CONDITIONAL MOMENT GENERATING FUNCTIONS FOR INTEGRALS AND STOCHASTIC INTEGRALS[*]

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Abstract. In this paper we present two methods for computing filtered estimates for moments of integrals and stochastic integrals of continuous-time nonlinear systems. The first method utilizes recursive stochastic partial differential equations. The second method utilizes conditional moment generating functions. An application of these methods leads to the discovery of new classes of finite-dimensional filters. For the case of Gaussian systems the recursive computations involve integrations with respect to Gaussian densities, while the moment generating functions involve differentiations of parameter dependent ordinary stochastic differential equations. These filters can be used in Volterra or Wiener chaos expansions and the expectation-maximization algorithm. The latter yields maximum-likelihood estimates for identifying parameters in state space models.

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1. Introduction. Conditional expectations of functionals of systems state processes given noisy observations require, in general, infinite-dimensional computations. To determine whether such conditional expectations are finite-dimensional, it is of interest to derive representations of the conditional distribution.

This paper discusses the following problem. We are given noisy observations $\{y_s; 0 \leq s \leq t\}$ of the system state process $\{x_s; 0 \leq s \leq t\}$, and we wish to derive filtered estimates for moments of integrals and stochastic integrals. The underlying mathematical system model can be diverse; for example, it includes continuous-time processes, discrete-time processes, jump point processes, or a combination of these processes. In this paper we focus our attention on continuous-time processes.

Here, our system state process $\{x_s; 0 \leq s \leq t\}$ and observation process $\{y_s; 0 \leq s \leq t\}$ are solutions of the Itô stochastic differential equations

$$(1.1) \qquad dx_t = f(t, x_t)dt + \sigma(t, x_t)dw_t, \quad x(0) \; \epsilon \; \mathbb{R}^n,$$

$$(1.2) \qquad dy_t = h(t, x_t)dt + \alpha_t dw_t + N_t^{1/2} db_t, \quad y(0) = 0 \; \epsilon \; \mathbb{R}^n,$$

in which $\{w_s; 0 \leq s \leq t\}$ and $\{b_s; 0 \leq s \leq t\}$, are, respectively, m-dimensional and d-dimensional, independent standard Wiener processes; $x(0)$ is a random variable

independent of the Wiener processes. The precise assumptions on the coefficients of our model are stated in section 2.

We are interested in conditional expectations (filtered estimates) of moments of integrals and stochastic integrals

$$(1.3) \qquad L_{0,t}^{\kappa,1} = \left(\int_0^t f^1(s,x_s)ds \right)^{\kappa}, \quad L_{0,t}^{\kappa,2} = \left(\int_0^t f^2(s,x_s)dw_s \right)^{\kappa},$$

$$L_{0,t}^{\kappa,3} = \left(\int_0^t f^3(s,x_s)db_s \right)^{\kappa}, \quad \kappa \geq 1,$$

for Borel measurable functions $f^1 : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $f^2 : [0,T] \times \mathbb{R}^n \to (\mathbb{R}^m)'$, $f^3 : [0,T] \times \mathbb{R}^n \to (\mathbb{R}^d)'$, which are continuous in t. Aside from their mathematical value, these estimates are important, for example, in least-squares estimation/filtering, Volterra series expansions of nonlinear realization theory [1], Wiener chaos expansions (of nonlinear filtering) [2], and maximum likelihood estimation through the expectation-maximization (EM) algorithm [3]. For the case $\kappa = 1$, these estimates are important in estimating parameters, a problem which arises in many disciplines, such as signal processing, communications, and control systems.

The first method, Theorem 3.1, utilizes a system of stochastic partial differential equations (SPDEs) that enable us to compute the above estimates recursively. The second method, Theorem 4.5, utilizes conditional moment generating functions for $L_{0,t}^{1,j}, j = 1,2,3$. That is, for a test function $\Phi : \mathbb{R}^n \to \mathbb{R}$, we use measure-valued conditional moment generating functions

$$(1.4) \quad \widetilde{\beta}_t^{\theta,j}(\Phi) = \widetilde{\mathbb{E}}[\Phi(x_t) \exp\left(\theta L_{0,t}^{1,j} \right) | \mathcal{F}_{0,t}^y], \qquad j = 1,2,3, \quad \theta = i\omega, \quad i = \sqrt{-1}.$$

Therefore, when the unnormalized versions of $\widetilde{\beta}_t^{\theta,j}(\Phi)$ have densities $\beta^{\theta,j}(x,t), j = 1,2,3$, the latter satisfy linear SPDEs. The computation of filtered estimates of moments (1.3) are obtained by simply differentiating the conditional densities with respect to the parameter θ.

For the case of Gaussian system models (i.e., $dx_t = Fx_t dt + Gw_t$, $dy_t = Hx_t dt + N^{\frac{1}{2}}b_t$), we derive filtered estimates for

$$(1.5) \qquad L_{0,t}^{1,1} = \int_0^t x_s' Q x_s ds \quad L_{0,t}^{1,2} = \int_0^t x_s' R dw_s, \quad L_{0,t}^{1,3} = \int_0^t x_s' S db_s.$$

Each filtered estimate is propagated by four statistics. Two of these are the conditional mean and error covariances of x_t given $\{y_s; 0 \leq s \leq t\}$ (Kalman filter), while the remaining two are modified versions of the Kalman filter; the latter are driven by the conditional mean and error covariance of the Kalman filter.

In the past, the computation of these filtered estimates was confined to integrals $L_{0,t}^{1,1}$, which are obtained using smoothing operations (e.g., [4]), and certain Lie algebraic techniques applied to Volterra expansions (e.g., [1]). However, for analogous discrete-time systems the filtering estimates in (1.5) are obtained using smoothing operations (e.g., [5]). Recently, conditional expectations for the items in (1.5) were obtained using filtering operations in [6]; the estimates were propagated by five statistics. The techniques in [6], which are different from ours, are only applicable to Gaussian systems, and they are confined to $\kappa = 1$.

2. The Duncan–Mortensen–Zakai (DMZ) equation.
Notation 2.1.

1. "t" denotes transposition of a matrix;
2. I_k denotes $k \times k$ identity matrices;
3. $(\cdot)_i$ denotes the ith component of a vector and $(\cdot)_{i,j}$ denotes the ijth component of a matrix;
4. $\mathcal{L}(V_1; V_2)$ denotes the space of linear transformations of a vector space V_1 into a vector space V_2;
5. $C_{x,t}^{p,q}(\mathbb{R}^n \times [0,T]) = \{\Phi : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n; \Phi(\cdot, t)$ is "p" times continuously differentiable in "x," and $\Phi(x, \cdot)$ is "q" times continuously differentiable in "t"$\}$;
6. $D_x = [\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}]'$, $\qquad D_x^2 = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2}, & \cdots, & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1}, & \cdots, & \frac{\partial^2}{\partial x_n^2} \end{bmatrix}$;
7. $\Phi : \mathbb{R}^n \to \mathbb{R}$ denotes an arbitrary test function which is $C_x^2(\mathbb{R}^n)$ and has compact support;
8. $\mathrm{E}, \widetilde{\mathrm{E}}$ denote expectations with respect to measures $P, \widetilde{P}$, respectively;
9. $N_t \doteq N_t^{\frac{1}{2}} N_t^{\frac{1}{2},'}$.

Assumption 2.2.

1. $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, h : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d, T > 0$, are bounded Borel measurable functions;
2. $N : [0,T] \to \mathcal{L}(\mathbb{R}^d; \mathbb{R}^d), \alpha : [0,T] \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^d), N, \alpha$ are bounded Borel measurable functions, and $\exists \beta_1 > 0, \beta_2 > 0$ such that $N_t \geq \beta_1 I_d \ \forall t \ \epsilon \ [0,T]$, $a(t,x) \doteq \sigma(t,x)\sigma(t,x)' \geq \beta_2 I_n \ \forall (t,x) \ \epsilon \ [0,T] \times \mathbb{R}^n$;
3. σ is continuous in x, uniformly on compact subsets of $[0,T] \times \mathbb{R}^n$, $\frac{\partial}{\partial x_i}\sigma_{i,j}$ is a bounded measurable function of $(t,x) \ \epsilon \ [0,T] \times \mathbb{R}^n, \ 1 \leq i, j \leq n$;
4. $|f(t,x) - f(t,z)| + \|\sigma(t,x) - \sigma(t,z)\| \leq k|x - z|$;
5. $x(0)$ has distribution $\Pi_0(dx) = p_0(x)dx$, where $p_0(\cdot) \ \epsilon \ L^2(\mathbb{R}^n)$.

The above assumptions, with the exception of statement 4, are assumed to hold throughout the manuscript.

Next, we start with a reference probability measure which is important in deriving certain conditional densities for the filtering problem discussed earlier. Let $(\Omega, \mathcal{F}, P)$ be a reference probability with complete filtration $\{\mathcal{F}_{0,t}; t \ \epsilon \ [0,T]\}$, on which we have the following:

(a) $w : [0,T] \times \Omega \to \mathbb{R}^n, b : [0,T] \times \Omega \to \mathbb{R}^d$, which are $\{\mathcal{F}_{0,t}; t \ \epsilon \ [0,T]\}$ adapted independent Wiener processes;

(b) $x(0) : \Omega \to \mathbb{R}^n$, an $\mathcal{F}_{0,0}$-measurable random variable, which is independent of $\{w_t, b_t; t \ \epsilon \ [0,T]\}$;

(c) processes $\{x_t; t \ \epsilon \ [0,T]\}, \{y_t; t \ \epsilon \ [0,T]\}$, which (in view of Assumption 2.2) are unique and continuous solutions of the stochastic differential equations

$$(2.1) \quad dx_t = f(t,x_t)dt - \sigma(t,x_t)\alpha_t' C_t^{-1} h(t,x_t)dt + \sigma(t,x_t)dw_t, \quad x(0) \ \epsilon \ \mathbb{R}^n,$$

$$(2.2) \quad dy_t = \alpha_t dw_t + N_t^{1/2} db_t, \quad y(0) = 0 \ \epsilon \ \mathbb{R}^n,$$

where

$$(2.3) \qquad\qquad\qquad C_t \doteq \alpha_t \alpha_t' + N_t.$$

Consider the P-martingale

$$(2.4) \qquad\qquad\qquad m_t = \int_0^t h'(s,x_s) C_s^{-1} dy_s,$$

and introduce the exponential martingale

$$(2.5) \qquad \varepsilon(m_t) = \exp\left(m_t - \frac{1}{2}\langle m, m\rangle_t\right) = \Lambda_{0,t},$$

where $\langle m, m\rangle_t = \int_0^t |C_s^{-1/2}h(s, x_s)|^2 ds$ is the quadratic variation of $\{m_t; t \in [0, T]\}$. By Assumption 2.2, we have $\mathrm{E}[\Lambda_{0,t}] = 1 \forall t \in [0, T]$ (see [7]). Consequently, we define a measure $\widetilde{P}$ through the Radon–Nikodým derivative

$$(2.6) \qquad \Lambda_{0,T} \doteq \mathrm{E}\left[\frac{d\widetilde{P}}{dP}|\mathcal{F}_{0,T}\right] = \varepsilon(m_T).$$

Since $\widetilde{P}(\Omega) = \int_\Omega \Lambda_{0,t}(\omega)dP(\omega) = 1 \ \forall t \in [0, T]$, the Girsanov theorem (see [7]) states that $\widetilde{P}$ is a probability measure on $(\Omega, \mathcal{A})$ and that

$$\begin{bmatrix} \overline{w}_t \\ \overline{b}_t \end{bmatrix} = \begin{bmatrix} w_t \\ b_t \end{bmatrix} - \begin{bmatrix} \langle w, m\rangle_t \\ \langle b, m\rangle_t \end{bmatrix}$$

$$(2.7) \qquad = \begin{bmatrix} w_t \\ b_t \end{bmatrix} - \begin{bmatrix} \int_0^t \alpha_s' C_s^{-1} h(s, x_s)ds \\ \int_0^t N_s^{1/2,\prime} C_s^{-1} h(s, x_s)ds \end{bmatrix}$$

are independent Wiener processes on $(\Omega, \mathcal{F}, \widetilde{P}; \mathcal{F}_{0,t})$. Substituting (2.7) into (2.1), (2.2), on the new probability space $(\Omega, \mathcal{F}, \widetilde{P}; \mathcal{F}_{0,t})$ we have constructed (weak) solutions $\{x_t; t \in [0, T]\}, \{y_t; t \in [0, T]\}$ of the stochastic equations

$$(2.8) \qquad dx_t = f(t, x_t)dt + \sigma(t, x_t)d\overline{w}_t, \qquad x(0) \in \mathbb{R}^n,$$

$$(2.9) \qquad dy_t = h(t, x_t)dt + \alpha_t d\overline{w}_t + N_t^{1/2} d\overline{b}_t, \qquad y(0) = 0 \in \mathbb{R}^d.$$

Since $\{\overline{w}_t; t \in [0, T]\}$ and $\{\overline{b}_t; t \in [0, T]\}$ are versions of Wiener processes (which are independent), (2.8), (2.9) constitute our original system model (simply by letting $\overline{w} \to w, \overline{b} \to b$). Note that we may remove the Lipschitz condition Assumption 2.2, statement 4, and employ the martingale approach to construct weak solutions.

