EECE251

Circuit Analysis I

Set 4: Capacitors, Inductors, and First-Order Linear Circuits

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Overview

- Passive elements that we have seen so far: resistors. We will look into two other types of passive components, namely capacitors and inductors.

- We have already seen different methods to analyze circuits containing sources and resistive elements.

- We will examine circuits that contain two different types of passive elements namely resistors and one (equivalent) capacitor (RC circuits) or resistors and one (equivalent) inductor (RL circuits)

- Similar to circuits whose passive elements are all resistive, one can analyze RC or RL circuits by applying KVL and/or KCL. We will see whether the analysis of RC or RL circuits is any different!

Note: Some of the figures in this slide set are taken from (R. Decarlo and P.-M. Lin, *Linear Circuit Analysis*, 2nd Edition, 2001, Oxford University Press) and (C.K. Alexander and M.N.O Sadiku, *Fundamentals of Electric Circuits*, 4th Edition, 2008, McGraw Hill)

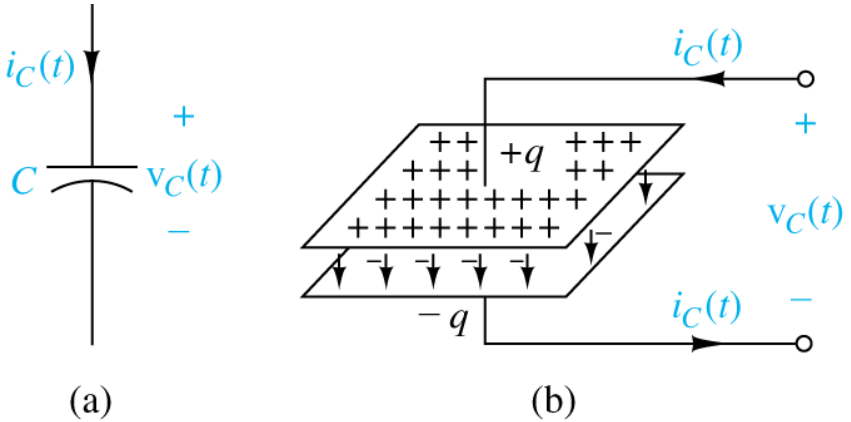
Reading Material

- Chapters 6 and 7 of the textbook
  - Section 6.1: Capacitors
  - Section 6.2: Inductors
  - Section 6.3: Capacitor and Inductor Combinations
  - Section 6.5: Application Examples
  - Section 7.2: First-Order Circuits

- Reading assignment:
  - Review Section 7.4: Application Examples (7.12, 7.13, and 7.14)

Capacitors

- A capacitor is a circuit component that consists of two conductive plate separated by an insulator (or dielectric).

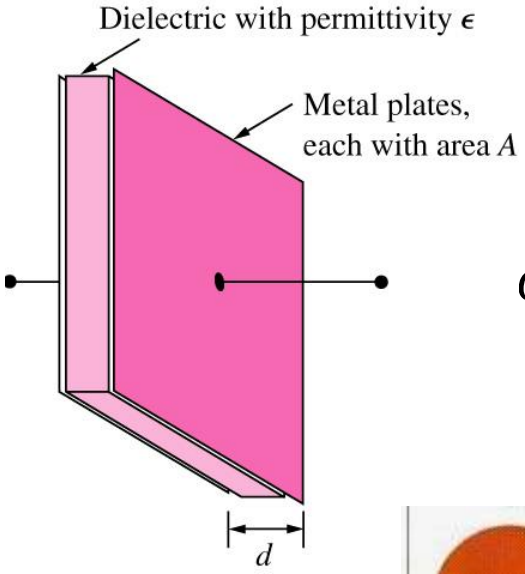


- Capacitors store charge and the amount of charge stored on the capacitor is directly proportional to the voltage across the capacitor. The constant of proportionality is the capacitance of the capacitor. That is:

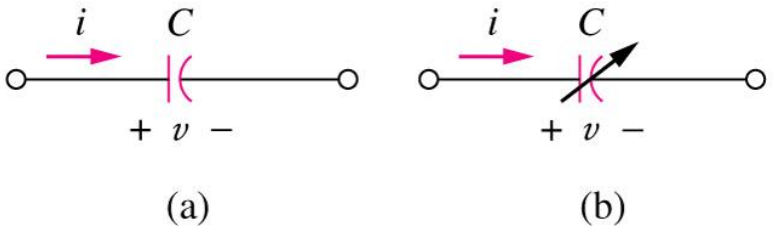
$$q_C(t) = Cv_C(t)$$

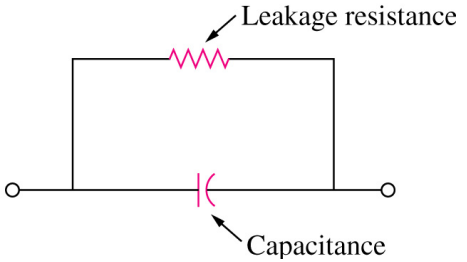
- Capacitor stores energy in its electric field.

