_{k+1})\right|^2 |\theta_l : l \le t/\mu\right] \\ &= \boldsymbol{E}\left[\left|\theta_{t/\mu} - \theta_{(t+s)/\mu}\right|^2 \Big| \theta_{t/\mu}\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} |\overline{\theta}_i - \overline{\theta}_j|^2 \\ &\quad \times P\left(\theta_{(t+s)/\mu} = \overline{\theta}_j | \theta_{t/\mu} = \overline{\theta}_i\right) I_{\{\theta_{t/\mu} = \overline{\theta}_i\}}. \end{aligned} \tag{36}$$

Using the transition matrix given by (3) and Newton's binomial expansions, the (s/μ)-step transition matrix

$$\begin{aligned}(I + \varepsilon Q)^{s/\mu} &= \sum_{l=0}^{s/\mu} \binom{(s/\mu)}{l} (\varepsilon Q)^l I^{(s/\mu)-l} \\ &= I + O\left(\sum_{l=1}^{s/\mu} (s/\mu)^l \varepsilon^l\right) \\ &= I + O\left(\sum_{l=1}^{s/\mu} (s\mu)^l\right) = I + O(s\mu) \end{aligned} \tag{37}$$

since $\varepsilon = O(\mu^2)$. Using (37) in (36) yields (35). □

In view of Lemma 12 together with (5), we have

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \mu\varphi_n\varphi_n'\widetilde{\theta}_n + \mu\varphi_n e_n + \rho_n \tag{38}$$

where $\boldsymbol{E}\rho_n = O(\mu)$. We state the assertion regarding the limit ODE as follows.

Theorem 13: Under (A1)–(A3) and assuming that $\widetilde{\theta}_0 = \widetilde{\theta}_0^\mu$ converges weakly to $\widetilde{\theta}^0$, then $\widetilde{\theta}^\mu(\cdot)$ defined in (33) converges weakly to $\widetilde{\theta}(\cdot)$, which is a solution of the ODE

$$\frac{d}{dt}\widetilde{\theta}(t) = -B\widetilde{\theta}(t), \qquad t \ge 0, \ \widetilde{\theta}(0) = \widetilde{\theta}^0. \tag{39}$$

Remark 14: This theorem provides us with the evolution of the tracking errors. It shows that $\widehat{\theta}_n - \theta_n$ evolves dynamically so that its trajectories follows a deterministic ODE. Since the ODE is asymptotically stable, the errors decay exponentially fast to 0 as time grows.

Since the proof of the diffusion approximation in the following section uses similar techniques and is more difficult, we omit the proof here. However, we prove the following corollary holds.

Corollary 15: Under the conditions of Theorem 13, for any sequence $t_\mu \to \infty$ as $\mu \to 0$, $\widetilde{\theta}^\mu(\cdot + t_\mu)$ converges to 0 in probability.

Proof: Consider the process $\breve{\theta}^\mu(\cdot) = \widetilde{\theta}^\mu(\cdot + t_\mu)$. Using the same argument as in the proof of Theorem 13, $\{\breve{\theta}^\mu(\cdot), \breve{\theta}^\mu(\cdot - T)\}$ is tight. Extract a convergent subsequence (still index it by μ) with limit $(\breve{\theta}(\cdot), \breve{\theta}_T(\cdot))$, which satisfies the ODE (39). Note that $\breve{\theta}(0) = \breve{\theta}_T(T)$. By virtue of Theorem 6, $\{\widetilde{\theta}_n\}$ is tight, so is $\{\breve{\theta}_T(0)\}$. Moreover, the limit ODE leads to

$$\breve{\theta}_T(T) = \exp(-BT)\breve{\theta}_T(0) \to 0, \quad \text{as } T \to \infty.$$

The desired assertion then follows. □

C. Diffusion Limit

We need one more condition.

(A4) $\sqrt{\mu}\sum_{j=n}^{n+t/\mu} \boldsymbol{E}_n \varphi_j e_j$ converges weakly to a Brownian motion with a covariance Σt for a positive-definite Σ. Moreover

$$\left| \sum_{j=n}^{n+t/\mu} \sum_{k=n}^{n+t/\mu} \boldsymbol{E}_n \varphi_j \varphi_k' e_j e_k \right| \le K.$$

To proceed, define $u_n = (\widehat{\theta}_n - \theta_n)/\sqrt{\mu}$. We obtain

$$u_{n+1} = u_n - \mu\varphi_n\varphi_n' u_n + \sqrt{\mu}\varphi_n e_n + \frac{\theta_n - \theta_{n+1}}{\sqrt{\mu}}. \quad (40)$$

The following lemma holds.

Lemma 16: Under conditions (A1)–(A4), there is an N_μ such that $\{u_n : n \ge N_\mu\}$ is tight.

Proof: This is a corollary of Theorem 6. □

Next, define the continuous-time interpolation $u^\mu(\cdot)$ as

$$u^\mu(t) = u_n \text{ for } t \in [\mu(n - N_\mu), \mu(n - N_\mu) + \mu). \quad (41)$$

Remark 17: Note that in (A4), we have assumed the convergence to a Brownian motion. Sufficient conditions guaranteeing the convergence of the scaled sequence to the Brownian motion in (A4) are readily available in the literature; see, for example, [2], [6] among others. For instance, if $\{\varphi_n e_n\}$ is a uniform mixing sequence with mixing rate ψ_n satisfying $\sum_k \psi_k^{1/2} < \infty$, the classical result in functional central limit theorem (see [2], [6]) implies that $\sqrt{\mu}\sum_j^{t/\mu - 1} \varphi_j e_j$ converges weakly to a Brownian motion process $\widetilde{w}(t)$ whose covariance is given by Σt, where

$$\Sigma = \boldsymbol{E}\varphi_0\varphi_0' e_0^2 + \sum_{j=1}^{\infty} \boldsymbol{E}\varphi_j\varphi_0' e_j e_0 + \sum_{j=1}^{\infty} \boldsymbol{E}\varphi_0\varphi_j' e_0 e_j. \quad (42)$$

Theorem 18: Assume that (A1)–(A4) hold, and that $u^\mu(0)$ converges weakly to u^0. Then the interpolated process $u^\mu(\cdot)$ defined in (41) converges weakly to $u(\cdot)$, which is the solution of the stochastic differential equation

$$du(t) = -Bu(t)dt + \Sigma^{1/2} dw(t), \qquad u(0) = u^0 \quad (43)$$

where $w(\cdot)$ is a standard r-dimensional Brownian motion, B and Σ are given in (A2), Section II and (A3), Section IV, and $\Sigma^{1/2}$ denotes the square root of Σ (i.e., $\Sigma = \Sigma^{1/2}(\Sigma^{1/2})'$).

Remark 19: Since B is symmetric positive definite, the matrix $-B$ is negative definite and hence Hurwitz (i.e., all of its eigenvalue being negative). It follows that

$$\widetilde{\Sigma} = \int_0^\infty \exp(-Bt)\Sigma\exp(-Bt)dt \quad (44)$$

is well defined. In fact, $\widetilde{\Sigma}$ is the stationary covariance of the diffusion process and can be obtained as the solution of the Liapunov equation

$$\widetilde{\Sigma}B + B\widetilde{\Sigma} = \Sigma; \quad (45)$$

see [17, Ch. 10] for further discussion. In view of Theorem 18, $\widehat{\theta}_n - \theta_n$ is asymptotically normally distributed with mean 0 and covariance $\mu\widetilde{\Sigma}$. This covariance is identical to that of the constant step size LMS algorithm estimating a constant parameter (i.e., $\varepsilon = 0$). The expression for the asymptotic covariance is not surprising. Since θ_n is a slowly varying Markov chain, its structure of transition probability matrix P^ε makes the chain act almost like a constant parameter, with infrequent jumps. As a result, the process with suitable scaling leads to a diffusion limit, whose stationary covariance is the solution of the Liapunov (45).

Remark 20: Before proceeding with the proof of Theorem 18, it is worthwhile considering the following intuitive explanation. Consider the first-order time discretization of the stochastic differential equation (43) with discretization interval μ (see (41)) equal to the step size of the LMS algorithm. The resulting discretized system is

$$u_{n+1} = u_n - \mu B u_n + \sqrt{\mu}\Sigma^{1/2} v_n \quad (46)$$

where $v_n = w_{n+1} - w_n$ is a discrete-time white Gaussian noise. By comparing (46) with (40), it is intuitively clear that they are equivalent in distribution. In particular, by stochastic averaging principle, the fast variable $\phi_n\phi_n'$ behaves as its average B yielding the second term in (46). The equivalence in distribution of the third terms in (46) and (40) can be seen similarly. Thus, discretizing the continuous-time process (43) yields (at least intuitively) the discrete-time process (46).