Notation 2.3.

1. $\{\mathcal{F}_{0,t}^y; t \in [0, T]\}$ denotes the complete filtration generated by the observations σ-algebra $\sigma\{y_\tau; 0 \le \tau \le t\}, \{\mathcal{F}_{0,t}^w; t \in [0, T]\}$ denotes that of $\sigma\{w_\tau; 0 \le \tau \le t\}$, and $\mathcal{F}^{x(0)} = \sigma\{x(0)\}$;

2. The measure-valued process $q_t(\Phi) = \mathrm{E}[\Phi(x_t)\Lambda_{0,t}|\mathcal{F}_{0,t}^y]$ is well defined.

The problem of least-squares filtering is concerned with estimating the conditional mean of x_t given the past and present measurements, i.e., $\mathcal{F}_{0,t}^y$. Thus, the least-squares filtering can be cast in terms of computing conditional expectations $\widetilde{\mathrm{E}}[\Phi(x_t)|\mathcal{F}_{0,t}^y]$.

LEMMA 2.4.

1. *A version of Bayes's formula yields*

$$(2.10) \qquad \widetilde{\mathrm{E}}[\Phi(x_t)|\mathcal{F}_{0,t}^y] = \frac{\mathrm{E}[\Phi(x_t)\frac{d\widetilde{P}}{dP}|\mathcal{F}_{0,t}^y]}{\mathrm{E}[\frac{d\widetilde{P}}{dP}|\mathcal{F}_{0,t}^y]} = \frac{q_t(\Phi)}{q_t(1)}.$$

2. *If the measure-valued process $q_t(\Phi)$ has an $\mathcal{F}_{0,t}^y$-measurable density function $q : \mathbb{R}^n \times [0, T] \times \Omega \to \mathbb{R}$, then*

$$(2.11) \qquad \widetilde{\mathrm{E}}[\Phi(x_t)|\mathcal{F}_{0,t}^y] = \frac{\int_{\mathbb{R}^n} \Phi(z)q(z, t)dz}{\int_{\mathbb{R}^n} q(z, t)dz}.$$

Proof. 1. A version of Bayes' rule yields the equality in (2.10).

2. The proof follows from the existence of the density $q(\cdot)$. □

The existence of the density $q(x, t)$ will follow from the existence and uniqueness of solutions of SPDEs [8, 9, 10], as it will be shown shortly.

We now derive an evolution equation for $q(\cdot)$. Note that $\{\Lambda_{0,t}; t \in [0, T]\}$ is a solution of the stochastic differential equation

$$(2.12) \qquad \Lambda_{0,t} = 1 + \int_0^t \Lambda_{0,s} h'(s, x_s) C_s^{-1} dy_s.$$

THEOREM 2.5. *The unnormalized density of the conditional distribution $\widetilde{P}(x_t \in A | \mathcal{F}_{0,t}^y)$, $A \in \mathcal{B}(\mathbb{R}^n)$ is $q(\cdot)$ and satisfies the SPDE*

$$(2.13) \qquad dq(z, t) = A(t)^* q(z, t) dt + B(t)^* q(z, t) dy_t, \qquad (z, t) \in (0, T] \times \mathbb{R}^n,$$

$$(2.14) \qquad q(z, 0) = p_0(z), \qquad z \in \mathbb{R}^n,$$

where

(2.15)

$$A(t)^* \Phi(x) = \frac{1}{2} \sum_{i,j=1}^n \left(\frac{\partial^2}{\partial x_i \partial x_j} \left((\sigma(t, x) \sigma'(t, x))_{i,j} \Phi(x) \right) \right) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(f_i(t, x) \Phi(x) \right),$$

(2.16)

$$B_k(t)^* \Phi(x) = \sum_{i=1}^d \left(C_t^{-1} \right)_{i,k} h_i(t, x) \Phi(x) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left((\sigma(t, x) \alpha_t' C_t^{-1})_{i,k} \Phi(x) \right).$$

Proof. Recall that under $P, \{x_t, y_t; t \in [0, T]\}$ are solutions of (2.1), (2.2). Define

$$(2.17) \qquad D_t \doteq I_m - \alpha_t' C_t^{-1} \alpha_t,$$

and introduce

$$(2.18) \qquad \widetilde{y}_t = \int_0^t C_s^{-1/2} dy_s, \qquad \widetilde{w}_t = \int_0^t D_s^{-1/2} (dw_s - \alpha_s' C_s^{-1} dy_s).$$

Substituting into (2.1) we have

$$(2.19) \qquad dx_t = \left(f(t, x_t) - \sigma(t, x_t) \alpha_t' C_t^{-1} h(t, x_t) \right) dt + \sigma(t, x_t) D_t^{1/2} d\widetilde{w}_t$$
$$+ \sigma(t, x_t) \alpha_t' C_t^{-1/2} d\widetilde{y}_t, \qquad x(0) \in \mathbb{R}^n.$$

Moreover, $\{\widetilde{y}_t; t \in [0, T]\}$ and $\{\widetilde{w}_t; t \in [0, T]\}$ are independent standard Wiener processes, and $\mathcal{F}_{0,t}^y = \mathcal{F}_{0,t}^{\widetilde{y}}$; that is, no information is gained or lost. By (2.12), (2.18) we deduce

$$(2.20) \qquad \Lambda_{0,t} = 1 + \int_0^t \Lambda_{0,s} h'(s, x_s) C_s^{-1/2} d\widetilde{y}_s.$$

By the Itô product rule

$$\Phi(x_t) \Lambda_{0,t} = \Phi(x(0)) + \int_0^t \Phi(x_s) d\Lambda_{0,s} + \int_0^t d\Phi(x_s) \Lambda_{0,s}$$
$$(2.21) \qquad \qquad + \int_0^t d\langle \Phi(x), \Lambda \rangle_s.$$

Since

$$\Phi(x_t) = \Phi(x(0)) + \frac{1}{2}\int_0^t \mathrm{Tr}\big(\sigma(s,x_s)D_s'\sigma'(s,x_s)D_x^2\Phi(x_s)\big)ds$$

$$+\frac{1}{2}\mathrm{Tr}\big(\sigma(s,x_s)\alpha_s'C_s^{-1}\alpha_s\sigma'(s,x_s)D_x^2\Phi(x_s)\big)ds$$

$$+\int_0^t D_x'\Phi(x_s)\big(f(s,x_s)-\sigma(s,x_s)\alpha_s'C_s^{-1}h(s,x_s)\big)ds$$

$$+\int_0^t D_x'\Phi(x_s)\sigma(s,x_s)D_s^{1/2}d\widetilde{w}_s + \int_0^t D_x'\Phi(x_s)\sigma(s,x_s)\alpha_s'C_s^{-1/2}d\widetilde{y}_s,$$

$$\langle\Phi(x),\Lambda\rangle_t = \int_0^t \Lambda_{0,s}D_x'\Phi(x_s)\sigma(s,x_s)\alpha_s'C_s^{-1}h(s,x_s)ds,$$

substituting into (2.21) we have

$$\Phi(x_t)\Lambda_{0,t} = \Phi(x(0)) + \frac{1}{2}\int_0^t \Lambda_{0,s}\mathrm{Tr}\big(\sigma(s,x_s)D_s\sigma'(s,x_s)D_x^2\Phi(x_s)\big)ds$$

$$+\frac{1}{2}\int_0^t \Lambda_{0,s}\mathrm{Tr}\big(\sigma(s,x_s)\alpha_s'C_s^{-1}\alpha_s\sigma'(s,x_s)D_x^2\Phi(x_s)\big)ds$$

$$+\int_0^t \Lambda_{0,s}D_x'\Phi(x_s)\big(f(s,x_s)-\sigma(s,x_s)\alpha_s'C_s^{-1}h(s,x_s)\big)ds$$

$$+\int_0^t \Lambda_{0,s}D_x'\Phi(x_s)\sigma(s,x_s)D_s^{1/2}d\widetilde{w}_s + \int_0^t \Lambda_{0,s}\Phi(x_s)h'(s,x_s)C_s^{-1/2}d\widetilde{y}_s$$

(2.22)
$$+\int_0^t \Lambda_{0,s}D_x'\Phi(x_s)\sigma(s,x_s)\alpha_s'C_s^{1/2}d\widetilde{y}_s$$

$$+\int_0^t \Lambda_{0,s}D_x'\Phi(x_s)\sigma(s,x_s)\alpha_s'C_s^{-1}h(s,x_s)ds.$$

Conditioning each side of (2.22) on $\mathcal{F}_{0,t}^y$ and then using the mutual independence of $x(0), \{\widetilde{w}_t; t\,\epsilon\,[0,T]\}, \{\widetilde{y}_t; t\,\epsilon\,[0,T]\}$ (see [11]) and a version of Fubinis theorem [7, 12], we conclude that

(2.23)
$$q_t(\Phi) = q_0(\Phi) + \int_0^t q_s(A(s)\Phi(x))ds + \int_0^t q_s(B(s)\Phi(x))C_s^{-1/2}d\widetilde{y}_s.$$

Integrating each term by parts and then substituting $\widetilde{y}_t = \int_0^t C_s^{-1/2}dy_s$ we obtain (2.13), (2.14).   □

Next, we employ certain results of variational methods of partial differential equations to show existence and uniqueness of solutions to (2.13) and (2.14).

Introduce the space $H(\mathbb{R}^n) = L^2(\mathbb{R})$ and the Sobolev space $H^1(\mathbb{R}^n)$ defined by

$$H^1(\mathbb{R}^n) = \left\{ u\,\epsilon\,L^2(\mathbb{R}^n),\ \frac{\partial}{\partial x_i}u\,\epsilon\,L^2(\mathbb{R}^n), 1 \leq i \leq n \right\}.$$

Furnish $H(\mathbb{R}^n), H^1(\mathbb{R}^n)$ with the norm topologies

$$\|u\|_H = \int_{\mathbb{R}^n}|u|^2dx,\ u\,\epsilon\,H(\mathbb{R}^n),$$

$$\|u\|_{H^1} = \left\{\int_{\mathbb{R}^n}|u|^2dx + \sum_{i=1}^n\int\left|\frac{\partial}{\partial x_i}u\right|^2dx\right\}^{1/2},\ u\,\epsilon\,H^1(\mathbb{R}^n).$$

$H(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ are Hilbert spaces with scalar products defined by

$$(\phi, \psi)_H = \int_{\mathbb{R}^n} \phi \psi dx, \ \phi, \psi \ \epsilon \ H(\mathbb{R}^n),$$

$$(\phi, \psi)_{H^1} = \int_{\mathbb{R}^n} \phi \psi dx + \sum_{i=1}^{n} \int \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_i} dx = (\phi, \psi)_{L^2(\mathbb{R}^n)}$$

$$+ \sum_{i=1}^{n} \left(\frac{\partial \phi}{\partial x_i}, \frac{\partial \psi}{\partial x_i} \right)_{L^2(\mathbb{R}^n)}, \ \phi, \psi \ \epsilon \ H^1(\mathbb{R}^n).$$

Let $H^{-1}(\mathbb{R}^n)$ denote the dual of $H^1(\mathbb{R}^n)$ (the space of continuous linear functionals on $H^1(\mathbb{R}^n)$). The norm of elements of $H^{-1}(\mathbb{R}^n)$ is denoted by $\|\cdot\|_*$, and the duality between $H^1(\mathbb{R}^n)$ and $H^{-1}(\mathbb{R}^n)$ is denoted by $\langle \cdot, \cdot \rangle$.