Capacitors



$$C = \frac{\varepsilon A}{d}$$





Model for a non-ideal capacitor

Capacitors

- In honor of Michael Faraday (1791-1867), an English chemist and physicist, the unit of capacitance is named Farad (F).

- The voltage-current relationship of the capacitor is:

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$ why?

$$v_C(t) = \frac{q(t)}{C} = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)\cdot d\tau \quad \text{or} \quad v_C(t) = \frac{1}{C}\int_{t_0}^{t} i_C(\tau)\cdot d\tau + v_C(t_0)$$

Capacitors

- Note that;
  - A capacitor acts as an open circuit when connected to a DC voltage source
  - A capacitor impede the abrupt change of its voltage

- The instantaneous power absorbed by the capacitor is:

$$p_C(t) = i_C(t)v_C(t) = C\frac{dv_C(t)}{dt}v_C(t)$$

and the total stored energy in the capacitor is:

$$W_C(t) = \int_{-\infty}^{t} p_C(\tau)\cdot d\tau = \int_{-\infty}^{t} Cv_C(\tau)dv_C(\tau) = \frac{1}{2}Cv_C^2(t)$$

Have we assumed anything in writing the above equation?!

Example

- Calculate the area of simple parallel plate 1 F capacitor. Assume that the plates are separated by air with a distance of the thickness one sheet of paper, i.e., 1.016×10^{-4} m.

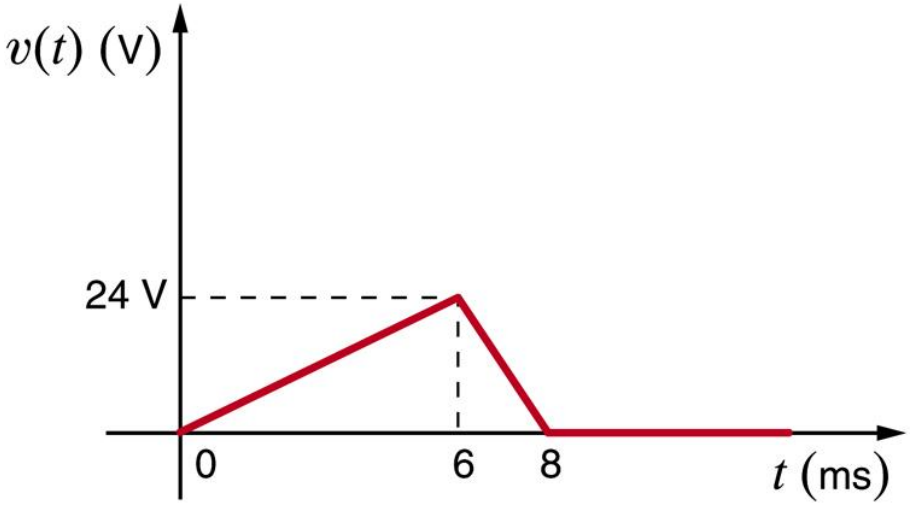
  The permitivity of free space is: $\varepsilon_0 = 8.85 \times10^{-12}$ F/m.

  $C = (\varepsilon_0 A)/d \Rightarrow A = C\times d/ \varepsilon_0 = 1\times1.016\times10^{-4} / 8.85 \times10^{-12}$

  $A = 1.148 \times10^{7}\ m^2 = 11.48\ km^2$!!!!!!

Example

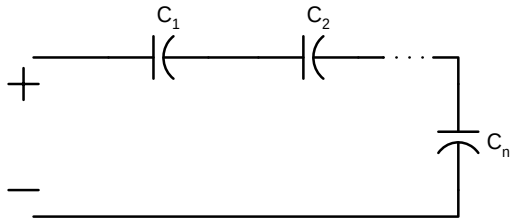
- The voltage across a 5-μF capacitor is given below. Determine the current of the capacitor.