The proof that follows goes in the reverse direction, i.e., it shows that as the discretization interval $\mu \to 0$, the discrete-time process (40) converges weakly (in distribution) to the continuous-time process (43) under suitable technical assumptions. The key technical assumption for this weak convergence of a discrete-time process to a continuous-time process is that between discrete-time sample points, the process should be well behaved in distribution. This well behavedness is captured by the tightness assumption which roughly speaking states that between sample points, the process is bounded in probability. The main tool used below to prove the weak convergence is the "martingale problem" formulation; see Step 2 of the proof.

Proof of Theorem 18: To establish the result, we use the martingale averaging techniques (see [17, Chs. 8 and 10]), which requires the tightness of the sequence be verified and the limit be identified as a solution of an appropriate martingale problem. The proof is divided into several steps.

Step 1: It is difficult to verify the tightness of $\{u^\mu(\cdot)\}$. To circumvent the difficulty, we use an N-truncation device. To be more specific, for each $N > 0$, let $S_N = \{u : |u| \le N\}$ be the sphere with radius N and let a smooth truncation function be defined by

$$q^N(u) = \begin{cases} 1, & \text{if } u \in S_N \\ 0, & \text{if } u \in \mathbb{R}^r - S_{N+1}. \end{cases}$$

Let

$$u_{n+1}^N = u_n^N - \mu\varphi_n\varphi_n' u_n^N q^N\left(u_n^N\right) + \sqrt{\mu}\varphi_n e_n + \frac{\theta_n - \theta_{n+1}}{\sqrt{\mu}}.$$

Define the corresponding interpolated process $u^{\mu,N}(\cdot)$ by

$$u^{\mu,N}(t) = u_n^N, \qquad \text{for } t \in [\mu(n - N_\mu), \mu(n - N_\mu) + \mu).$$

Then $u^{\mu,N}(\cdot)$ is an N-truncation for $u^\mu(\cdot)$ [17, p. 284].

Lemma 21: Under (A1)–(A3), $\{u^{\mu,N}(\cdot)\}$ is tight.

Proof of Lemma 21: For each $\delta > 0$, $t > 0$, and $0 < s < \delta$, we have

$$\begin{aligned} &\boldsymbol{E}_t^\mu |u^{\mu,N}(t+s) - u^{\mu,N}(t)|^2 \\ &\quad\le K\mu^2 \boldsymbol{E}_t^\mu \left| \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j\varphi_j' u_j^N q^N\left(u_j^N\right)\right|^2 \\ &\qquad + K\mu \left| \sum_{j=t/\mu}^{(t+s)/\mu-1} \sum_{k=t/\mu}^{(ts+s)/\mu-1} \boldsymbol{E}_t^\mu \varphi_j'\varphi_k e_j e_k \right| \\ &\qquad + \frac{K}{\mu}\boldsymbol{E}_t^\mu \left| \sum_{j=t/\mu}^{(t+s)/\mu-1} [\theta_j - \theta_{j+1}] \right|^2. \end{aligned} \tag{47}$$

By virtue of the boundedness of $\{u_n^N\}$ and $\{\varphi_n\varphi_n'\}$, the first term on the right-hand side of (47) is bounded by $O(s^2)$. In view of (A3) part a), the second term is bounded by $O(s)$. Using (35) and noting $0 < s < \delta$

$$\lim_{\delta\to 0}\limsup_{\mu\to 0} \frac{1}{\mu} E_t^\mu |\theta_{(t+s)/\mu} - \theta_{t/\mu}|^2 = \lim_{\delta\to 0} O(\delta) = 0.$$

Consequently, it is easily verified that

$$\lim_{\delta\to 0}\limsup_{\mu\to 0} \boldsymbol{E}|u^{\mu,N}(t+s) - u^{\mu,N}(t)|^2 = 0.$$

Thus $\{u^{\mu,N}(\cdot)\}$ is tight. □

Step 2: Extract a weakly convergent subsequence, and still denote it by $u^{\mu,N}(\cdot)$ for notational simplicity. Denote the limit by $u^N(\cdot)$. We have

$$\begin{aligned} u^{\mu,N}(t+s) - u^{\mu,N}(t) &= -\mu \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j\varphi_j' u_j^N q^N\left(u_j^N\right) \\ &\quad + \sqrt{\mu} \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j e_j + \sum_{j=t/\mu}^{(t+s)/\mu-1} \frac{\theta_j - \theta_{j+1}}{\sqrt{\mu}}. \end{aligned} \tag{48}$$

By virtue of (34) in Lemma 12

$$\boldsymbol{E}\left| \sum_{j=t/\mu}^{(t+s)/\mu-1} \frac{\theta_j - \theta_{j+1}}{\sqrt{\mu}} \right| = O(\mu)/\sqrt{\mu} = O(\sqrt{\mu}).$$

Thus, it does not contribute anything to the limit, and

$$\begin{aligned} u^{\mu,N}(t+s) - u^{\mu,N}(t) &= -\mu \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j\varphi_j' u_j^N q^N\left(u_j^N\right) \\ &\quad + \sqrt{\mu} \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j e_j + o(1) \end{aligned} \tag{49}$$

where $o(1) \to 0$ in probability as $\mu \to 0$ uniformly in t. Therefore, we can ignore the $o(1)$ term in (49) henceforth.

We proceed to use the martingale averaging techniques [17, Ch. 8] to complete the proof. We will show that the $u^{\mu,N}(\cdot)$ converges weakly to $u^N(\cdot)$, which is a solution of the truncated stochastic differential equation

$$du^N = -Bu^N q^N(u^N)dt + \Sigma^{1/2}dw \tag{50}$$

where $w(\cdot)$ is a standard Brownian motion, and B and Σ given by (A3). The key result used below is that proving $u^{\mu,N}(\cdot)$ converges weakly to $u^N(\cdot)$ in (50) is equivalent to verifying that $u^N(\cdot)$ is a solution of the *martingale problem* with operator

$$\mathcal{L}^N f(u^N) = \frac{1}{2}\text{tr}[\Sigma\nabla^2 f(u^N)] - \nabla f'(u^N)Bu^N q^N(u^N) \tag{51}$$

for any C^2 function with compact support. Note that $\mathcal{L}^N$ is the generator of the diffusion (50), and ∇f and $\nabla^2 f$ denote the gradient and Hessian of $f(\cdot)$, respectively. The phrase "$u^N(\cdot)$ is a solution of the martingale problem" means that we need to show that

$$f(u^N(t+s)) - f(u^N(t)) - \int_t^{t+s} \mathcal{L}^N f(u^N(\tau))d\tau$$

is a continuous-time martingale. A necessary and sufficient condition for $u^N(\cdot)$ to satisfy the above continuous martingale property is that (see [6, p. 174]) for any arbitrary positive integer n_0, for all $j \le n_0$, bounded and continuous functions $h_j(\cdot)$, and $0 < t_j \le t \le t+s$

$$\boldsymbol{E}\prod_{j=1}^{n_0} h_j(u^N(t_j)) \left[f(u^N(t+s)) - f(u^N(t)) - \int_t^{t+s} \mathcal{L}^N f(u^N(\tau))d\tau \right] = 0. \tag{52}$$

Thus, to prove $u^{\mu,N}(\cdot)$ converges weakly to $u^N(\cdot)$, we only need to show that (52) holds. Note that (52) relates to distribution of $u^N(\cdot)$ at times $t_1, \ldots, t_{n_0}$, i.e., finite-dimensional distributions.

To verify (52), we work with the sequence indexed by μ, namely, $\{u^{\mu,N}(\cdot)\}$. Since the development is along the line of [17, Ch. 10], we will not provide the details; we will also suppress the function $f(\cdot)$ for simplicity.