Let

$$B(\cdot)^* u = \begin{bmatrix} B_1(\cdot)^* u \\ \vdots \\ B_d(\cdot)^* u \end{bmatrix}, \ u \ \epsilon \ H^1(\mathbb{R}^n),$$

and write the adjoint operators of $A(\cdot)^*$ and $B(\cdot)^*$ as

$$\langle u, A(t)^* v \rangle = \langle \ A(t)u, v \rangle = -\frac{1}{2} \sum_{i,j=1}^{n} \left(a_{i,j}(t, \cdot) \frac{\partial}{\partial x_i} u, \frac{\partial}{\partial x_j} v \right)_{L^2(\mathbb{R}^n)}$$

$$+ \sum_{i=1}^{n} \left(\tilde{f}_i(t, \cdot) \frac{\partial}{\partial x_i} u, v \right)_{L^2(\mathbb{R}^n)}, \qquad u, v \ \epsilon \ H^1(\mathbb{R}^n),$$

where

$$\tilde{f}_i(t, x) = f_i(t, x) - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_{i,j}(t, x)$$

$$\langle u, B(t)^* v \rangle = \langle \ B(t)u, v \rangle = \sum_{i,k=1}^{d} ((C_t^{-1})_{i,k} h_i(t, \cdot) u, v)_{L^2(\mathbb{R}^n)}$$

$$+ \sum_{k=1}^{d} \sum_{i=1}^{n} \left((\sigma(t, \cdot) \alpha_t' C_t^{-1})_{i,k} \frac{\partial}{\partial x_i} u, v \right)_{L^2(\mathbb{R}^n)}, \qquad u, v \ \epsilon \ H^1(\mathbb{R}^n).$$

In view of Assumption 2.2, statements 1, 2, 3, and 5, it can be shown that

$$A(\cdot), A(\cdot)^* \ \epsilon \ L^\infty((0, T); \mathcal{L}(H^1(\mathbb{R}^n); H^{-1}(\mathbb{R}^n))),$$
(2.24)
$$B(\cdot), B(\cdot)^* \ \epsilon \ L^\infty((0, T); \mathcal{L}(H^1(\mathbb{R}^n); (L^2(\mathbb{R}^n))^d)).$$

Moreover, $A(t) \ \epsilon \ \mathcal{L}(H^1(\mathbb{R}^n); H^{-1}(\mathbb{R}^n))$, $B(t) \ \epsilon \ \mathcal{L}(H^1(\mathbb{R}^n); (L^2(\mathbb{R}^n))^d$ satisfy the following coercivity condition. There exist $\lambda_1, \lambda_2 > 0$ such that

(2.25) $\quad -2\langle A(t)u, u \rangle + \lambda_1 \|u\|_{L^2(\mathbb{R}^n)}^2 \geq \lambda_2 \|u\|_{H^1(\mathbb{R}^n)}^2$

$$+ \|Bu\|_{(L^2(\mathbb{R}^n))^d}^2 \ \forall u \ \epsilon \ H^1(\mathbb{R}^n), \ \forall t \ \epsilon \ [0, T].$$

Define the space

$$L_y^2((0,T); H^1)$$
$$\doteq \{u \epsilon L^2(\Omega, \mathcal{F}, P; L^2((0,T); H^1); \text{ a.e. on } [0,T], u(t) \epsilon L^2(\Omega, \mathcal{F}_{0,t}^y, P; H^1)\}.$$

LEMMA 2.6. *There exists one and only one solution $q(\cdot)$ of (2.13), (2.14) in the space*

$$q(\cdot) \epsilon L_y^2((0,T); H^1) \cap L^2(\Omega, \mathcal{F}, P; C((0,T); H)).$$

Proof. Assumption 2.2 statements 1, 2, 3, and 5 imply the coercivity condition (2.25), which is then employed to show existence and uniqueness of solutions to (2.13), (2.14) (see [8, 9, 10]).   □

The next tool employed in subsequent sections is the concept of fundamental solutions to stochastic differential equations.

DEFINITION 2.7. *A fundamental solution of (2.13), (2.14) is an $\mathcal{F}_{0,t}^y$-measurable function $q(z,t;x,s)$, with $(z,x) \epsilon \mathbb{R}^n \times \mathbb{R}^n, 0 \le s < t \le T$, such that the following hold:*

1. *$q(\cdot,\cdot;x,s)$ is a solution of*

(2.26)
$$dq(z,t;x,s) = A(t)^* q(z,t;x,s)dt + B(t)^* q(z,t;x,s)dy_t, \quad 0 < s < t \le T,$$
(2.27)
$$\lim_{t \downarrow s} q(z,t;x,s) = \delta(z-x).$$

2. *For fixed $(s,x) \epsilon (0,t) \times \mathbb{R}^n$, $q(\cdot,t;x,s) \epsilon C_z^2(\mathbb{R}^n)$.*
3. *For $\varphi : \mathbb{R}^n \to \mathbb{R}$, which is continuous with compact support,*

(2.28)
$$\lim_{t \downarrow s} \int_{-\infty}^{\infty} q(z,t;x,s)\varphi(x)dx = \varphi(z).$$

That is, $\lim_{t \downarrow s} q(z,t;x,s) = \delta(z-x)$ is a Dirac delta function.

Unfortunately, Assumption 2.2 is too weak to imply that $q(\cdot,t;x,s) \epsilon C_z^2(\mathbb{R}^n)$. However, if there is no correlation between the state noise and the observation noise (e.g., $\alpha_t = 0 \ \forall t \epsilon [0,T]$), and we impose additional smoothness and continuity conditions on (f,σ,h), then by considering the pathwise version of (2.13), (2.14), it can be shown that for each $y \epsilon C([0,T]; \mathbb{R}^d)$ there exists a unique solution, which is a fundamental solution [13]. For the correlated case, we have the following result which is found in [9, 10] (see also [14] for alternative conditions).

THEOREM 2.8. *Suppose the coefficient of A and B_k, $k = 1,\dots,d$ have bounded partial derivatives in x of any order. Then*

1. *$\{q(z,t;x,s); 0 \le s < t \le T\}$, $(z,x) \epsilon \mathbb{R}^n \times \mathbb{R}^n$, is a unique fundamental solution of the unnormalized condition density equation (2.13), (2.14), and $q(\cdot,t;x,s) \epsilon C_b^\infty(\mathbb{R}^n)$, $P - a.s. \forall t \epsilon (s,T]$.*
2. *A version of the conditional distribution $\widetilde{P}(x_t \epsilon A|\mathcal{F}_{0,t}^y), A \epsilon \mathcal{B}(\mathbb{R}^n)$, is*

(2.29)
$$\widetilde{E}[\Phi(x_t)|\mathcal{F}_{0,t}^y] = \frac{q_t(\Phi)}{q_t(1)} = \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \Phi(z)q(z,t;x,0)p_0(x)dxdz}{\int_{\mathbb{R}^n \times \mathbb{R}^n} q(z,t;x,0)p_0(x)dxdz}.$$

Proof. 1. This is shown in [10, pp. 227–228].

2. Let $q(z, t; x, s)$ be a solution of (2.26), (2.27); set $\widetilde{q}(z, t) = \int_{\mathbb{R}^n} q(z, t; x, 0)p_0(x)dx$. Then

$$d\widetilde{q}(z, t) = \int_{\mathbb{R}^n} dq(z, t; x, 0)p_0(x)dx$$

$$= \int_{\mathbb{R}^n} A(t)^*q(z, t; x, 0)p_0(x)dxdt + \int_{\mathbb{R}^n} B(t)^*q(z, t; x, 0)p_0dxdy_t$$

$$= A(t)^*\widetilde{q}(z, t)dt + B(t)^*\widetilde{q}(z, t)dy_t.$$

This shows that $\widetilde{q}(z, t)$ satisfies (2.13) for $(z, t) \in \mathbb{R}^n \times (0, T]$. Since $\lim_{t \downarrow 0} \widetilde{q}(z, t) = \lim_{t \downarrow 0} \int_{\mathbb{R}^n} q(z, t; x, 0)p_0(x)dx = p_0(z)$, we also have (2.14). By Lemma 2.4 we establish (2.29). $\square$

DEFINITION 2.9. *Let* $f^1 : [0, T] \times \mathbb{R}^n \to \mathbb{R}, f^2 : [0, T] \times \mathbb{R}^n \to (\mathbb{R}^n)', f^3 : [0, T] \times \mathbb{R}^n \to (\mathbb{R}^d)'$ *be Borel measurable and bounded functions.*

1. *The integrals*

$$(2.30) \quad L_{0,t}^{\kappa,1} = \left(\int_0^t f^1(s, x_s)ds \right)^\kappa, \quad L_{0,t}^{\kappa,2} = \left(\int_0^t f^2(s, x_s)dw_s \right)^\kappa,$$

$$L_{0,t}^{\kappa,3} = \left(\int_0^t f^3(s, x_s)db_s \right)^\kappa, \quad \kappa \geq 1,$$

are well defined.

2. *The measure-valued processes*

$$(2.31) \qquad M_t^{\kappa,j}(\Phi) = \mathrm{E}[\Phi(x_t)\Lambda_{0,t}L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y], \quad \kappa \geq 0, \quad j = 1, 2, 3,$$

are well defined.

We are interested in filtered estimates of κth moments $(\kappa \geq 1)$ of integrals and stochastic integrals. That is, we wish to derive expressions for $\widetilde{\mathrm{E}}[L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y]$. An application of Bayes's theorem yields

$$(2.32) \qquad \widetilde{\mathrm{E}}[L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y] = \frac{\mathrm{E}[\Lambda_{0,t}L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y]}{\mathrm{E}[\Lambda_{0,t}|\mathcal{F}_{0,t}^y]}, \quad \kappa \geq 1, \quad j = 1, 2, 3.$$

3. Recursive equations. Here we prove that the filtered estimates (2.32) can be expressed in terms of the fundamental solution of the DMZ equation; namely, $q(z, t; x, s), 0 \leq s < t \leq T$, which satisfies (2.13), (2.14). This enables us to conclude that if $q(z, t; x, s)$ is a finite-dimensional statistic, then these filtered estimates can be described in terms of solutions of a finite-number of stochastic differential equations.

THEOREM 3.1. *Suppose* $M_t^{\kappa,j}(\cdot)$ *have* $\mathcal{F}_{0,t}^y$*-measurable density functions* $M^{\kappa,j} : \mathbb{R}^n \times [0, T] \times \Omega \to \mathbb{R}, j = 1, 2, 3.$
Then

$$(3.1) \qquad M^{\kappa,j}(x, t)dx = \mathrm{E}[I_{x_t \in dx}\Lambda_{0,t}L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y], \quad \kappa \geq 1, \quad j = 1, 2, 3,$$

satisfy the following recursive system of SPDEs:

$$dM^{\kappa,1}(x,t) = A(t)^* M^{\kappa,1}(x,t)dt + B(t)^* M^{\kappa,1}(x,t)dy_t$$

(3.2)
$$+ \kappa f^1(t,x) M^{k-1,1}(x,t)dt, \quad \kappa \geq 1, \quad (t,x) \in (0,T] \times \mathbb{R}^n,$$

$$dM^{\kappa,2}(x,t) = A(t)^* M^{\kappa,2}(x,t)dt + B(t)^* M^{\kappa,2}(x,t)dy_t$$

$$+ \frac{1}{2}\kappa(\kappa-1)|f^{2,\prime}(t,x)|^2 M^{\kappa-2,2}(x,t)dt$$

$$- \kappa \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(M^{\kappa-1,2}(x,t) \left(\sigma(t,x) f^{2,\prime}(x,t) \right)_i \right) dt$$

(3.3)
$$+ \kappa f^2(t,x) M^{\kappa-1,2}(x,t)\alpha'_t C_t^{-1}dy_t, \quad \kappa \geq 1, \quad (t,x) \in (0,T] \times \mathbb{R}^n,$$

$$dM^{\kappa,3}(x,t) = A(t)^* M^{\kappa,3}(x,t)dt + B(t)^* M^{\kappa,3}(x,t)dy_t$$

$$+ \frac{1}{2}\kappa(k-1)|C^{1/2}N^{-1/2}f^{3,\prime}(t,x)|^2 M^{\kappa-2,3}(x,t)dt$$

(3.4)
$$+ \kappa f^3(t,x) M^{\kappa-1,3}(x,t)N^{1/2}C^{-1}dy_t, \quad \kappa \geq 1, \quad (t,x) \in (0,T] \times \mathbb{R}^n,$$

where the convention $M^{p,j}(x,t) = 0$ *for* $p < 0$ *is used. The initial conditions are*

(3.5)
$$M^{\kappa,j}(x,0) = 0, \quad \kappa \geq 1, \quad j = 1,2,3,$$

and for $\kappa = 0$

(3.6)
$$M^{0,j}(x,t) = q(x,t), \quad j = 1,2,3.$$

Proof. We shall use induction. Consider (3.2). Now, the case $\kappa = 1$ is easily verified, so it is omitted. Suppose (3.2) holds for $\kappa \to k-1$. We shall show that it also holds for κ. To this end, consider $\Phi(x_t)\Lambda_{0,t}L_{0,t}^{\kappa,1}$, where $\{x_t; t \in [0,T]\}$ and $\{\Lambda_{0,t}; t \in [0,T]\}$ are solutions of (2.19), (2.20), respectively. By the Itô product rule