Choose a sequence of integers $\{n_\mu\}$ such that $n_\mu \to \infty$ as $\mu \to 0$ but $\mu n_\mu = \delta_\mu \to 0$. By virtue of the continuity and the smoothness of $q^N(\cdot)$

$$\mu \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j \varphi_j' u_j^N q^N\left(u_j^N\right) = \sum_{in_\mu=t/\mu}^{(t+s)/\mu-1} \delta_\mu \frac{1}{n_\mu} \sum_{j=in_\mu}^{in_\mu+n_\mu-1} \varphi_j \varphi_j' u_{in_\mu}^N q^N\left(u_{in_\mu}^N\right) + o(1) \tag{53}$$

where $o(1) \to 0$ in probability as $\mu \to 0$ uniformly in t. By virtue of (A3) part b)

$$\frac{1}{n_\mu} \sum_{j=in_\mu}^{in_\mu+n_\mu-1} \boldsymbol{E}_{in_\mu} \varphi_j \varphi_j' \to B \text{ in probability as } \mu \to 0.$$

Thus, as $\mu \to 0$, $i\delta_\mu \to \tau$ (and, hence, $i\delta_\mu + \delta_\mu \to \tau$), we obtain that

$$\mu \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j \varphi_j' u_j^N q^N\left(u_j^N\right) \to \int_t^{t+s} Bu^N(\tau) q^N(u^N(\tau)) d\tau \tag{54}$$

by virtue of the weak convergence of $u^{\mu,N}(\cdot)$, and the Skorohod representation. Likewise, detailed estimates yield

$$\sqrt{\mu} \sum_{j=t/\mu}^{(t+s)/\mu-1} \varphi_j e_j \to \int_t^{t+s} \Sigma^{1/2} dw, \qquad \text{as } \mu \to 0. \tag{55}$$

Step 3: Using the technique of martingale problem approach as in [17, Chs. 8 and 10], we have demonstrated that the limit $u^N(\cdot)$ is a diffusion process satisfying (50).

Note that (43) is a linear stochastic differential equation and hence has a unique solution for each initial condition. By a piecing together argument (see [17, pp. 284–285]), letting $N \to \infty$, we obtain that the untruncated process $u^\mu(\cdot)$ also converges weakly. □

D. Iterate Averaging and Minimal Window of Averaging

In this subsection, we illustrate the use of iterate averaging for tracking the Markov parameter θ_n. Iterate averaging was originally proposed in [25] for accelerating the convergence of stochastic approximation algorithms. It is well known [17] that for a constant parameter, i.e., $\varepsilon = 0$ in (3), and decreasing step size (e.g., $\mu = 1/n^\gamma$ with $\gamma < 1$ in (4)), iterate averaging results in asymptotically optimal convergence rate, i.e., identical to recursive least squares algorithm (which is a matrix-step-size algorithm). In the tracking case for a random-walk time-varying parameter, in general, iterate averaging does not result in an optimal tracking algorithm [19]. In light of Theorem 18, it is shown later for the slow Markov chain parameter that iterate averaging results in an asymptotically optimal tracking algorithm.

The rationale in using iterate averaging is to reduce the stationary covariance. To see how we may incorporate this into the current setup, we begin with a related algorithm

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \frac{\Gamma}{n} \varphi_n (y_n - \varphi_n' \widehat{\theta}_n)$$

where Γ is an $r \times r$ matrix. Redefine $u_n = \sqrt{n}(\widehat{\theta}_n - \theta_n)$. Set $t_n = \sum_{j=1}^n (1/j)$ and let $u^0(t)$ be the piecewise-constant interpolation of u_n on $[t_n, t_{n+1})$ and $u^n(t) = u^0(t + t_n)$. Then using analogues argument, as previously, we arrive at $u^n(\cdot)$ converging weakly to $u(\cdot)$, which is the solution of the stochastic differential equation

$$du = (-\Gamma B + I/2) u dt + \Gamma \Sigma^{1/2} dw. \tag{56}$$

Note that $(-\Gamma B + I/2)$ and $\Gamma \Sigma^{1/2}$ replace $-B$ and Σ in (43). The additional term $I/2$ is due to the use of step size $O(1/n)$ [17, p. 329]. Minimizing the stationary covariance

$$\int_0^\infty \exp[(-\Gamma B + I/2)t] \Gamma \Sigma \Gamma' \exp[(-B\Gamma' + I/2)t] dt$$

of the diffusion given in (56) with resect to the matrix parameter Γ leads to the "optimal" covariance $B^{-1} \Sigma B^{-1}$.

Recall from (30) that our interpolation is taken with $t \in [\mu n, \mu n + \mu)$. Consider the LMS algorithm (4) together with an iterate average

$$\Theta_n = \frac{\mu}{t} \sum_{j=\widetilde{n}_0}^{\widetilde{n}_0 + t/\mu - 1} \widehat{\theta}_j \tag{57}$$

for a sufficiently large $\widetilde{n}_0$. That is, the average is over a window with window width $O(1/\mu)$, and is the so-called minimal window width of averaging (see [17, Ch. 11]).

The analysis of the iterate averaged LMS algorithm (4) together with (57) for tracking the slow Markov chain proceeds as follows: Letting t_1 and t_2 be nonnegative real numbers that satisfy $t_1 + t_2 = t$, define

$$\theta_n = \frac{\mu}{t} \sum_{j=t_\mu - \frac{t_2}{\mu}}^{t_\mu + \frac{t_1}{\mu} - 1} \widehat{\theta}_j \quad \text{and} \quad \widetilde{\theta}_n = \frac{\mu}{t} \sum_{j=t_\mu - \frac{t_2}{\mu}}^{t_\mu + \frac{t_1}{\mu} - 1} (\widehat{\theta}_j - \theta_j) \tag{58}$$

where $t_\mu \to \infty$ as $\mu \to 0$. Then consider the scaled cumulative tracking error

$$\overline{u}^\mu(t) = \frac{\sqrt{\mu}}{t} \sum_{j=t_\mu - \frac{t_2}{\mu}}^{t_\mu + \frac{t_1}{\mu} - 1} (\widehat{\theta}_j - \theta_j). \tag{59}$$

Using a similar argument as [17, p. 379], we obtain the following.

Theorem 22: Assume (A1)–(A3) hold. For each fixed $t > 0$, the scaled cumulative tracking error $\overline{u}^\mu(t)$ of the iterate averaged LMS algorithm (i.e., (4) together with averaging (58)) converges in distribution to a normal random vector $\overline{u}(t)$ with mean 0 and covariance $B^{-1} \Sigma B^{-1}/t + O(1/t^2)$.

Remark 23: Note that $B^{-1} \Sigma B^{-1}/t$ is the optimal asymptotic covariance of recursive least squares when estimating a constant parameter. Note that the number of iterates within the window of averaging is t/μ ($t/\mu = n$ is an integer by our convention) and $\mu/t = 1/n$. Then Theorem 22 indicates that within this window $\widetilde{\Theta}_n \sim N(0, B^{-1}\Sigma B^{-1}(\mu/t)) = N(0, B^{-1}\Sigma B^{-1}/(\# \text{ of iterates within the window}))$.

Instead of using the arithmetic mean in (57), another possible iterate averaging LMS algorithm is obtained by using (4) together with a weighted average of the form

$$\Theta_{n+1} = (1 - \widetilde{\alpha}\mu) \Theta_n + \widetilde{\alpha} \mu \widehat{\theta}_{n+1}.$$

Here, the forgetting factor $\widetilde{\alpha}$ is a small positive constant. The tracking analysis of this algorithm is similar to Theorem 22 with $1/\widetilde{\alpha}$ replacing the role of t. Ignoring an exponentially decaying term, we redefine the scaled cumulative tracking error as

$$\overline{u}^{\mu}(t) = \widetilde{\alpha}\sqrt{\mu} \sum_{j=t_\mu - \frac{t_2}{\mu}}^{t_\mu + \frac{t_1}{\mu} - 1} (1-\widetilde{\alpha}\mu)^{\frac{1}{\widetilde{\alpha}\mu} - j} (\widehat{\theta}_j - \theta_j)$$

where $t_1 + t_2 = 1/\widetilde{\alpha}$. In this case, $1/(\widetilde{\alpha}\mu)$ (with integer part convention used) is the effective number of iterates within the averaging window; see [7, pp. 379–380] for further discussion.

E. Hypermodel With Deterministic Holding Times and Random Jumps

In this paper, the hypermodel considered so far is a Markov chain that has a nonzero (but small) probability of jump to a new state at any given instant. We now briefly consider a simpler problem where the parameter is subject to infrequent but random jump changes once every ℓ time units, where the holding time ℓ is deterministic but unknown. Such problems arise in access controllers with periodic policies, see Section V-C.