(3.7)
$$L_{0,t}^{\kappa,1} = \kappa \int_0^t L_{0,s}^{\kappa-1,1} f^1(s,x_s)ds, \quad \kappa \geq 1.$$

Employing the Itô product rule once again, we have

$$\Phi(x_t)\Lambda_{0,t}L_{0,t}^{\kappa,1} = \int_0^t \Phi(x_s)d(\Lambda_{0,s}L_s^{\kappa,1}) + \int_0^t d\Phi(x_s)\Lambda_{0,s}L_{0,s}^{\kappa,1}$$

(3.8)
$$+ \int_0^t \langle \Phi(x), \Lambda L^{\kappa,1} \rangle_s.$$

Now, from (3.7), (2.20) we compute

$$\Lambda_{0,t}L_{0,t}^{\kappa,1} = \int_0^t \Lambda_{0,s}dL_{0,s}^{\kappa,1} + \int_0^t L_{0,s}^{\kappa,1}d\Lambda_{0,s} + \int_0^t d\langle \Lambda, L^{\kappa,1} \rangle_t$$

(3.9)
$$= \kappa \int_0^t f^1(s,x_s)\Lambda_{0,s}L_{0,s}^{\kappa-1,1}ds + \int_0^t \Lambda_{0,s}L_{0,s}^{\kappa,1}h'(s,x_s)C_s^{-1/2}d\widetilde{y}_s.$$

Substituting (3.9) into (3.8) and then proceeding as in the derivation of Theorem

2.5, we obtain

$$
\Phi(x_s)\Lambda_{0,t}L_{0,t}^{\kappa,1} = \frac{1}{2}\int_0^t \Lambda_{0,s}L_{0,s}^{\kappa,1}\mathrm{Tr}\left(\sigma(s,x_s)\sigma'(s,x_s)D_x^2\Phi(x_s)\right)ds
$$

$$
+ \int_0^t \Lambda_{0,s}L_{0,s}^{\kappa,1}D_x'\Phi(x_s)\sigma(s,x_s)D_s^{1/2}d\widetilde{w}_s
$$

$$
+ \int_0^t \Lambda_{0,s}L_{0,s}^{\kappa,1}\Phi(x_s)h'(s,x_s)C_s^{-1/2}d\widetilde{y}_s
$$

$$
(3.10) \qquad + \int_0^t \Lambda_{0,s}L_{0,s}^{\kappa,1}D_x'\Phi(x_s)\sigma(s,x_s)\alpha_s'C_s^{1/2}d\widetilde{y}_s + \kappa\int_0^t \Lambda_{0,s}L_{0,s}^{\kappa-1,1}f^1(s,x_s)ds.
$$

Conditioning each side of (3.10) on $\mathcal{F}_{0,t}^y$ using (3.1), and then integrating by parts, we deduce (3.2). When $\kappa = 0, j = 1$, we have $M^{0,1}(x,t)dx = \mathrm{E}[I_{x_t\epsilon dx}\Lambda_{0,t}|\mathcal{F}_{0,t}^y]$, and thus $M^{0,1}(x,t)$ satisfies the DMZ equation.

The derivation (3.3) is done similarly; therefore we shall outline only the important steps. Under measure P,

$$
(3.11) \qquad L_{0,t}^{\kappa,2} = \left[\int_0^t f^2(s,x_s)(dw_s - \alpha_s'C_s^{-1}h(s,x_s)ds)\right]^\kappa.
$$

Substituting $w_t = \int_0^t D_s^{1/2}d\widetilde{w}_s + \int_0^t \alpha_s'C_s^{-1/2}d\widetilde{y}_s$ into (3.11),

$$
(3.12) \qquad L_{0,t}^{\kappa,2} = \left[\int_0^t f^2(s,x_s)(D_s^{1/2}d\widetilde{w}_s + \alpha_s'C_s^{-1/2}d\widetilde{y}_s - \alpha_s'C_s^{-1}h(s,x_s)ds)\right]^\kappa.
$$

By the Itô product rule

$$
L_{0,t}^{\kappa,2} = \kappa\int_0^t L_{0,s}^{k-1,2}f^2(s,x_s)(D_s^{1/2}d\widetilde{w}_s + \alpha_s'C_s^{-1/2}d\widetilde{y}_s - \alpha_s'C_s^{-1}h(s,x_s)ds)
$$

$$
+ \frac{1}{2}\kappa(k-1)\int_0^t L_{0,s}^{k-2,2}f^2(s,x_s)D_s^{1/2}D_s^{1/2,'}f^{2,'}(s,x_s)ds
$$

$$
(3.13) \qquad + \frac{1}{2}\kappa(k-1)\int_0^t L_{0,s}^{k-2,2}f^2(s,x_s)\alpha_s'C_s^{-1}\alpha_s f^{2,'}(s,x_s)ds.
$$

Employing the Itô product rule to $\Phi(x_t)\Lambda_{0,t}L_{0,t}^{\kappa,2}$, as in (3.9), (3.10), and then invoking $M^{\kappa,2}(z,t)dx = \mathrm{E}[I_{x_t\epsilon dx}\Lambda_{0,t}L_{0,t}^{\kappa,2}|\mathcal{F}_{0,t}^y]$, after some algebra we derive (3.3), and (3.5) for $j = 2, \kappa \geq 2$. The special case $\kappa = 1, 2$ is done similarly. Also, to derive (3.4), we start with

$$
(3.14) \qquad L_{0,t}^{\kappa,3} = \left[\int_0^t f^3(s,x_s)\left(db_s - N_s^{1/2}C_s^{-1}h(s,x_s)ds\right)\right]^\kappa,
$$

which is defined under measure P, and then we follow the above procedure to obtain (3.4), and (3.5), for $j = 3$.   □

Next, we establish existence and uniqueness of the moment processes $M^{\kappa,j}(\cdot), \kappa \geq 1, j = 1, 2, 3$, using the variational methods of SPDEs, similar to Theorem 3.1.

Clearly, (3.2)–(3.4) with their corresponding boundary conditions (3.5), (3.6) are

of the general form

$$M(x,t) = \int_0^t A(s)^* M(x,s)ds + \int_0^t B(s)^* M(x,s)dy_s + \int_0^t \psi(s)ds$$

(3.15)
$$+ \int_0^t \phi(s)dy_s + \int_0^t \eta(s)ds,$$

where $\eta(t) \in L_y^2((0,T);H^1), \psi(t) \in L_y^2((0,T);H^{-1}), \phi(t) \in L_y^2((0,T);(L^2(\mathbb{R}^n))^d)$. For example, the fourth right side term of (3.3) belongs to $L_y^2((0,T);(L^2(\mathbb{R}^n))^d)$. Therefore, for finite κ, an application of variational methods of SPDEs (see [8, 9, 10]) implies there exists one and only one solution to (3.15) in the space $M(\cdot) \in L_y^2((0,T);H^1) \cap L_y^2(\Omega,\mathcal{F},P;C((0,T);H))$. Consequently, the moment processes of Theorem 3.1 have unique solutions as well.

Notice that the filtered estimates for $L_{0,t}^{\kappa,j}, \kappa \geq 1, j = 1,2,3$, can be computed from

(3.16)
$$\widetilde{E}[L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y] = \frac{\int_{\mathbb{R}^n} M^{\kappa,j}(z,t)dz}{\int_{\mathbb{R}^n} q(z,t)dz}, \quad \kappa \geq 1, \quad j = 1,2,3.$$

Clearly, if the fundamental solution of the DMZ equation $q(t,t;x,s)$ is finite-dimensional, then according to Lemma 3.2, (3.16) can be computed explicitly in terms of finite numbers of statistics.

LEMMA 3.2. *Suppose the coefficients of $A, B_k, \ k = 1,\dots,d$, and $f^j, \ 1 \leq j \leq 3$, have bounded partial derivatives in x of any order. Then $M_t^{\kappa,j}(\cdot)$ have $\mathcal{F}_{0,t}^y$-measurable density functions given by*

(3.17)
$$M^{\kappa,1}(z,t) = \kappa \int_0^t \int_{\mathbb{R}^n} f^1(s,x)M^{\kappa-1,1}(x,s)q(z,t;x,s)dxds, \quad \kappa \geq 1,$$

$$M^{\kappa,2}(z,t) = \frac{1}{2}\kappa(\kappa-1)\int_0^t \int_{\mathbb{R}^n} |f^2(s,x)|^2 M^{k-2,2}(x,s)q(z,t;x,s)dxds$$

$$- \kappa \int_0^t \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(M^{k-1,2}(x,s)\left(\sigma(s,x)f^{2,\prime}(s,x)\right)_i\right)q(z,t;x,s)dxds$$

(3.18)
$$+ \kappa \int_0^t \int_{\mathbb{R}^n} f^2(s,x)M^{k-1,2}(x,s)\alpha_s' C_s^{-1}q(z,t;x,s)dxdy_s, \quad \kappa \geq 1,$$

$$M^{\kappa,3}(z,t) = \frac{1}{2}\kappa(k-1)\int_0^t \int_{\mathbb{R}^n} |C_s^{1/2}N_s^{-1/2}f^3(s,x)|^2 M^{k-2,3}(x,s)q(z,t;x,s)dxds$$

(3.19)
$$+ \kappa \int_0^t \int_{\mathbb{R}^n} f^3(s,x)M^{k-1,3}(x,s)N^{1/2}C_s^{-1}q(z,t;x,s)C_s^{-1}dxdy_s, \quad \kappa \geq 1,$$

with the convention $M^{p,j}(x,t) = 0$ for $p < 0, \ j = 1,2,3$.

Proof. Theorem 2.8 establishes the existence and uniqueness of a fundamental solution to the DMZ equation. Let $\widehat{M}^{\kappa,1}(z,t)$ denote the right side of (3.17). For $\kappa = 1$, we have

$$\widehat{M}^{1,1}(z,t) = \int_0^t \int_{\mathbb{R}^n} f^1(x,s)q(x,s)q(z,t;x,s)dxds,$$

because $M^{0,1}(\cdot,\cdot) = q(\cdot,\cdot)$. Then

$$
\begin{aligned}
d\widehat{M}^{1,1}(z,t) &= f^1(t,z)q(z,t)dt + \int_0^t \int_{\mathbb{R}^n} f^1(s,x)q(x,s)dq(z,t;x,s)dxds \\
&= f^1(t,z)q(z,t)dt + A(t)^* \int_0^t \int_{\mathbb{R}^n} f^1(s,x)q(x,s)q(z,t;x,s)dxdsdt \\
&\quad + B(t)^* \int_0^t \int_{\mathbb{R}^n} f^1(s,x)q(x,s)q(z,t)dxdsdy_t \\
&= A(t)^* \widehat{M}^{1,1}(z,t)dt + B(t)^* \widehat{M}^{1,1}(z,t)dy_t + f^1(t,z)q(z,t)dt.
\end{aligned}
$$

Thus, $\widehat{M}^{1,1}(\cdot,\cdot)$ satisfies (3.2); for $t = 0$, $\widehat{M}^{1,1}(z,0)$, and so (3.17) holds for $\kappa = 1$. Let

$$
\tag{3.20} \widehat{M}^{\kappa,1}(z,t) = \kappa \int_0^t \int_{\mathbb{R}^n} f^1(s,x)\widehat{M}^{k-1,1}(x,s)q(z,t;x,s)dxds
$$

and assume it satisfies (3.2) for $(t,z) \in (0,T] \times \mathbb{R}^n$, and (3.5) for $t = 0$. We shall show that

$$
\tag{3.21} \widehat{M}^{k+1,1}(z,t) = (k+1) \int_0^t \int_{\mathbb{R}^n} f^1(s,x)\widehat{M}^{k,1}(x,s)q(z,t;x,s)dxds
$$

satisfies (3.2), with $k \to k+1$, for $(t,z) \in (0,T] \times \mathbb{R}^n$. Clearly, $\widehat{M}^{k+1,1}(z,0) = 0$, so (3.5) holds (with $j = 1$). Now,

$$
\begin{aligned}
d\widehat{M}^{k+1,1}(z,t) &= (k+1)f^1(t,z)\widehat{M}^{k,1}(z,t)dt \\
&\quad + (k+1)\int_0^t \int_{\mathbb{R}^n} f^1(s,x)\widehat{M}^{k,1}(x,s)dq(z,t;x,s)dxds \\
&= (k+1)f^1(t,z)\widehat{M}^{k,1}(z,t)dt \\
&\quad + (k+1)A^*(t)\int_0^t \int_{\mathbb{R}^n} f^1(s,x)\widehat{M}^{k,1}(x,s)q(z,t;x,s)dxds \\
&\quad + (k+1)B(t)^*\int_0^t \int_{\mathbb{R}^n} f^1(s,x)\widehat{M}^{k,1}(x,s)q(z,t;x,s)dxdy_s \\
&= (k+1)f^1(t,z)\widehat{M}^{k,1}(z,t)dt + A^*(t)\widehat{M}^{k+1,1}(z,t)dt + B(t)^*\widehat{M}^{k+1,1}(z,t)dy_t.
\end{aligned}
$$

Hence (3.17) satisfies (3.2), (3.5) with $k \to k+1$. Similarly, one may use induction to verify the representations (3.18), (3.19). □

Next, we introduce an example to demonstrate the computations described in (3.17).

3.1. Specific application. Consider the system

$$
\begin{aligned}
dx_t &= Fx_t dt + Gdw_t, &\quad x(0) \in \mathbb{R}^n, \\
dy_t &= Hx_t + N^{\frac{1}{2}}db_t, &\quad y(0) = 0 \in \mathbb{R}^d.
\end{aligned}
$$

The random variable $x(0)$ is Gaussian. Although the above linear system does not satisfy Assumption 2.2, statements 1, 2, 3, and 5, the fundamental solution of the DMZ equation exists, and it is given by

$$
\tag{3.22} q(z,t;x,s) = \frac{1}{(2\pi)^{n/2}|P_{s,t}|^{1/2}} \exp\left(-\frac{1}{2}|P_{s,t}^{-1/2}(z - r_{s,t}(x))|^2\right) \times \Lambda_{s,t}(x),
$$

$$(3.23) \quad dr_{s,t}(x) = \left(F - P_{s,t}H'N^{-1}H\right)r_{s,t}(x)dt + P_{s,t}H'N^{-1}dy_t, \quad r_{s,s}(x) = x,$$

$$(3.24) \quad \dot{P}_{s,t} = FP_{s,t} + P_{s,t}F' - P_{s,t}H'N^{-1}HP_{s,t} + GG', \quad P_{s,s} = 0,$$

$$(3.25) \quad \Lambda_{s,t}(x) = \exp\left(\int_s^t (Hr_{s,\tau})'N^{-1}dy_\tau - \frac{1}{2}\int_s^t |N_\tau^{-\frac{1}{2}}Hr_{s,\tau}(x)|^2 d\tau\right).$$

Let

$$r_{s,t}(x) = \Phi_{s,t}x + \beta_{s,t},$$

where

$$\dot{\Phi}_{s,t} = \left(F - P_{s,t}H'N^{-1}H\right)\Phi_{s,t}, \quad \Phi_{s,s} = I_n,$$

$$d\beta_{s,t} = \left(F - P_{s,t}H'N^{-1}H\right)\beta_{s,t}dt + P_{s,t}H'N^{-1}dy_t, \quad \beta_{s,s} = 0.$$

Then

$$\Lambda_{s,t}(x) = \gamma_{s,t}\exp\left(x'\rho_{s,t} - \frac{1}{2}x'S_{s,t}x\right),$$

where

$$\gamma_{s,t} = \exp\left(\int_s^t \beta'_{s,\tau}H'N^{-1}dy_\tau - \frac{1}{2}\int_s^t |N^{-1/2}H\beta_{s,\tau}|^2 d\tau\right),$$

$$S_{s,t} = \int_s^t \Phi'_{s,\tau}H'N^{-1}H\Phi_{s,\tau}d\tau,$$

$$\rho_{s,t} = \int_s^t \Phi'_{s,\tau}H'N^{-1}\left(dy_\tau - H\beta_{s,\tau}d\tau\right).$$

Moreover, the unnormalized conditional density of x_s given $\mathcal{F}_{0,s}^y$ is

$$q(x,s) = \frac{1}{(2\pi)^{n/2}|\Sigma_{0,s}|^{1/2}}\exp\left(-\frac{1}{2}|\Sigma_{0,s}^{-1/2}(x - \widehat{x}_{0,s})|^2\right) \times \widehat{\Lambda}_{0,s},$$

where $\widehat{x}_{0,s}$ is a solution of (3.23) with $\widehat{x}_{0,0} = \xi$, $\Sigma_{0,s}$ is a solution of (3.24) with $\Sigma_{0,0} = \Sigma_0$, and $\widehat{\Lambda}_{0,s}$ is given by (3.25) with $r \to \widehat{x}, P \to \Sigma$. Notice that

$$\int_{\mathbb{R}^n} q(x,s)q(z,t;x,s)dx = \frac{1}{(2\pi)^{n/2}|P_{s,t}|^{1/2}} \times \frac{1}{(2\pi)^{n/2}|\Sigma_{0,s}|^{1/2}} \times \widehat{\Lambda}_{0,s} \times \gamma_{s,t}$$

$$\times \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}|P_{s,t}^{-1/2}(z - \Phi_{s,t}x - \beta_{s,t})|^2 - \frac{1}{2}|\Sigma_{0,s}^{-1/2}(x - \widehat{x}_{0,s})|^2 + x'\rho_{s,t} - \frac{1}{2}x'S_{s,t}x\right)dx.$$

Therefore, the integral with respect to x is computed by completing the squares. Consequently, we deduce that $\tilde{E}\left[\int_0^t f(x_s)ds|\mathcal{F}_{0,t}^y\right]$ is finite-dimensional computable for large classes of functions $f(x)$ such as $f(x) = x^p, p \geq 1, p$ an integer.

4. Conditional moment generating functions. Next we introduce moment generating functions for computing the conditional moments of integrals and stochastic integrals (2.32).

DEFINITION 4.1. *Let $\theta = i\omega, i = \sqrt{-1}$.*

1. *The measure-valued conditional moment generating functions of the stochastic processes $\{L_{0,t}^{1,j}; t \,\epsilon\, [0,T]\}, j = 1, 2, 3$, given by*

$$(4.1) \qquad \widetilde{\beta}_t^{\theta,j}(\Phi) = \widetilde{\mathrm{E}}[\Phi(x_t)\exp\left(\theta L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y], \qquad j = 1, 2, 3,$$

   are well defined.

2. *The measure-valued unnormalized conditional moment generating functions of the stochastic processes $\{L_{0,t}^{1,j}; t \,\epsilon\, [0,T]\}, j = 1, 2, 3$, given by*

$$(4.2) \qquad \beta_t^{\theta,j}(\Phi) = \mathrm{E}[\Phi(x_t)\Lambda_{0,t}\exp\left(\theta L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y], \qquad j = 1, 2, 3,$$

   are well defined.

LEMMA 4.2. *Suppose $\beta_t^{\theta,j}(\cdot)$ have $\mathcal{F}_{0,t}^y$-measurable density function $\beta^{\theta,j} : \mathbb{R}^n \times [0,T] \times \Omega \to \mathbb{R}, j = 1, 2, 3$.*

1. *Then*

(4.3)
$$\widetilde{\mathrm{E}}[\Phi(x_t)\exp\left(\theta L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y] = \frac{\beta_t^{\theta,j}(\Phi)}{q_t(1)} = \frac{\int_{\mathbb{R}^n}\Phi(z)\beta^{\theta,j}(z,t)dz}{\int_{\mathbb{R}^n} q(z,t)dz}, \qquad j = 1, 2, 3.$$

2. *The conditional characteristic functions of the stochastic processes $\{L_{0,t}^{1,j}; t \,\epsilon\, [0,T]\}, j = 1, 2, 3$, are given by*

$$(4.4) \quad \widetilde{\mathrm{E}}\left[\exp\left(i\omega L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y\right] = \frac{\beta_t^{i\omega,j}(1)}{q_t(1)} = \frac{\int_{\mathbb{R}^n}\beta^{i\omega,j}(z,t)dz}{\int_{\mathbb{R}^n} q(z,t)dz}, \quad j = 1, 2, 3.$$

Proof. The proof is similar to Lemma 2.4.   □

THEOREM 4.3. *Suppose $\beta_t^{\theta,j}(\cdot)$ have $\mathcal{F}_{0,t}^y$-measurable density functions $\beta^{\theta,j}(\cdot), j = 1, 2, 3$.*

The densities of the measure-valued unnormalized conditional moment generating functions, namely,

$$(4.5) \qquad \beta^{\theta,j}(x,t)dx = \mathrm{E}\left[I_{x_t \epsilon dx}\Lambda_{0,t}\exp\left(\theta L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y\right], \qquad j = 1, 2, 3,$$

satisfy the following system of SPDEs:

$$d\beta^{\theta,1}(x,t) = A(t)^*\beta^{\theta,1}(x,t)dt + B(t)^*\beta^{\theta,1}(x,t)dy_t$$
$$(4.6) \qquad + \theta f^1(t,x)\beta^{\theta,1}(x,t)dt, \qquad (t,x) \,\epsilon\, (0,T] \times \mathbb{R}^n,$$

$$d\beta^{\theta,2}(x,t) = A(t)^*\beta^{\theta,2}(x,t)dt + B(t)^*\beta^{\theta,2}(x,t)dy_t$$
$$+ \frac{\theta^2}{2}|f^{2,\prime}(t,x)|^2\beta^{\theta,2}(x,t)dt - \theta\sum_{i=1}^n\frac{\partial}{\partial x_i}\left((\sigma(t,x)f^{2,\prime}(x,t))_i\beta^{\theta,2}(x,t)\right)dt$$

$$(4.7) \qquad\qquad + \theta f^2(t,x)\beta^{\theta,2}(x,t)\alpha'_t C_t^{-1} dy_t, \qquad (t,x) \in (0,T] \times \mathbb{R}^n,$$

$$
\begin{aligned}
d\beta^{\theta,3}(x,t) = {} & A(t)^* \beta^{\theta,3}(x,t)dt + B(t)^* \beta^{\theta,3}(x,t)dy_t \\
& + \frac{\theta^2}{2}|C^{1/2}N^{-1/2}f^{3,\prime}(t,x)|^2 \beta^{\theta,3}(x,t)dt \\
& + \theta f^3(t,x)\beta^{\theta,3}(x,t)N^{1/2}C^{-1}dy_t, \qquad (t,x) \in (0,T] \times \mathbb{R}^n.
\end{aligned}
$$
(4.8)

The initial conditions are

$$(4.9) \qquad\qquad \beta^{\theta,j}(x,0) = p_0(x), \qquad x \in \mathbb{R}^n, \qquad j = 1,2,3.$$

Proof. First, absorb $\exp\left(\theta L_{0,t}^{1,j}\right)$ in the exponential term $\Lambda_{0,t}$ by setting

$$\widehat{\Lambda}_{0,t}^j = \Lambda_{0,t} \exp\left(\theta L_{0,t}^{1,j}\right).$$

Second, apply the Itô product rule as in Theorem 3.1. This derivation is along the lines of information state equations in [13]. □

Equations (4.6), (4.7), (4.8) with their respective boundary conditions (4.9) are of the general form

$$(4.10) \qquad M^\theta(x,t) = p_0(x) + \int_0^t A^\theta(s)^* M(x,s)ds + \int_0^t B^\theta(s)^* M(x,s)dy_s,$$

where the operators $A^\theta(t)^*, B^\theta(t)^*$ and their adjoints $A^\theta(t), B^\theta(t)$ depend on θ. Moreover, for sufficiently small $\theta \in \mathbb{R}$, these operators are bounded linear operators as described in (2.24), and there exist $\lambda_1^\theta, \lambda_2^\theta$, which depend on $\theta \in \mathbb{R}$ such that they satisfy the coercivity condition (2.25). Consequently, similar to Lemma 2.6, there exists one and only one solution of (4.6)–(4.9) in the space $\beta^{\theta,j}(\cdot) \in L_y^2((0,T);H^1) \cap L^2(\Omega,\mathcal{F},P;C((0,T);H))$.