Suppose that the parameters only change at multiples of $\ell = \lfloor 1/\varepsilon \rfloor$, where $\lfloor z \rfloor$ denotes the integer part of z. With n as usual representing the global discrete time, denote the *local* time in an interval $i\ell \le n < (i+1)\ell$ for $i = 0, 1 \ldots$, as k. Hence $k = n (\bmod \ell)$. Denote θ_n for $i\ell \le n < (i+1)\ell$ by $\theta_k^{i\ell}$. Instead of (A1), assume that the hypermodel evolves as

$$\theta_k^{i\ell} = U^i, \qquad i = 0, 1, 2, \ldots, \ i\ell \le n < (i+1)\ell$$

where $\{U^i\}$ is a sequence of independent bounded random variables. For example, $\{U^i\}$ may be a process taking finitely many values. In this case, $\theta_k^{i\ell}$ is a finite-state process that changes state every ℓ time points.

Relax (A2) so that $\{\varphi_n, e_n\}$ is θ_n dependent. In terms of the local time k denote $\varphi_n = \varphi_k^{i\ell}$, define B^i as in (6). Define $\Theta_i = \theta_k^{i\ell}$. Then on the slow time scale $i = 0, 1, \ldots$, the evolution of the hypermodel is $\Theta_i = U^i$.

Assume that the LMS algorithm (4) is used to track the parameter θ_n with $\mu = O(\varepsilon^2)$. Denote the LMS estimates as $\widehat{\theta}_k^{\mu, i\ell} = \widehat{\theta}_{i\ell + k}$. Similar to (33), define the tracking error $\widetilde{\theta}_k^{i\ell} = \widehat{\theta}_k^{i\ell} - \theta_k^{i\ell}$ and its continuous-time interpolation $\widetilde{\theta}^{\mu,i}(t) = \widetilde{\theta}_k^{i\ell}$ for each $t \in [\mu k, \mu k + \mu)$. Then, by a straightforward application of the results in [16] (see also [17]) for each of the iterations within $i\ell \le n < (i+1)\ell$, it can be shown that $\widetilde{\theta}^{\mu,i}(\cdot)$ converges weakly to $\widetilde{\theta}^i(\cdot)$, which is the solution of the following two-time-scale system: $\widetilde{\theta}^i(\cdot)$ evolves as a deterministic ODE (cf. (39))

$$\frac{d\widetilde{\theta}^i(t)}{dt} = -B^i \widetilde{\theta}^i(t) \tag{60}$$

which is coupled with the random algebraic equation $\Theta_i = U^i$ (that determines B^i in (60)). Since the ODE (60) is asymptotically stable, the tracking error decays to zero exponentially fast. Similarly, using the results in [16], the diffusion approximation (43) holds with B and Σ replaced by B^i and Σ^i. Thus, for the above hypermodel, iterate averaging as in Section IV-D also results in a tracking algorithm with asymptotically optimal convergence rate.

V. Examples: Tracking Slow HMMs and Adaptive Multiuser Detection

As in Section IV, throughout this section we assume that $\varepsilon = O(\mu^2)$ or $\mu = O(\sqrt{\varepsilon})$, i.e., the adaptation speed of the LMS algorithm (4) is faster than the Markov chain dynamics. To gain further insight, Section V-A is concerned with probability bounds; Section V-B deals with estimating the underlying state of an HMM when the Markov chain has slow dynamics; finally, Section V-C studies the effect of Markovian admission/access policies on adaptive multiuser detection in DS/CDMA systems.

A. Probability Bounds

Here, we assume that $\{\varphi_n, e_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. The estimates indicate that $\widehat{\theta}_n - \theta_n$ is asymptotically normally distributed with mean 0 and covariance $\mu\widetilde{\Sigma}$ with $\widetilde{\Sigma}$ given by (44). Note that $\widetilde{\Sigma}$ is identical to the error covariance matrix of the LMS algorithm estimate when the underlying parameter θ_n is a constant for all time n. Thus, we have shown that the LMS algorithm for tracking a slow Markov chain behaves identically (in terms of asymptotic covariance) to the LMS algorithm estimating a constant parameter. In the application examples to follow, we require Gaussian approximations for finite values of n. We consider two cases.

i) Gaussian $\{\varphi_n, e_n\}$: Then at time n, $\widehat{\theta}_n - \theta_n \sim N(0, \mu\widetilde{\Sigma})$. Denote by $\widehat{\theta}_{n,i}$ and $\theta_{n,i}$ the ith component of $\widehat{\theta}_n$ and θ_n, respectively. For any $\alpha > 0$ and each $i \le r$

$$\begin{aligned} P(|\widehat{\theta}_{n,i} - \theta_{n,i}| \ge \alpha) &= P\left(\frac{\widehat{\theta}_{n,i} - \theta_{n,i}}{\sqrt{\mu\widetilde{\Sigma}_{ii}}} \ge \frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right) \\ &\quad + P\left(\frac{\widehat{\theta}_{n,i} - \theta_{n,i}}{\sqrt{\mu\widetilde{\Sigma}_{ii}}} \le -\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right) \\ &= 2P\left(\frac{[\widehat{\theta}_{n,i} - \theta_{n,i}]}{\sqrt{\mu\widetilde{\Sigma}_{ii}}} \ge \frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right). \end{aligned} \tag{61}$$

A rough exponential-type upper (see [7]) bound gives us

$$P(|\widehat{\theta}_{n,i} - \theta_{n,i}| \ge \alpha) \le 2\exp\left(\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}\right) \boldsymbol{E}\exp\left(\frac{\widehat{\theta}_{n,i} - \theta_{n,i}}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right).$$

Likewise, we obtain

$$P(|\widehat{\theta}_{n,i} - \theta_{n,i}| < \alpha) \ge 1 - 2\exp\left(\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}\right) \boldsymbol{E}\exp\left(\frac{\widehat{\theta}_{n,i} - \theta_{n,i}}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right).$$

We can further fine tune the above approximation by use an idea of Feller as follows. Denote the standard normal distribution function by $\Phi(x)$ and the corresponding density function by $z(x)$. Then

$$\Phi(x) = \int_{-\infty}^{x} z(s)ds, \text{ and } z(x) = (1/\sqrt{2\pi})\exp(-x^2/2). \tag{62}$$

Then (61) implies that

$$P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\geq \alpha)=2-2\Phi\left(\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right) \quad (63)$$

via a normal approximation.

By virtue of [7, p. 175]

$$\left[\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}-\left(\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}\right)^{-3}\right] z\left(\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right) < 1-\Phi\left(\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right) < \frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha} z\left(\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right).$$

Owing to (63), it follows that

$$P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\geq \alpha)\leq \sqrt{\frac{2}{\pi}}\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}\exp\left(-\frac{\alpha^2}{2\mu\widetilde{\Sigma}_{ii}}\right). \quad (64)$$

We also obtain the lower bound

$$P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\geq \alpha) \geq \sqrt{\frac{2}{\pi}}\left[\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}-\left(\frac{\sqrt{\mu\widetilde{\Sigma}_{ii}}}{\alpha}\right)^{-3}\right]\exp\left(-\frac{\alpha^2}{2\mu\widetilde{\Sigma}_{ii}}\right). \quad (65)$$

Thus, the following result is obtained.

Remark 24: Suppose that the conditions (A1)–(A3) are satisfied. In addition, assume that $\{\varphi_n, e_n\}$ is a sequence of jointly normally distributed random variables. Then both (64) and (65) hold.

ii) Non-Gaussian $\{\varphi_n, e_n\}$: In view of Theorem 18, $(\widehat{\theta}_n-\theta_n)/\sqrt{\mu}$ is asymptotically normal with mean zero and covariance $\widetilde{\Sigma}=(\widetilde{\Sigma}_{ij})\in\mathbb{R}^{r\times r}$. Thus, $(\widehat{\theta}_{n,i}-\theta_{n,i})/\sqrt{\mu}$ is approximately normal with mean 0 and variance $\widetilde{\Sigma}_{ii}$ and

$$(\widehat{\theta}_{n,i}-\theta_{n,i})/\sqrt{\mu\widetilde{\Sigma}_{ii}}$$

is approximately a standard normal random variable. This section aims to derive further bounds without assuming normal distribution on the signals $\{\varphi_n, e_n\}$ with the help of the Berry–Essen estimates, given in the following lemma, whose proof is in [8, Theorem 1, p. 524].