LEMMA 4.4. *For $j = 1,2,3$,*

(4.11)

$$E\left[\Phi(x_t)\Lambda_{0,t}\exp\left(\theta L_{0,t}^{1,j}\right)|\mathcal{F}_{0,t}^y\right] = E\left[\Phi(x_t)\Lambda_{0,t}|\mathcal{F}_{0,t}^y\right] + \sum_{\kappa=1}^\infty \frac{\theta^\kappa}{\kappa!}E\left[\Phi(x_t)\Lambda_{0,t}L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y\right],$$

where the infinite series converges in $L^1(\Omega,\mathcal{F}_{0,t}^y,P)$. Moreover,

$$(4.12) \qquad\qquad \beta_t^{\theta,j}(\Phi) = q_t(\Phi) + \sum_{\kappa=1}^\infty \frac{\theta^\kappa}{\kappa!}M_t^{\kappa,j}(\Phi), \qquad j = 1,2,3.$$

Proof. We shall invoke the following estimate found in [15, p. 353]:

$$\left| e^{\theta x} - \sum_{k=0}^n \frac{(\theta x)^k}{k!} \right| \leq \min\left\{ \frac{|\theta x|^{n+1}}{(n+1)!}, \frac{2|\theta x|^n}{n!} \right\}, \quad \theta \in \mathbb{R}, \ x \in \mathbb{R}.$$

The first right side term is an estimate for $|\theta x|$ small and the second for $|\theta x|$ large. Using the above estimate

$$
\mathrm{E}\Big\{\Big|\mathrm{E}[\Phi(x_t)\Lambda_{0,t}\exp(\theta L_{0,t}^{1,j})|\mathcal{F}_{0,t}^y] - \sum_{k=0}^{n}\frac{\theta^k}{k!}\mathrm{E}[\Phi(x_t)\Lambda_{0,t}L_{0,t}^{k,j}|\mathcal{F}_{0,t}^y]\Big|\Big\}
$$

$$
\leq \mathrm{E}\Big\{|\Phi(x_t)|\Lambda_{0,t}\Big(\Big|\exp(\theta L_{0,t}^{1,j}) - \sum_{k=0}^{n}\frac{\theta^k}{k!}L_{0,t}^{k,j}\Big|\Big)\Big\}
$$

$$
\leq \mathrm{E}\Big\{|\Phi(x_t)|\Lambda_{0,t}\min\Big\{\frac{|\theta|^{n+1}L_{0,t}^{n+1,j}}{(n+1)!}, \frac{2|\theta|^n L_{0,t}^{n,j}}{n!}\Big\}\Big\}
$$

(4.13)

$$
\leq \Big(\mathrm{E}\big\{|\Phi(x_t)|^2\Lambda_{0,t}^2\big\}\Big)^{1/2}\Big(\mathrm{E}\min\Big\{\Big(\frac{|\theta|^{n+1}L_{0,t}^{n+1,j}}{(n+1)!}\Big)^2, \Big(\frac{2|\theta|^n L_{0,t}^{n,j}}{n!}\Big)^2\Big\}\Big)^{1/2}.
$$

The first right side term of (4.13) is bounded for any $(t,x) \, \epsilon \, [0,T] \times \mathbb{R}^n$, and the second is bounded because $f^j(t,x)$, $j=1,2,3$, are bounded for any $(t,x) \, \epsilon \, [0,T] \times \mathbb{R}^n$.

Moreover, $\lim_{n\to\infty}\big(\frac{\theta^n}{n!}\big)^2\mathrm{E}L_{0,t}^{2n,j} = 0$, and therefore in the limit, as $n \to \infty$, the right side of (4.13) converges to zero. Consequently, the following expansion must hold:

$$
\mathrm{E}[\Phi(x_t)\Lambda_{0,t}\exp(\theta L_{0,t}^{1,j})|\mathcal{F}_{0,t}^y] = \sum_{k=0}^{\infty}\frac{\theta^k}{k!}\mathrm{E}[\Phi(x_t)\Lambda_{0,t}L_{0,t}^{k,j}|\mathcal{F}_{0,t}^y],
$$

which is equivalent to (4.11) and, by Definition 2.9, to (4.12). □

At this stage, we may formally differentiate both sides of (4.12) with respect to θ, and then take the limit as $\theta \to 0$, to obtain relations between $\lim_{\theta\to 0}\frac{d^\kappa}{d\theta^\kappa}\beta_t^{\theta,j}(\Phi)$ and $M_t^{\kappa,j}(\Phi)$, $j=1,2,3$. These results are presented next.

THEOREM 4.5. *We have the following:*

1. $\widetilde{\beta}^{i\omega,j}(1), \beta^{i\omega,j}(1), j=1,2,3$ *have* κ *continuous derivatives with respect to* ω, *w.p.1.*

2.

$$
(4.14)\quad \lim_{\theta\to 0}\frac{d^\kappa}{d\theta^\kappa}\widetilde{\beta}_t^{\theta,j}(\Phi) = \lim_{\theta\to 0}\frac{d^\kappa}{d\theta^\kappa}\frac{\beta_t^{\theta,j}(\Phi)}{q_t(1)} = \widetilde{\mathrm{E}}[\Phi(x_t)L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y]\quad w.p.1,
$$
$$
\theta = i\omega, \quad \kappa \geq 0, \quad j = 1,2,3.
$$

3.

$$
(4.15)\quad \lim_{\theta\to 0}\frac{d^\kappa}{d\theta^\kappa}\widetilde{\beta}_t^{\theta,j}(1) = \lim_{\theta\to 0}\frac{d^\kappa}{d\theta^\kappa}\frac{\beta_t^{\theta,j}(1)}{q_t(1)} = \widetilde{\mathrm{E}}[L_{0,t}^{\kappa,j}|\mathcal{F}_{0,t}^y]\quad w.p.1,
$$
$$
\theta = i\omega, \quad \kappa \geq 0, \quad j = 1,2,3.
$$

Proof. Recall that

$$
\widetilde{\beta}_t^{i\omega,j}(\Phi) = \widetilde{\mathrm{E}}\big[\Phi(x_t)\exp\big(i\omega L_{0,t}^{1,j}\big)|\mathcal{F}_{0,t}^y\big] = \frac{E\big[\Lambda_{0,t}\exp\big(i\omega L_{0,t}^{1,j}\big)|\mathcal{F}_{0,t}^y\big]}{q_t(1)}.
$$

Here $q_t(1)$ is independent of θ. The numerator $\mathrm{E}[\Phi(x_t)\Lambda_{0,t}\exp(i\omega L_{0,t}^{1,j})|\mathcal{F}_{0,t}^y]$ admits the power series expansion of Lemma 4.4, which implies 1, 2, 3. □

4.1. Specific application. Consider the system

$$dx_t = Fx_t dt + Gdw_t, \qquad x(0) \in \mathbb{R}^n,$$

$$dy_t = Hx_t + N^{\frac{1}{2}} db_t, \qquad y(0) = 0,$$

$$f^1(t,x) = \frac{1}{2}x'Qx, \quad f^2(t,x) = x'R, \quad f^3(t,x) = x'S, \qquad Q = Q'.$$

We assume $x(0)$ is a Gaussian random variable.

Suppose F, H are random matrices which we wish to identify or estimate. In [6] an algorithm is presented for estimating these matrices. This involves filtered estimates of the processes $\int_0^t f^1(s, x_s)ds, \int_0^t f^2(s, x_s)dw_s, \int_0^t f^3(s, x_s)db_s$. Here we apply Theorem 4.5 to obtain these estimates.

A solution of (2.13), (2.14) is

$$(4.16) \qquad q(x,t) = \frac{1}{(2\pi)^{n/2}|P_t^0|^{1/2}} \exp\left(-\frac{1}{2}|P_t^{0,-1/2}\left(x - \widehat{x}_t^0\right)|^2\right) \times \widehat{\Lambda}_{0,t}^0,$$

where $\widehat{x}^0(\cdot), P^0(\cdot), \widehat{\Lambda}^0(\cdot)$ are given by

$$(4.17) \qquad d\widehat{x}_t^0 = F\widehat{x}_t^0 dt + P_t^0 H'N^{-1}\left(dy_t - H\widehat{x}_t^0 dt\right), \qquad \widehat{x}^0(0) = \xi,$$

$$(4.18) \qquad \dot{P}_t^0 = FP_t^0 + P_t^\theta F' - P_t^0 H'N^{-1}HP_t^0 + GG', \qquad P^0(0) = P_0,$$

$$(4.19) \qquad \widehat{\Lambda}_{0,t}^0 = \exp\left(\int_0^t \left(H\widehat{x}_s^0\right)' N^{-1} dy_s - \frac{1}{2}\int_0^t \left(H\widehat{x}_s^0\right)' N^{-1} H\widehat{x}_s^0 ds\right).$$

These computations are easily verified by substitution into the DMZ equation. Recall also that $q(x,t) = M^{0,j}(x,t), j = 1, 2, 3$.

1. *Computation of* $\widehat{L}_{0,t}^{1,1} = \widetilde{\mathbb{E}}[\frac{1}{2}\int_0^t x_s'Qx_s ds | \mathcal{F}_{0,t}^y]$:

A solution of (4.6), (4.9) is (see, for example, [11, 13])

$$(4.20)$$

$$\beta^{\theta,1}(x,t) = \frac{1}{(2\pi)^{n/2}|P_t^\theta|^{1/2}} \exp\left(-\frac{1}{2}|P_t^{\theta,-1/2}\left(x - \widehat{x}_t^\theta\right)|^2\right) \times \widehat{\Lambda}_{0,t}^\theta \times \exp\left(\frac{\theta}{2}\int_0^t \mathrm{Tr}(P_s^\theta Q)ds\right),$$

where

$$(4.21) \quad d\widehat{x}_t^\theta = \left(F + \theta P_t^\theta Q\right)\widehat{x}_t^\theta dt + P_t^\theta H'N^{-1}\left(dy_t - H\widehat{x}_t^\theta dt\right), \qquad \widehat{x}(0) = \xi,$$

$$(4.22) \quad \dot{P}_t^\theta = FP_t^\theta + P_t^\theta F' - P_t^\theta\left(H'N^{-1}H - \theta Q\right)P_t^\theta + GG', \qquad P^\theta(0) = P_0,$$

$$(4.23) \quad \widehat{\Lambda}_{0,t}^\theta = \exp\left(\int_0^t \left(H\widehat{x}_s^\theta\right)' N^{-1} dy_s - \frac{1}{2}\int_0^t \left(H\widehat{x}_s^\theta\right)' N^{-1} H\widehat{x}_s^\theta ds\right).$$

In fact, we can show that $\lim_{\theta \to 0} P_t^\theta = P_t^0$, uniformly on compact subsets of $[0, T]$, and $\lim_{\theta \to 0} \widehat{x}_t^\theta = \widehat{x}_t^0$ a.s.

According to Theorem 4.5 we need

$$(4.24) \qquad \frac{d}{d\theta}\frac{\beta_t^{\theta,1}(1)}{\widehat{\Lambda}_{0,t}^0} = \frac{d}{d\theta}\left[\widehat{\Lambda}_{0,t}^\theta\left(\widehat{\Lambda}_{0,t}^0\right)^{-1}\exp\left(\frac{\theta}{2}\int_0^t \mathrm{Tr}(P_s^\theta Q)\right)\right].$$

Let

$$r_t^\theta = \frac{d}{d\theta}\widehat{x}_t^\theta, \qquad \Sigma_t^\theta = \frac{d}{d\theta}P_t^\theta.$$

Then from the differentiability of parameter dependent solutions of stochastic differential equations we know that

$$r_t^\theta = \int_0^t \left(F + \theta P_s^\theta Q - P_s^\theta H' N^{-1} H r_t^\theta \right) r_s^\theta ds$$

(4.25)
$$+ \int_0^t \theta \Sigma_s^\theta Q \widehat{x}_s^\theta ds + \int_0^t \Sigma_s^\theta H' N^{-1} \left(dy_s - H \widehat{x}_s^\theta ds \right) + \int_0^t P_s^\theta Q \widehat{x}_s^\theta ds,$$

$$\Sigma_t^\theta = \int_0^t F \Sigma_s^\theta ds + \int_0^t \Sigma_s^\theta F' ds - \int_0^t \Sigma_s^\theta \left(H' N^{-1} H - \theta Q \right) P_s^\theta ds$$

(4.26)
$$- \int_0^t P_s^\theta \left(H' N^{-1} H - \theta Q \right) \Sigma_s^\theta ds + \int_0^t P_s^\theta Q P_s^\theta ds$$

are well defined. Similarly as before we have $\lim_{\theta \to 0} r_t^\theta = r_t^0$ (a.s.), $\lim_{\theta \to 0} \Sigma_t^\theta = \Sigma_t^0$, where

$$r_t^0 = \int_0^t P_s^0 Q \widehat{x}_s^0 ds + \int_0^t \Sigma_s^0 H' N^{-1} \left(dy_s - H \widehat{x}_s^0 ds \right)$$

(4.27)
$$+ \int_0^t \left(F - P_s^0 H' N^{-1} H \right) r_s^0 ds,$$

(4.28)
$$\Sigma_t^0 = \int_0^t \left(F - P^0 H' N^{-1} H \right) \Sigma_s^0 ds + \int_0^t \Sigma_s^0 \left(F - P^0 H' N^{-1} H \right)' ds + \int_0^t P_s^0 Q P_s^0 ds.$$