Lemma 25: Suppose that $\{X_n\}$ is a sequence of i.i.d. random variables satisfying $\boldsymbol{E}X_n=0$, $\boldsymbol{E}X_n^2=\sigma^2>0$, and $\rho=\boldsymbol{E}|X_n|^3<\infty$. Then the distribution function of

$$S_n=\sum_{i=1}^{n}X_i/(\sigma\sqrt{n})$$

denoted by $F_{S_n}(x)$, satisfies

$$|F_{S_n}(x)-\Phi(x)|\leq\frac{3\rho}{\sigma^3\sqrt{n}}$$

where $\Phi(x)$ is the standard normal distribution.

Remark 26: Suppose that the conditions (A1)–(A3) are satisfied, and that $\{\varphi_n, e_n\}$ is a sequence of (not necessarily Gaussian) i.i.d. random variables such that $\boldsymbol{E}\varphi_n e_n=0$, $\boldsymbol{E}\left[\varphi_n\varphi_n' e_n^2\right]>0$, and $\boldsymbol{E}|\varphi_n e_n|^3<\infty$. Then for any $\alpha>0$ and for some $K>0$

$$\begin{aligned} P\left(-\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\leq Z\leq\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right)-K\sqrt{\mu} &\leq P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\leq\alpha)\\ &\leq P\left(-\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\leq Z\leq\frac{\alpha}{\sqrt{\mu\widetilde{\Sigma}_{ii}}}\right)+K\sqrt{\mu} \end{aligned} \quad (66)$$

where Z denotes a standard normal random variable. Then we can proceed as in the Gaussian case to obtain the upper and lower bounds for $P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\leq\alpha)$. In addition, we can also obtain bounds on $P(|\widehat{\theta}_{n,i}-\theta_{n,i}|\geq\alpha)$. Moreover, large deviations upper bounds may also be obtained. Furthermore, for correlated signals, we may proceed to obtain Berry–Essen estimate. We will omit the details, however.

B. Tracking Analysis of Slow HMM

In this subsection, we analyze the usefulness of the LMS algorithm (4) for estimating the state of a HMM with slow dynamics. As mentioned in Section I, given the computational efficiency of the LMS algorithm for estimating the state of an HMM, there is strong motivation to analyze the performance of the algorithm. Given that the underlying parameter θ_n is a finite-state Markov chain, the probability of error of the estimate quantized to the nearest state value is a more meaningful performance measure than the asymptotic covariance.

A conventional HMM [5], [28] comprising of a finite-state Markov chain observed in noise is of the form (1) where $\varphi_n=1$ for all n and the states of the Markov chain $\overline{\theta}_i$, $1\leq i\leq m$ are real-valued scalars. For this HMM case, the LMS algorithm (4) has complexity O(1), i.e., independent of m.

Let $\widehat{\theta}_n^H$ denote the estimate $\widehat{\theta}_n$ of (4) quantized to the nearest Markov state, i.e.,

$$\widehat{\theta}_n^H=\overline{\theta}_{i^*}, \qquad \text{where } i^*=\arg\min_{1\leq i\leq m}|\overline{\theta}_i-\widehat{\theta}_n|. \quad (67)$$

Assume that the zero mean scalar noise process $\{e_n\}$ in (1) with variance σ_e^2 satisfies the condition in Lemma 25.

Error Probability for Slow HMM: For notational convenience assume that the states of the above HMM are ordered in ascending order and are equally spaced, i.e., $\overline{\theta}_1<\overline{\theta}_2<\cdots<\overline{\theta}_m$, and $d=\overline{\theta}_{i+1}-\overline{\theta}_i$ is a positive constant.

Equation (45) implies that $\widetilde{\Sigma}=\sigma_e^2/2$. The probability of error can be computed as follows:

$$\begin{aligned} P\left(\widehat{\theta}_n^H\neq\theta_n\right)=&P(\widehat{\theta}_n-\theta_n>d/2\mid\theta_n=\overline{\theta}_1)P(\theta_n=\overline{\theta}_1)\\ &+\sum_{i=2}^{m-1}P(|\widehat{\theta}_n-\theta_n|>d/2\mid\theta_n=\overline{\theta}_i)P(\theta_n=\overline{\theta}_i)\\ &+P(\theta_n-\widehat{\theta}_n>d/2\mid\theta_n\neq\overline{\theta}_m)P(\theta_n=\overline{\theta}_m). \end{aligned} \quad (68)$$

We summarize this in the following result.

Theorem 27: Suppose that the HMM satisfies (A1), (A2), and $\{e_n\}$ is a sequence of zero mean i.i.d. random variables that satisfies the condition in Lemma 25. Then the probability of error of the LMS algorithm estimate θ_n^H (67) in estimating the state θ_n of the HMM is

$$\begin{aligned} P\left(\widehat{\theta}_n^H \neq \theta_n\right) &= \pi_1(\varepsilon n)\Phi^c\left(\frac{d}{2\sqrt{\widetilde{\Sigma}\mu}}\right) + 2(\pi_2(\varepsilon n) + \cdots \\ &\quad + \pi_{m-1}(\varepsilon n))\Phi^c\left(\frac{d}{2\sqrt{\widetilde{\Sigma}\mu}}\right) \\ &\quad + \pi_m(\varepsilon n)\Phi^c\left(\frac{d}{2\sqrt{\widetilde{\Sigma}\mu}}\right) \\ &= (2-\pi_1(\varepsilon n)-\pi_m(\varepsilon n))\Phi^c\left(\frac{d}{2\sqrt{\widetilde{\Sigma}\mu}}\right) \end{aligned} \quad (69)$$

where $\Phi^c(\cdot)$ is the complementary Gaussian cumulative distribution function $\Phi^c(x) = \int_x^\infty z(s)ds$.

Note that the complement Gaussian distribution above is commonly referred to as $Q(\cdot)$ function in the communications literature. Since Q is used as generator in our formulation, we use $\Phi^c(x)$ to denote the complementary Gaussian distribution. The above result (based on weak convergence) is to be contrasted with the following computation of the error probability using the mean-square convergence in Corollary 7 above. Using Chebyshev's inequality that

$$P(|\widehat{\theta}_n - \theta_n| > d/2 | \theta_n = \overline{\theta}_i) \leq K\mu/d^2$$

(where K is a positive constant independent of μ and d) together with (68) and Corollary 7 yields

$$P\left(\widehat{\theta}_n^H \neq \theta_n\right) \leq K\mu/d^2. \quad (70)$$

Expression (70) is less useful than Theorem 27. However, it serves as a consistency check. As μ and $\varepsilon \to 0$, the probability of error of the tracking algorithm goes to zero.

Comparison of Error Probabilities: It is instructive to compare the probability of error expression (69) for the LMS algorithm (4) tracking a slow Markov chain parameter with that of the asymptotic HMM filter derived in [9]. By virtue of [35, eq. 3.35], the following upper bound holds for Φ^c in (69):

$$\Phi^c\left(\frac{d/2}{\sqrt{\mu\widetilde{\Sigma}}}\right) \leq \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\mu\widetilde{\Sigma}}}{d/2}\exp\left(-\frac{(d/2)^2}{2\mu\widetilde{\Sigma}}\right). \quad (71)$$

As described in [35], the above is an excellent approximation for small μ. In [9], it is shown that the steady-state asymptotic HMM filter with $O(m)$ computational complexity has error probability $K\mu^2\log(1/\mu^2)$ where the constant K depends on the steady-state probabilities of the Markov chain. These error probabilities converge to zero much faster than that of the $O(1)$ complexity LMS algorithm with error probability upper bound in (71).