Consequently,

$$\lim_{\theta \to 0} \frac{d}{d\theta} \left\{ \widehat{\Lambda}_{0,t}^\theta \left(\widehat{\Lambda}_{0,t}^0 \right)^{-1} \exp \left(\frac{\theta}{2} \int_0^t \mathrm{Tr}(P_s^\theta Q) ds \right) \right\}$$

$$= \lim_{\theta \to 0} \left\{ \left(\int_0^t (H r_s^\theta)' N^{-1} dy_s - \int_0^t (H r_s^\theta)' N^{-1} H \widehat{x}_s^\theta ds + \frac{1}{2} \int_0^t \mathrm{Tr} \left(P_s^\theta Q + \theta \Sigma_s^\theta Q \right) ds \right) \right.$$

$$\left. \times \widehat{\Lambda}_{0,t}^\theta \left(\widehat{\Lambda}_{0,t}^0 \right)^{-1} \exp \left(\frac{\theta}{2} \int_0^t \mathrm{Tr}(P_s^\theta Q) ds \right) \right\}$$

(4.29)
$$= \frac{1}{2} \int_0^t \mathrm{Tr} \left(P_s^0 Q \right) ds + \int_0^t (H r_s^0)' N^{-1} \left(dy_s - H \widehat{x}_s^0 \right) ds.$$

Finally, $\widehat{L}_{0,t}^{1,1} = \widetilde{\mathrm{E}}[\frac{1}{2} \int_0^t x_s' Q x_s ds | \mathcal{F}_{0,t}^y]$ is a solution of the stochastic equation

(4.30)
$$d\widehat{L}_{0,t}^{1,1} = \frac{1}{2} \mathrm{Tr} \left(P_t^0 Q \right) dt + \left(H r_t^0 \right)' N^{-1} \left(dy_t - H \widehat{x}_t^0 ds \right), \qquad \widehat{L}_{0,t}^{1,1} = 0.$$

2. *Computation of* $\widehat{L}_{0,t}^{1,2} = \widetilde{\mathrm{E}}[\int_0^t x_s' R dw_s | \mathcal{F}_{0,t}^y]$:
A solution of (4.7), (4.9) is (see [11])

(4.31)
$$\beta^{\theta,2}(x,t) = \frac{1}{(2\pi)^{n/2} |P_t^\theta|^{1/2}} \exp \left(-\frac{1}{2} |P_t^{\theta,-1/2} \left(x - \widehat{x}_t^\theta \right)|^2 \right)$$

$$\times \widehat{\Lambda}_{0,t}^\theta \times \exp \left(\frac{\theta^2}{2} \int_0^t \mathrm{Tr}(P_s^\theta R R') ds \right),$$

where

(4.32)
$$d\widehat{x}_t^\theta = \left(F + \theta^2 P_t^\theta RR' + \theta GR'\right)\widehat{x}_t^\theta dt + P_t^\theta H'N^{-1}\left(dy_t - H\widehat{x}_t^\theta dt\right), \quad \widehat{x}(0) = \xi,$$

(4.33)
$$\dot{P}_t^\theta = \left(F + \theta GR'\right)P_t^\theta + P_t^\theta\left(F + \theta RG'\right)' - P_t^\theta\left(H'N^{-1}H - \theta^2 RR'\right)P_t^\theta + GG',$$
$$P^\theta(0) = P_0,$$

(4.34)
$$\widehat{\Lambda}_{0,t}^\theta = \exp\left(\int_0^t \left(H\widehat{x}_s^\theta\right)' N^{-1}dy_s - \frac{1}{2}\int_0^t \left(H\widehat{x}_s^\theta\right)' N^{-1}H\widehat{x}_s^\theta ds\right).$$

By Theorem 4.5 we need

(4.35)
$$\frac{d}{d\theta}\frac{\beta_t^{\theta,2}(1)}{\widehat{\Lambda}_{0,t}^0} = \frac{d}{d\theta}\left[\widehat{\Lambda}_{0,t}^\theta\left(\widehat{\Lambda}_{0,t}^0\right)^{-1}\exp\left(\frac{\theta^2}{2}\int_0^t \operatorname{Tr}(P_s^\theta RR')\right)\right].$$

Computing $\lim_{\theta\to 0} r_t^\theta = \lim_{\theta\to 0}\frac{d}{d\theta}\widehat{x}_t^\theta = r_t^0$, $\lim_{\theta\to 0}\Sigma_t^\theta = \lim_{\theta\to 0}\frac{d}{d\theta}P_t^\theta = P_t^0$, similarly as before, we have

$$r_t^0 = \int_0^t GR'\widehat{x}_s^0 ds + \int_0^t \Sigma_s^0 H'N^{-1}\left(dy_s - H\widehat{x}_s^0 ds\right)$$

(4.36)
$$+ \int_0^t \left(F - P_s^0 H'N^{-1}H\right) r_s^0 ds,$$

$$\Sigma_t^0 = \int_0^t \left(F - P^0 H'N^{-1}H\right)\Sigma_s^0 ds + \int_0^t \Sigma_s^0\left(F - P^0 H'N^{-1}H\right)' ds$$

(4.37)
$$+ \int_0^t GR'P_s^0 ds + \int_0^t P_s^0 RG' ds.$$

Hence

$$\lim_{\theta\to 0}\frac{d}{d\theta}\left\{\widehat{\Lambda}_{0,t}^\theta\left(\widehat{\Lambda}_{0,t}^0\right)^{-1}\exp\left(\frac{\theta^2}{2}\int_0^t Tr(P_s^\theta RR')ds\right)\right\}$$

(4.38)
$$= \int_0^t \left(Hr_s^0\right)' N^{-1}\left(dy_s - H\widehat{x}_s^0\right)ds.$$

Finally, $\widehat{L}_{0,t}^{1,2} = \widetilde{\mathbb{E}}[\int_0^t x_s'Rdw_s|\mathcal{F}_{0,t}^y]$ is a solution of the stochastic equation

(4.39)
$$d\widehat{L}_{0,t}^{1,2} = \left(Hr_t^0\right)' N^{-1}\left(dy_t - H\widehat{x}_t^0 ds\right), \quad \widehat{L}_{0,t}^{1,2} = 0.$$

3. *Computation of* $\widehat{L}_{0,t}^{1,3} = \widetilde{\mathbb{E}}[\int_0^t x_s'Sdb_s|\mathcal{F}_{0,t}^y]$:
A solution of (4.7), (4.9) is (see [11])

(4.40)
$$\beta^{\theta,3}(x,t) = \frac{1}{(2\pi)^{n/2}|P_t^\theta|^{1/2}}\exp\left(-\frac{1}{2}|P_t^{\theta,-1/2}\left(x - \widehat{x}_t^\theta\right)|^2\right)$$
$$\times \widehat{\Lambda}_{0,t}^\theta \times \exp\left(\frac{\theta^2}{2}\int_0^t \operatorname{Tr}(P_s^\theta SS')ds\right),$$

where

$$(4.41) \quad d\widehat{x}_t^\theta = \left(F + \theta^2 P_t^\theta SS'\right)\widehat{x}_t^\theta dt + P_t^\theta H^{\theta,'} N^{-1}\left(dy_t - H^\theta \widehat{x}_t^\theta dt\right), \quad \widehat{x}(0) = \xi,$$

$$(4.42) \quad \dot{P}_t^\theta = FP_t^\theta + P_t^\theta F' - P_t^\theta \left(H^{\theta,'} N^{-1} H^\theta - \theta^2 SS'\right) P_t^\theta + GG', \quad P^\theta(0) = P_0,$$

$$(4.43) \quad \widehat{\Lambda}_{0,t}^\theta = \exp\left(\int_0^t \left(H^\theta \widehat{x}_s^\theta\right)' N^{-1} dy_s - \frac{1}{2}\int_0^t \left(H^\theta \widehat{x}_s^\theta\right)' N^{-1} H^\theta \widehat{x}_s^\theta ds\right),$$

$$(4.44) \quad H^\theta = H + \theta N^{-1/2,'} S'.$$

From Theorem 4.5 we need

$$(4.45) \qquad \frac{d}{d\theta}\frac{\beta_t^{\theta,3}(1)}{\widehat{\Lambda}_{0,t}^0} = \frac{d}{d\theta}\left[\widehat{\Lambda}_{0,t}^\theta \left(\widehat{\Lambda}_{0,t}^0\right)^{-1}\exp\left(\frac{\theta^2}{2}\int_0^t \mathrm{Tr}(P_s^\theta SS')\right) ds\right].$$

This can be done as in the previous cases.

Finally, $\widehat{L}_{0,t}^{1,3} = \widetilde{\mathrm{E}}[\int_0^t x_s' S db_s | \mathcal{F}_{0,t}^y]$ is a solution of the stochastic equation

$$(4.46)$$

$$d\widehat{L}_{0,t}^{1,3} = \left(Hr_t^0\right)' N^{-1}\left(dy_t - H\widehat{x}_t^0 dt\right) + \left(N^{-\frac{1}{2}} S'\widehat{x}_t^0\right)' N^{-1}\left(dy_t - N^{-\frac{1}{2},'} S'\widehat{x}_t^0 dt\right),$$

$$\widehat{L}_{0,t}^{1,3} = 0,$$

where

$$r_t^0 = \int_0^t \left(\Sigma_s^0 H' N^{-1} + P_s^0 (SN^{-1/2})' N^{-1}\right)\left(dy_s - H\widehat{x}_s^0 ds\right)$$

$$(4.47) \qquad + \int_0^t P_s^0 H' N^{-1}\left(dy_s - (SN^{-1/2})'\widehat{x}_s^0 ds\right) + \int_0^t \left(F - P_s^0 H' N^{-1}\right) r_s^0 ds,$$

$$\Sigma_t^0 = \int_0^t \left(F - P^0 H' N^{-1} H\right)\Sigma_s^0 ds + \int_0^t \Sigma_s^0 \left(F - P^0 H' N^{-1} H\right)' ds$$

$$(4.48) \qquad - \int_0^t P_s^0 \left((SN^{-1/2})N^{-1}H + (N^{-1}H)'(SN^{-1/2})'\right)P_s^0 ds. \qquad \square$$

Remark 4.6.   The above methodology can be generalized to joint conditional moment generating functions of $L_{0,t}^{1,j}, L_{0,t}^{1,\ell}, 1 \le j, \ell \le 3$.

5. Applications to nonlinear filtering problems.   Both methods introduced in section 3 can be used in Wiener chaos expansions of nonlinear filtering [2] problems.

Consider the nonlinear filtering problem

$$(5.1) \qquad\qquad dx_t = f(t, x_t)dt + \sigma(t, x_t)dw_t, \quad x(0) \in \mathbb{R}^n,$$

$$(5.2) \qquad\qquad dy_t = h(t, x_t)dt + N_t^{1/2}db_t, \quad y(0) = 0 \in \mathbb{R}^d.$$

Here $\{x_t; t \in [0, T]\}$ and $\{y_t; t \in [0, T]\}$ are the state and observation processes, respectively. Throughout, we assume Assumption 2.2 holds. Similar to section 2, under the reference probability space $(\Omega, \mathcal{A}, P, \mathcal{F}_{0,t})$, processes $\{x_t; t \in [0, T]\}$, $\{y_t; t \in [0, T]\}$, are independent; the former is a solution of (5.1), while the latter is a solution of

$$(5.3) \qquad\qquad dy_t = N_t^{1/2}db_t, \quad y(0) = 0 \in \mathbb{R}^d.$$

The Radon–Nikodým derivative is

$$(5.4) \qquad \Lambda_{0,T} \doteq \left[\frac{d\widetilde{P}}{dP} | \mathcal{F}_{0,T} \right] = \exp\left(m_t - \frac{1}{2}\langle m, m \rangle_t \right),$$

where $m_t = \int_0^t h'(s, x_s) N_s^{-1} dy_s$. Thus, $\{\Lambda_{0,t}; t \in [0, T]\}$ is a solution of the stochastic differential equation

$$(5.5) \qquad \Lambda_{0,t} = 1 + \int_0^t \Lambda_{0,s} h'(s, x_s) N_s^{-1} dy_s.$$

Moreover, if the measured-valued processes $q_t(\Phi) = E[\Phi(x_t)\Lambda_{0,t}|\mathcal{F}_{0,t}^y]$ have a density $q(x,t)$, then

$$(5.6) \qquad dq(z,t) = A(t)^* q(z,t) dt + h'(t,z) q(z,t) N_t^{-1} dy_t, \qquad (z,t) \in [0,T] \times \mathbb{R}^n,$$

$$(5.7) \qquad q(z,0) = p_0(z), \qquad z \in \mathbb{R}^n.$$

In what follows, we employ some of the recursive systems derived in section 3 to obtain representations for certain asymptotic expansions of $E[\Phi(x_t)|\mathcal{F}_{0,t}^y]$.