Error Probability and Iterate Averaging: Iterate averaging can be used for vector state HMMs to reduce the error probability of estimates generated by the LMS algorithm. Consider the following vector state HMM: $\overline{\theta}_i \in \mathbb{R}^q$, $\varphi_n \in \mathbb{R}^q$ is a sequence of i.i.d. random variables statistically independent of θ_n and e_n, $\mathcal{M} = \{\overline{\theta}_1, \overline{\theta}_2\}$. Define $d = \overline{\theta}_2 - \overline{\theta}_1$ The probability of error of the LMS algorithm in tracking this vector state HMM is

$$\begin{aligned} P\left(\widehat{\theta}_n^H \neq \theta_n\right) &= P\left(|\widehat{\theta}_n - \overline{\theta}_1| > |\widehat{\theta}_n - \overline{\theta}_2| | \theta_n = \overline{\theta}_1\right) \\ &\quad \times P(\theta_n = \overline{\theta}_1) \\ &\quad + P\left(|\widehat{\theta}_n - \overline{\theta}_2| > |\widehat{\theta}_n - \overline{\theta}_1| | \theta_n = \overline{\theta}_2\right) \\ &\quad \times P(\theta_n = \overline{\theta}_2). \end{aligned} \quad (72)$$

The weak convergence result Theorem 18 implies that $[\widehat{\theta}_n - \theta_n]$ is asymptotically normal with mean 0 and variance $\mu\widetilde{\Sigma}$. This implies that conditional on $\theta_n = \overline{\theta}_1$, $(\widehat{\theta}_n - \theta_n)'d \sim N(0, \mu d'\widetilde{\Sigma}d)$. Substituting in (72) yields

$$P\left(\widehat{\theta}_n^H \neq \theta_n\right) = \Phi^c\left(\frac{d'd/2}{\sqrt{\mu d'\widetilde{\Sigma}d}}\right). \quad (73)$$

In the case of iterate averaging, see Theorem 22, $\widetilde{\Sigma}$ in (73) is replaced with $B^{-1}\Sigma B^{-1}$ defined in (A3) of Section IV-C. Since $\widetilde{\Sigma} - B^{-1}\Sigma B^{-1} > 0$ (positive definite) and Φ^c is monotonically decreasing, iterate averaging yields a lower probability of error.

C. Effect of Admission/Access Control on Adaptive Multiuser Detector

In this subsection, we examine the tracking performance of an adaptive linear multiuser detector in a cellular DS/CDMA system when the profile of active users changes due to an admission or access (scheduling) controller at the base station. The main point is to show that in many cases, the optimal LMMSE multiuser detector varies according to a finite Markov chain—hence, the above weak convergence analysis for the LMS algorithm directly applies to the corresponding adaptive linear multiuser detector which aims to track the LMMSE detector.

Consider a synchronous DS-CDMA system with a maximum of K users and an additive white Gaussian noise channel. After the received continuous-time signal is preprocessed and sampled at the CDMA receiver (the received signal is passed through a chip-matched filter followed by a chip-rate sampler), the resulting discrete-time received signal at time n, denoted by r_n, is given by (see [23] for details)

$$r_n = A(1)b_n(1)s(1) + \sum_{k\in\overline{K}_n} A(k)b_n(k)s(k) + \varpi_n. \quad (74)$$

Here, the user of interest is user 1, $\overline{K}_n$ denotes the active users (interferers) at time n, r_n is an N-dimensional vector; N is called the processing (spreading) gain; $s(k)$ is an N-vector denoting the normalized signature sequence of the kth user, so that $s'(k)s(k) = 1$; $b_n(k)$ denotes the data bit of the kth user transmitted at time n; $A(k)$ is the received amplitude of the kth user; ϖ_n is a white Gaussian vector with mean zero and covariance matrix σI where I denotes the $N \times N$ identity matrix and $\sigma > 0$ is a scalar. It is assumed that the discrete-time stochastic processes $\{b_n(k)\}$, and $\{\varpi_n\}$ are mutually independent, and that

$\{b_n(k)\}$ is a collection of independent equiprobable ± 1 random variables.

Specification of Active Users: Let $\mathcal{P}(X)$ denote the power set of an arbitrary finite set X. For user 1, the set of all possible combinations of active users (interferers) is (where $\emptyset$ denotes the null set, i.e., no interferer)

$$\mathcal{P}(\{2,\dots,K_0\}) = \{\emptyset, \{2\}, \{2,3\}, \{2,4\}, \dots \{2,3,4,\dots,K_0\}\}.$$

Then $\mathcal{P}(\{2,\dots,K_0\})$ denotes the state space of the finite-state process $\overline{K}_n$. Each time n, $\overline{K}_n$ assumes one of 2^{K_0-1} possible states in $\mathcal{P}(\{2,\dots,K_0\})$.

We assume that the network admission/access controller operates on a slower time scale (e.g., multiframe-by-multiframe basis) than the bit duration, i.e., the finite-state process $\{\overline{K}_n\}$ evolves according to a slower time scale than the bits $\{b_n[k]\}$. This is usual in DS/CDMA systems, where typically a user arrives or departs after several multiframes (i.e., several hundreds of bits). Then $\overline{K}_n$ can be modeled as a slow finite-state Markov chain with transition probability matrix $I+\varepsilon Q$ in the following examples.

i) Consider a single class of users (e.g., voice) with Poisson arrival rate λ and exponential departure rate μ. Then the active users form a continuous-time Markov chain (birth death process) with state space $\mathcal{P}(\{2,\dots,K_0\})$ and generator Q. The time-sampled version, sampled at the chip rate ε, is then a slow Markov chain with $P^\varepsilon = I+\varepsilon Q$.

ii) Markov decision-based admission control of multiclass users: The formulation in [33] considers admission control in a multiservice CDMA network comprising voice and data users. Assuming a linear multiuser detector at the receiver, the admission controller aims to minimize the blocking probability of users seeking to access the network subject to signal-to-interference ratio (quality-of-service) constraints on the active users $\overline{K}_n$. Assuming that the arrival rate of voice and data users are Poisson and departure rates of active users is exponential, the problem of devising the optimal admission policy is formulated in [33] as a semi-Markov decision process with exponential holding times. Again, assuming that the arrival and departure of users are at a slower time scale (e.g., several frames) than the bit duration, a time-sampled version of this continuous-time process at chip rate ε results in a slow Markov chain.

iii) Periodic access control of multiclass users: In access (scheduling) control [32], typically two classes of users are considered—Class R of real-time users (e.g., voice) and Class NR of non-real-time users (e.g., data users). Access control is required when the network admission controller admits more users than it has the capacity to accommodate in order to reduce call blocking and handoff blocking probabilities. The access controller regulates the transmission rates of NR users since they are tolerant to delays. By formulating the access control problem as an average reward constrained Markov decision process, [32] shows that a Markov access control strategy is optimal in the sense that the transmission rate of NR users is maximized subject to signal-to-interference ratio (SIR) and outage probability constraints. The Markov access control strategy determines on a multiframe-by-multiframe basis which subset of users in NR are allowed to transmit in conjunction with users in R. The term Markov strategy means that the particular subset $\overline{K}_n$ of users in NR that are allowed to transmit at each frame are determined by the trajectory of a homogeneous Markov chain. Since $\overline{K}_n$ evolves on a multiframe basis, in the bit-interval time scale n, $\overline{K}_n$ is a slow Markov chain. As described in [14], regulating the use of a wireless network by a Markov chain is easily implemented by broadcasting the transition probability matrix of the Markov chain to all mobile stations in the network. For example. in the SEEDEX medium access protocol [14], each mobile station generates a local version of the Markov chain $\overline{K}_n$ by using the same seed for its pseudorandom number generator. Another widely used access control policy is the periodic access control policy where user groups in NR transmit at period intervals. If the length ℓ of these intervals is large, i.e., $\ell = O(1/\varepsilon)$, then the hypermodel of Section IV-E applies.

In the following, we consider the effect of the above Markovian or periodic admission/access control strategies on three types of adaptive multiuser detectors: decision-directed receiver, blind receiver, and precombining receiver. In all three cases, the optimal receiver weight coefficients evolve according to a finite-state Markov chain and the adaptive linear receiver is an LMS algorithm which attempts to track this Markovian weight vector.

Adaptive Decision-Directed Multiuser Detection: We assume that user 1 is the user of interest. Assuming knowledge of the active user set $\overline{K}_n$, the optimal linear multiuser detector seeks to compute the weight vector c_n^* such that

$$c_n^* = \arg\min_c \boldsymbol{E}_{\overline{K}_n} \{Ab_n(1) - c' r_n\}^2 \tag{75}$$

where $b_n(1)$ is a training data sequence (or else the estimates of the bits when the receiver is operating in the decision directed mode). As shown in [26], $c_n^* = R_n^{-1} s_1$ where $R_n = \boldsymbol{E}_{\overline{K}_n}\{r_n r_n'\}$. Given $\overline{K}_n$ is a slow Markov chain, it follows from (74) that R_n and thus the optimal weight vector c_n^* are also $2^{(K_0-1)}$-state slow finite-state Markov chains, respectively. It is clear that the above formulation is identical to the signal model (1) with $y_n = Ab_n(1)$ (observation), $\theta_n = c_n^*$ (slow Markov chain parameter), $\varphi_n = r_n$ (regression vector). Indeed, $\{e_n\}$, with $e_n = y_n - \varphi_n' \theta_n$, is a sequence of i.i.d. random variables due to the orthogonality principle of Wiener filters.