DEFINITION 5.1. *Suppose* $E[\int_0^T |N_s^{-1/2} h(s, x_s)|^2 ds]^p < \infty$. *Then the multiple stochastic integrals*

$$(5.8) \quad I_t^p[h] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} h'(s_p, x_{s_p}) N_{s_p}^{-1} dy_{s_p} h'(s_{p-1}, x_{s_{p-1}}) N_{s_{p-1}}^{-1} dy_{s_{p-1}}$$
$$\cdots h'(s_1, x_{s_1}) N_{s_1}^{-1} dy_{s_1},$$

$$(5.9) \quad I_t^{[p]}[h] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_p} \Lambda_{s_{p+1}} h'(s_{p+1}, x_{s_{p+1}}) N_{s_{p+1}}^{-1} dy_{s_{p+1}} h'(s_p, x_{s_p}) N_{s_p}^{-1} dy_{s_p}$$
$$\cdots h'(s_1, x_{s_1}) N_{s_1}^{-1} dy_{s_1}$$

are well defined.

Consider the exponential martingale $\{\Lambda_{0,t}; t \in [0, T]\}$. Iterating (5.5) we have

$$\Lambda_{0,t} = 1 + \int_0^t \Lambda_{0,s} h'(s, x_s) N_s^{-1} dy_s$$

$$= 1 + \int_0^t h'(s, x_s) N_s^{-1} dy_s + \int_0^t \int_0^{s_1} \Lambda_{0,s_2} h'(s_2, x_{s_2}) N_{s_2}^{-1} dy_{s_2} h'(s_1, x_{s_1}) N_{s_1}^{-1} dy_{s_1}$$

$$+ \vdots$$

$$= 1 + \int_0^t h'(s, x_s) N_s^{-1} dy_s + \int_0^t \int_0^{s_1} \Lambda_{0,s_2} h'(s_2, x_{s_2}) N_{s_2}^{-1} dy_{s_2} h'(s_1, x_{s_1}) N_{s_1}^{-1} dy_{s_1}$$

$$+ \cdots + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} h'(s_p, x_{s_p}) N_{s_p}^{-1} dy_{s_p} h'(s_{p-1}, x_{s_{p-1}}) N_{s_{p-1}}^{-1} dy_{s_{p-1}}$$

$$(5.10) \qquad\qquad\qquad \cdots h'(s_1, x_{s_1}) N_{s_1}^{-1} dy_{s_1} + I_t^{[p]}[h].$$

If we now assume $E[\Phi^2(x_t) \int_0^T |N_s^{-1} h(s, x_s)|^2 ds]^p < \infty$ (which is satisfied by As-

sumption 2.2) and then substitute (5.10) into $q_t^0(\Phi) = E[\Lambda_{0,t}\Phi(x_t)|\mathcal{F}_{0,t}^y]$ we have

$$q_t(\Phi) = E[\Lambda_{0,t}\Phi(x_t)|\mathcal{F}_{0,t}^y] = E[\Phi(x_t)|\mathcal{F}_{0,t}^y]$$

$$+ \sum_{k=1}^p E\left[\ \Phi(x_t)\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}} h'(s_k,x_{s_k})N_{s_k}^{-1}dy_{s_k}\cdots h'(s_1,x_{s_1})N_{s_1}^{-1}dy_{s_1}|\mathcal{F}_{0,t}^y\ \right]$$

$$+ E\left[\Phi(x_t)I_t^{[p]}[h]|\mathcal{F}_{0,t}^y\right].$$

(5.11)

Note that under measure P, the processes $\{x_t; t \in [0,T]\}$ and $\{y_t; t \in [0,T]\}$ are independent; therefore $E[\Phi(x_t)|\mathcal{F}_{0,t}^y] = E[\Phi(x_t)]$. In addition, the increments $dy_{s_1}, dy_{s_2}, \dots, dy_{s_k}$ are measurable with respect to $\mathcal{F}_{0,t}^y$; in the scalar case, $d=1$, the second right side term of (5.11) becomes

$$\sum_{k=1}^p \int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-1}} E[\Phi(x_t)h(s_k,x_{s_k})N_{s_k}\cdots h(s_1,x_{s_1})N_{s_1}]dy_{s_k}\cdots dy_{s_1}.$$

Formally, letting $p=\infty$ in (5.11) we derive the full expansion, which is made rigorous in the next theorem.

THEOREM 5.2.
  1. *Suppose* $E[\int_0^t |N_s^{-1/2}h(s,x_s)|^2ds]^p < \infty$ *and* $E[\Phi(x_t)^2\int_0^t |N_s^{-1/2}h(s,x_s)|^2ds]^p < \infty$. *Then*

$$\widetilde{E}[\Phi(x_t)|\mathcal{F}_{0,t}^y] = \frac{q_t(\Phi)}{q_t(1)}$$

(5.12)

$$= \frac{E[\Phi(x_t)] + \sum_{k=1}^p E[\Phi(x_t)I_t^k[h]|\mathcal{F}_{0,t}^y] + E[\Phi(x_t)I_t^{[p]}[h]|\mathcal{F}_{0,t}^y]}{1 + \sum_{k=1}^p E[I_t^p[h]|\mathcal{F}_{0,t}^y] + E[I_t^{[p]}[h]|\mathcal{F}_{0,t}^y]}.$$

  2. *Suppose* $E[\exp\int_0^t |N_s^{-1/2}h(s,x_s)|^2ds] < \infty$ *and*

$$E[\Phi(x_t)\exp\int_0^t |N_s^{-1/2}h(s,x_s)|^2ds] < \infty.$$

  Then

(5.13) $$\widetilde{E}[\Phi(x_t)|\mathcal{F}_{0,t}^y] = \frac{q_t(\Phi)}{q_t(1)} = \frac{E[\Phi(x_t)] + \sum_{k=1}^\infty E[\Phi(x_t)I_t^k[h]|\mathcal{F}_{0,t}^y]}{1 + \sum_{k=1}^\infty E[I_t^k[h]|\mathcal{F}_{0,t}^y]}$$

and the infinite series of (5.13) *converges in* $L^1(\Omega,\mathcal{F}_{0,t}^y,P)$.
  Proof. See [2]. □

5.1. Recursive equations.
  DEFINITION 5.3. *Suppose* $E[\Phi^2(x_t)\int_0^t |N_s^{-1/2}h(s,x_s)|^2]^p < \infty$.
  The measure-valued processes

(5.14) $$M_t^0(\Phi) \doteq E[\Phi(x_t)] = \int_{\mathbb{R}^n} \Phi(z)p(z,t;x,0)dx,$$

(5.15) $$M_t^k(\Phi) \doteq E[\Phi(x_t)I_t^k[h]|\mathcal{F}_{0,t}^y], \quad k \geq 1,$$

are well defined.
  THEOREM 5.4. *Suppose* $M_t^k(\cdot), k \geq 0$, *have density functions* $M^k(z,t)$.

1. *The density of the distribution $\widetilde{P}(x_t \in A), A \in \mathcal{B}(\mathbb{R}^n)$, satisfies the Kolmogorov equation*

$$(5.16) \quad dp(z,t) = A(t)^* p(z,t)dt, \quad (z,t) \in \mathbb{R}^n \times (0,T]; \quad \lim_{t \to 0} p(z,t) = p_0(z).$$

2. *The densities of $M_t^k(\cdot), k \geq 1$ satisfy the following recursive system of SPDEs:*

$$(5.17)$$
$$dM^k(z,t) = A(t)^* M^k(z,t)dt + h^*(t,z)M^{k-1}(z,t)N_t^{-1}dy_t, \quad (z,t) \in \mathbb{R}^n \times [0,T],$$
$$(5.18)$$
$$M^k(z,0) = 0, \quad z \in \mathbb{R}^n.$$

Proof. The distribution of $\{x_t; t \in [0,T]\}$ is the same under measure $\widetilde{P}$ and P. Hence, the density $p(\cdot,\cdot)$ satisfies (5.16).

Now, for $k = 1$ consider

$$(5.19) \qquad\qquad I_t^1[h] = \int_0^t h'(s,x_s)N_s^{-1}dy_s.$$

By the Itô product rule

$$\Phi(x_t)I_t^1[h] = \int_0^t A(s)^* \Phi(x_s)I_s^1[h]ds$$
$$+ \int_0^t D_x' \Phi(x_s)I_s^1[h]\sigma(s,x_s)dw_s$$
$$(5.20) \qquad\qquad + \int_0^t \Phi(x_s)h'(s,x_s)N_s^{-1}dy_s.$$

Conditioning both sides of (5.20) on $\mathcal{F}_{0,t}^y$, and then using the independence of $\{w_t; t \in [0,T]\}$ and $\{y_t; t \in [0,T]\}$ (and a version of Fubini's theorem [12]), we have

$$M_t^1(\Phi) = \int_0^t \int_{\mathbb{R}^n} A(s)\Phi(z)M^1(z,s)dzds$$
$$(5.21) \qquad\qquad + \int_0^t \int_{\mathbb{R}^n} \Phi(z)h'(s,z)M^0(z,s)N_s^{-1}dzdy_s.$$

Hence (5.17), (5.18) hold for $k = 1$.

Now, for $k = \ell$ consider

$$(5.22) \qquad I_t^\ell[h] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{\ell-1}} h'(s_\ell, x_{s_\ell})N_{s_\ell}^{-1}dy_{s_\ell} \cdots h'(s, x_{s_1})N_{s_1}^{-1}dy_{s_1}.$$

Then

$$dI_t^\ell[h] = I_t^{\ell-1}[h]h'(t,x_t)N_t^{-1}dy_t.$$

By the Itô product rule

$$\Phi(x_t)I_t^\ell[h] = \int_0^t A(s)\Phi(x_s)I_s^\ell[h]ds$$
$$(5.23) \qquad\qquad + \int_0^t D_x'\Phi(x_s)I_s^\ell[h]\sigma(s,x_s)dw_s + \int_0^t \Phi(x_s)h'(s,x_s)I_s^{\ell-1}[h]N_s^{-1}dy_s.$$

Similarly as before, conditioning both sides of (5.2) on $\mathcal{F}_{0,t}^y$ we have

$$M_t^\ell(\Phi) = \int_0^t \int_{\mathbb{R}^n} A(s)\Phi(z)M^\ell(z,t)dzds$$

(5.24)
$$+ \int_0^t \int_{\mathbb{R}^n} \Phi(z)h'(s,z)M^{\ell-1}(z,s)N_s^{-1}dzdy_s.$$

Hence, (5.17), (5.18) holds for any $k \geq 1$.   □

COROLLARY 5.5.   *Let* $\{p(z,t;x,s); 0 \leq s < t \leq T\}, (z,x) \; \epsilon \; \mathbb{R}^n \times \mathbb{R}^n$ *be the fundamental solution of the density equation* (5.16):

(5.25)
$$dp(z,t;x,s) = A(t)^*p(z,t;x,s)dt, \quad (z,t) \; \epsilon \; \mathbb{R}^n \times (0,T]; \quad \lim_{t \to s} P(z,t;x,s) = \delta(z-x).$$

Then the solutions of (5.16)–(5.18) *are represented by*

(5.26) $$M^k(z,t) = \int_0^t \int_{\mathbb{R}^n} h'(s,x)N_s^{-1}M^{k-1}(x,s)p(z,t;x,s)dxdy_s, \quad k \geq 1,$$

(5.27) $$M^0(z,t) = \int_{\mathbb{R}^n} p(z,t;x,0)p_0(x)dx.$$

Proof. Follow the procedures of Theorem 2.5.   □

Remark 5.6. Because of the linearity of (5.16), (5.18), any finite expansion of both numerator and denominator of (5.13), say,

(5.28) $$q_t^\ell(\Phi) = \mathrm{E}[\Phi(x_t)] + \sum_{k=1}^{\ell} \mathrm{E}[\Phi(x_t)I_t^k[h]|\mathcal{F}_{0,t}^y], \quad \ell \geq 1,$$

is a solution of the SPDE

(5.29) $$dq_t^\ell(\Phi) = q_t^\ell(A(t)\Phi)dt + \sum_{k=1}^{\ell} M_t^{k-1}(h'(t,z)\Phi)N_t^{-1}dy_t.$$

From Theorem 5.2, we know that in order to approximate $\widetilde{\mathrm{E}}[\Phi(x_t)|\mathcal{F}_{0,t}^y]$ through a finite series we need to compute $M_t^k(\Phi), 0 \leq k \leq p$. The latter can be computed from the joint-moment generating function of the random processes $\{\int_0^t h'(s,x_s)N^{-1}dy_s; t \; \epsilon \; [0,T]\}$ and $\{\int_0^t |h'(s,x_s)|^2ds; t \; \epsilon \; [0,T]\}$.

6. Conclusion. This paper presents two methods for computing conditional moments of integrals and stochastic integrals for general diffusion processes. The first method employs recursive SPDEs; the second method employs conditional moment generating functions. An application of the first method results in new finite-dimensional filters. An application of the second method to the EM algorithm results in a significant reduction in the sufficient statistics required in the computation of the parameters.

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