Now consider the *adaptive* multiuser detection problem where the active user set $\overline{K}_n$ is not known. Thus, $\{y_n\}$ is the observation sequence of an HMM. Hence, in principle, the optimal (conditional mean) estimate of $\overline{K}_n$ and therefore c_n^* given the observation history $(y_1,\dots,y_n)$ can be computed using the HMM state filter. However, due to the large state space (exponential in the number of users K_0), this is computationally prohibitive. For this reason, the adaptive linear multiuser detector [35] uses the LMS algorithm (4) to minimize (75) without taking into account the Markovian dynamics of $\theta_n = c_n^*$, i.e., θ_n is the hypermodel (see Remark 4). For such an adaptive linear multiuser detector, Remark 26 holds implying that if $\varepsilon = \mu^2$, the estimate $\widehat{c}_n$ of the adaptive multiuser detector is approximately normally distributed with mean c_n^* and covariance $\widetilde{\Sigma}$.

Adaptive Blind Multiuser Detectors: The tracking analysis of the so-called "blind" adaptive multiuser detection case considered in [12], [13] proceeds similarly. Assuming knowledge of the active users $\overline{K}_n$, the blind linear multiuser detector uses the optimal weight vector c_n^* which is the solution of the following constrained optimization:

$$\min_c \boldsymbol{E}_{\overline{K}_n}\{c' r_n\}^2 \quad \text{subject to } c' s_1 = 1. \tag{76}$$

As shown in [12], $c_n^* = R_n^{-1} s_1 / s_1' R_n^{-1} s_1$ which is a slow finite-state Markov chain since R_n is a slow finite-state Markov chain. The constrained optimization problem (76) may be transformed into an unconstrained optimization problem by solving for one of the elements $c_{n,i}$, $i \in [1, \dots, N]$ using the constraint (76). With no loss of generality, we solve for the first element $c_{n,1}$ and obtain

$$c_{n,1} = \left(1 - \sum_{i=2}^{N} s_{1,i} c_{n,i}\right) / s_{1,1}.$$

By defining the $(N-1)$-dimensional vector $\theta_n = (c_{n,2}, \dots, c_{n,N})'$, we obtain the equivalent unconstrained optimization problem:

$$\text{Compute } \min_\theta J_n \quad \text{where } J_n = \boldsymbol{E}_{\overline{K}_n}(y_n - \theta' \varphi_n)^2. \tag{77}$$

Here, $y_n = -r_{n,1}/s_{1,1}$ and φ_n denotes the $(N-1)$-dimensional vector

$$\varphi_n = \left(r_{n,2} - r_{n,1} s_{1,2}/s_{1,1}, \dots, r_{n,N} - r_{n,1} s_{1,N}/s_{1,1}\right)'. \tag{78}$$

In the adaptive case, when $\overline{K}_n$ is unknown, i.e., θ_n is the hypermodel (see Remark 4), the adaptive blind multiuser detector in [12], [13] is an LMS algorithm of the form (4), implying that the above weak convergence results and Remark 26 apply. In [12], approximate expressions are presented for the covariance matrix $\widetilde{\Sigma}$.

Precombining Adaptive Multiuser Detectors for Fading Channels: A performance analysis of MMSE receivers for frequency-selective fading channels is presented in [21]. In general, the optimal receiver weight coefficient c_n^* of the LMMSE receiver varies rapidly in time depending on the instantaneous channel values. Here we consider a particular receiver structure, developed in [18], called a precombining LMMSE receiver (also called LMMSE-RAKE receiver) which results in the optimal receiver weight vector c_n^* evolving according to a slow finite-state Markov chain.

The continuous-time received signal for a frequency-selective fading channel has the form

$$r(t) = \sum_{n=0}^{N_b-1} \sum_{k \in \overline{K}_n} \sum_{l=1}^{L} A(k) b_n(k) c_n(k,l) s_{t-nT-\tau_{k,l}}(k) + \varpi(t)$$

where T denotes the symbol interval, L is the number of propagation paths, $\varpi(t)$ is complex zero mean additive white Gaussian noise with variance σ^2, $c_n(k,l)$ is the complex attenuation factor for the kth user and lth path, and $\tau_{k,l}$ is the propagation delay. The received discrete-time signal over a data block of N_b symbols after antialias filtering and sampling at the rate $T_s = T/(SG)$ (where S is the number of samples per chip, G is the number of chips per symbol) is (see [18] for details)

$$r_n = \boldsymbol{S} \boldsymbol{C}_n \boldsymbol{A}_n \boldsymbol{b}_n + \boldsymbol{n}_n \in \mathbb{C}^{SGN_b}$$

where $\boldsymbol{S}$ is the sampled spread sequence matrix, $\boldsymbol{C}_n$ is the channel coefficient matrix, $\boldsymbol{A}_n$ is a matrix of received amplitudes (the time variation in the notation is because inactive users i.e., $k \notin \overline{K}_n$ are considered to have zero amplitude—thus, $\boldsymbol{A}_n$ is a slow finite-state Markov chain), $\boldsymbol{b}_n$ is the data vector, and $\boldsymbol{n}_n$ is the complex-valued channel noise vector. Assuming knowledge of the active users $\overline{K}_n$, the precombining LMMSE receiver seeks to find c_n^* to minimize $\boldsymbol{E}_{\overline{K}_n}\{\boldsymbol{C}_n \boldsymbol{A}_n \boldsymbol{b}_n - c' r_n\}^2$. The optimal receiver is

$$c_n^* = \boldsymbol{S}\left(\boldsymbol{S}'\boldsymbol{S} + \sigma^2 \Sigma_n^{-1}\right)^{-1} \in \mathbb{R}^{SGN_b \times KLN_b}$$

where Σ_n is the covariance matrix of $\boldsymbol{C}_n \boldsymbol{A}_n \boldsymbol{b}_n$ which consists of transmitted user powers and average channel tap powers. As remarked in [18], this shows that the precombining LMMSE receiver no longer depends on the instantaneous values of the channel complex coefficients but on the average power profiles of the channel Σ_n. Thus, c_n^* is a finite-state Markov chain.

In the case when the active users $\overline{K}_n$ are unknown, i.e., θ_n is the hypermodel (see Remark 4), the adaptive precombining LMMSE receiver uses the LMS algorithm to optimize $\boldsymbol{E}\{\boldsymbol{C}_n \boldsymbol{A}_n \boldsymbol{b}_n - c' r_n\}^2$. This is again of the form (4) and the weak convergence tracking analysis of Theorem 26 applies.

Iterate Averaging: In all three cases above, providing $\mu = O(\varepsilon^2)$, iterate averaging (Section IV-D) over a window of $O(1/\mu)$ results in an adaptive receiver with asymptotically optimal convergence rate.

Periodic Access Control: As described in Section IV-E, for periodic policies, the assumption that the LMMSE receiver weight vector c_n^* jumps among a finite number of states can be relaxed. Indeed, as long as c_n^* is bounded and jumps only every $\lfloor 1/\varepsilon \rfloor$ time points, the above results hold.

VI. Further Remarks and Extensions

1) Projection Algorithm (8): Using the idea given in [17], we can develop a projection algorithm. Rewrite the recursive algorithm as

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \mu \varphi_n (y_n - \varphi_n' \widehat{\theta}_n) + \mu z_n$$

where μz_n is the vector with the shortest Euclidean length needed to take $\widehat{\theta}_n + \mu \varphi_n (y_n - \varphi_n' \widehat{\theta}_n)$ back to the set H if the iterates ever escape from it. Then we can carry out the analysis as in the previous sections. We can obtain the following.

Suppose that (A1)–(A4) are satisfied but without the boundedness assumption on the signals. Suppose that $\{\varphi_n \varphi_n', \varphi_n e_n\}$ is uniformly integrable. Suppose that $\widehat{\theta}_n \in H^0$, the interior of H. Then the conclusion of Theorem 18 continue to hold.

Without assuming the normal distribution of $\{\varphi_n, e_n\}$ and the independence assumption as in Theorem 26, we can proceed to obtain the upper and lower bounds as in [17, Secs. 6.9 and 6.10]; see also the original paper [3]. We omit the details and refer the reader to these references.

2) Semi-Markov Hypermodels: Finally, in communications network applications, often θ_n evolves according to a semi-Markov process instead of a discrete-time Markov chain. The semi-Markov processes can be considered as Markov chains on the general state space $(\mathcal{M} \times \mathbb{R}^+)$ where $\mathcal{M}$ is the finite set defined in (2). It would be of interest to extend the tracking analysis of this paper to such semi-Markov processes.

3) $\varepsilon = O(\mu)$ Case: In the weak convergence analysis of this paper we assumed $\varepsilon = O(\mu^2)$, i.e., the parameter evolves as a Markov chain on a slower time scale compared to the LMS algorithm.

Recently, in [37], we have analyzed the tracking capabilities of the LMS algorithm with $\varepsilon = O(\mu)$, i.e., the parameter evolves as a Markov chain on the same time scale as the LMS algorithm. Somewhat remarkably, in this case, as shown in [37], instead of having a deterministic ODE limit for the averaged system, we obtain a Markovian switched ODE limit. Markov swtiching ODEs also arise in analyzing the tracking performance of a discrete stochastic approximation algorithm; see [36].

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References

[1] A. Benveniste, M. Metivier, and P. Priouret, *Adaptive Algorithms and Stochastic Approximations*. Berlin, Germany: Springer-Verlag, 1990.

[2] P. Billingsley, *Convergence of Probability Measures*. New York: Wiley, 1968.

[3] P. Dupuis and H. J. Kushner, "Stochastic approximation via large deviations: Asymptotic properties," *SIAM J. Control Optim.*, vol. 23, pp. 675–696, 1985.

[4] R. Evans, V. Krishnamurthy, G. Nair, and L. Sciacca, "Sensor and data rate control for tracking maneuvering targets," *IEEE Trans. Signal Process.*, vol. 53, no. 6, June 2005, to be published.

[5] Y. Ephraim and N. Merhav, "Hidden Markov processes," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1518–1569, Jun. 2002.

[6] S. N. Ethier and T. G. Kurtz, *Markov Processes—Characterization and Covergence*. New York: Wiley, 1986.

[7] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. New York: Wiley, 1968, vol. I.

[8] ——, *An Introduction to Probability Theory and Its Applications*, 2nd ed. New York: Wiley, 1971, vol. II.

[9] G. Golubev and R. Khasminskii, "Asymptotic optimal filtering for a hidden Markov model," *Math. Methods Statist.*, vol. 7, no. 2, pp. 192–208, 1998.

[10] L. Guo, "Stability of recursive stochastic tracking algorithms," *SIAM J. Control Optim.*, vol. 32, no. 5, pp. 1195–1225, Sep. 1994.

[11] L. Guo and L. Ljung, "Performance analysis of general tracking algorithms," *IEEE Trans. Automatic Contr.*, vol. 40, no. 8, pp. 1388–1402, Aug. 1995.

[12] M. L. Honig, U. Madhow, and S. Verdú, "Adaptive blind multiuser detection," *IEEE Trans. Inf. Theory*, vol. 41, no. 4, pp. 944–960, Jul. 1995.

[13] V. Krishnamurthy, "Averaged stochatic gradient algorithms for adaptive blind multiuser detection in DS/CDMA systems," *IEEE Trans. Commun.*, vol. 48, no. 1, pp. 125–134, Jan. 2000.

[14] P. R. Kumar, "New technological vistas for systems and control: The example of wireless networks," *IEEE Contr. Syst. Mag.*, vol. 21, no. 1, pp. 24–37, Feb. 2001.

[15] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes, With Applications to Stochastic Systems Theory*. Cambridge, MA: MIT Press, 1984.

[16] H. J. Kushner and A. Shwartz, "Weak convergence and asymptotic properties of adaptive filters with constant gains," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 2, pp. 177–182, May 1984.

[17] H. J. Kushner and G. Yin, *Stochastic Approximation Algorithms and Recursive Algorithms and Applications*, 2nd ed. New York: Springer-Verlag, 2003.

[18] M. Latva-Aho, "LMMSE receivers for DS-CDMA systems in frequency selective fading channels," in *CDMA Techniques for Third Generaion Mobile Systems*. Norwell, MA: Kluwer, 1999, pp. 145–183.

[19] L. Ljung, "Recursive least-squares and accelerated convergence in stochastic approximation schemes," *Int. J. Adaptive Control Signal Process.*, vol. 15, no. 2, pp. 169–178, Mar. 2001.

[20] L. Ljung and S. Gunnarsson, "Adaptive tracking in system identification—A survey," *Automatica*, vol. 26, no. 1, pp. 7–22, 1990.

[21] S. Miller, H. Honig, and L. B. Milstein, "Performance analysis of MMSE receivers for DS-CDMA in frequency selective fading channels," *IEEE Trans. Commun.*, vol. 48, no. 11, pp. 1919–1929, Nov. 2000.

[22] U. Mitra and H. V. Poor, "Activity detection in a multi-user environment," *Wireless Personal Commun.*, vol. 3, no. 1–2, pp. 149–174, Jan. 1996.

[23] T. Oskiper and H. V. Poor, "On line activity detection in a multiuser environment using the matrix CUSUM algorithm," *IEEE Trans. Inf. Theory*, vol. 48, no. 2, pp. 477–491, Feb. 2002.

[24] B. T. Polyak, "New stochastic approximation type procedures," *Automat. Remote Contr.*, vol. 7, pp. 98–107, 1990.

[25] B. T. Polyak and A. B. Juditsky, "Acceleration of stochastic approximation by averaging," *SIAM J. Contr. Optimiz.*, vol. 30, no. 4, pp. 838–855, Jul. 1992.

[26] H. V. Poor and X. Wang, "Code-aided interference suppression for DS/CDMA communications-Part I: Interference suppression capability," *IEEE Trans. Commun.*, vol. 45, no. 9, pp. 1101–1111, Sep. 1997.

[27] ——, "Code-aided interference suppression for DS/CDMA communications-Part II: Parallel blind adaptive implementations," *IEEE Trans. Commun.*, vol. 45, no. 9, pp. 1112–1122, Sep. 1997.

[28] L. R. Rabiner, "A tutorial on hidden Markov models and selected applications in speech recognition," *Proc. IEEE*, vol. 77, no. 2, pp. 257–285, Feb. 1989.

[29] K. W. Ross, *Multiservice Loss Models for Broadband Telecommunication Networks*. Berlin, Germany: Springer-Verlag, 1995.

[30] D. Ruppert, "Stochastic approximation," in *Handbook in Sequential Analysis*, B. K. Ghosh and P. K. Sen, Eds. New York: Marcel Dekker, 1991, pp. 503–529.

[31] L. Shue, B. D. O. Anderson, and F. De Bruyne, "Asymptotic smoothing errors for hidden Markov models," *IEEE Trans. Signal Process.*, vol. 48, no. 12, pp. 3289–3302, Dec. 2000.

[32] S. Singh and V. Krishnamurthy, "Optimal access control for DS-CDMA cellular networks," in *Proc. Int. Conf. Acoustics, Speech, and Signal Processing*, Orlando, Fl, 2002, pp. 2749–2752.

[33] S. Singh, V. Krishnamurthy, and H. V. Poor, "Integrated voice/data call admission control for wireless DS-CDMA systems with fading," *IEEE Trans. Signal Process.*, vol. 50, no. 6, pp. 1483–1495, Jun. 2002.

[34] V. Solo and X. Kong, *Adaptive Signal Processing Algorithms—Stability and Performance*. Englewood Cliffs, N.J.: Prentice-Hall, 1995.

[35] S. Verdú, *Multiuser Detection*. Cambridge, U.K.: Cambridge Univ. Press, 1998.

[36] G. Yin, V. Krishnamurthy, and C. Ion, "Regime switching stochastic approximation algorithms with application to adaptive discrete stochastic optimization," *SIAM J. Optimiz.*, vol. 14, pp. 1187–1215, 2004.

[37] G. Yin and V. Krishnamurthy, "Least mean squares algorithms with Markov regime-switching limit," *IEEE Trans. Automatic Contr.*, vol. 50, no. 5, pp. 577–593, May 2005.

[38] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*. New York: Springer-Verlag, 1998.

[39] ——, "Singularly perturbed discrete-time Markov chains," *SIAM J. Appl. Math.*, vol. 61, pp. 833–854, 2000.

[40] G. Yin, Q. Zhang, and G. Badowski, "Discrete-time singularly perturbed Markov chains: Aggregation, occupation measures, and switching diffusion limit," *Adv. Appl. Probab.*, vol. 35, no. 2, pp. 449–476, 2003.