Series and Parallel Capacitors

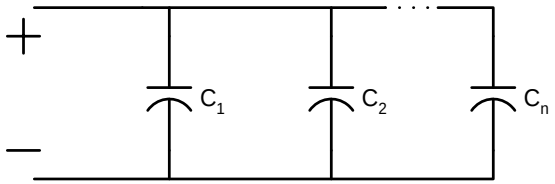
- The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances. Why?

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$



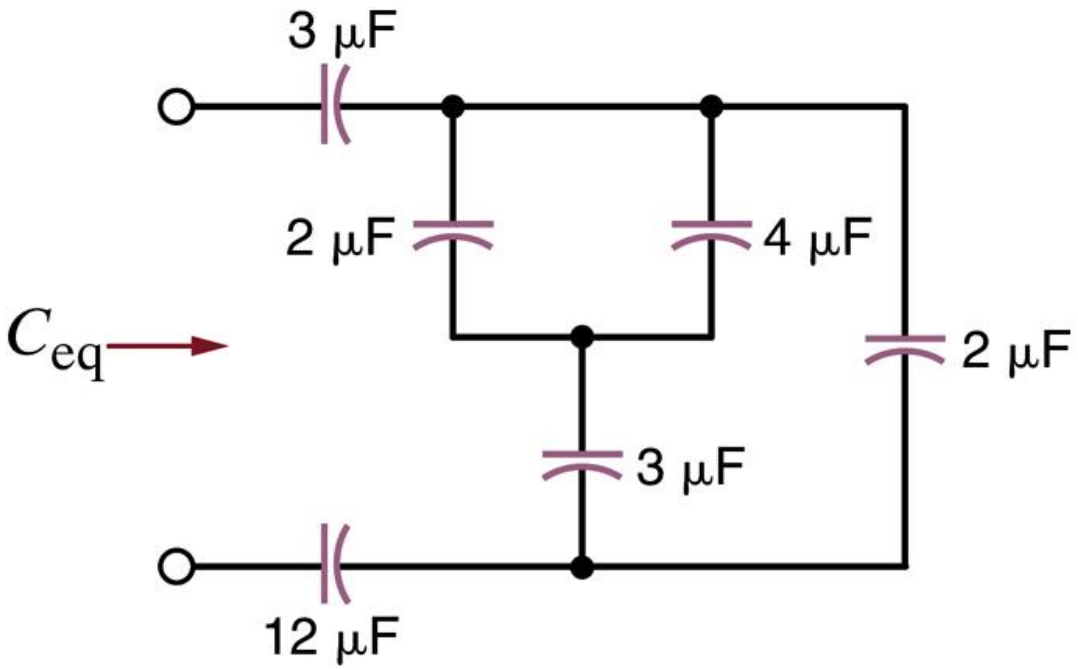
- The equivalent capacitance of parallel capacitors is the sum of the individual capacitances. Why?

$$C_{eq} = C_1 + C_2 + \cdots + C_n$$



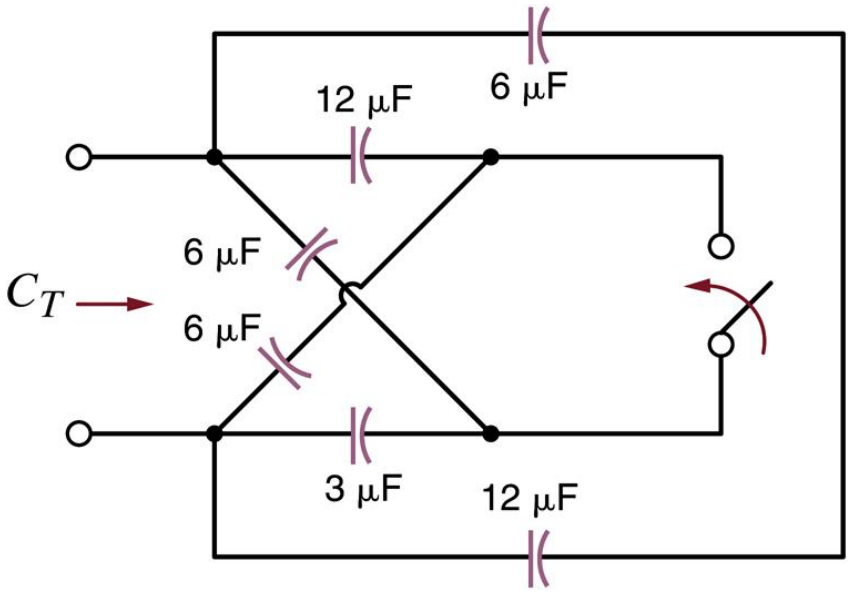
Example

- Compute the equivalent capacitance of the following network:



Example

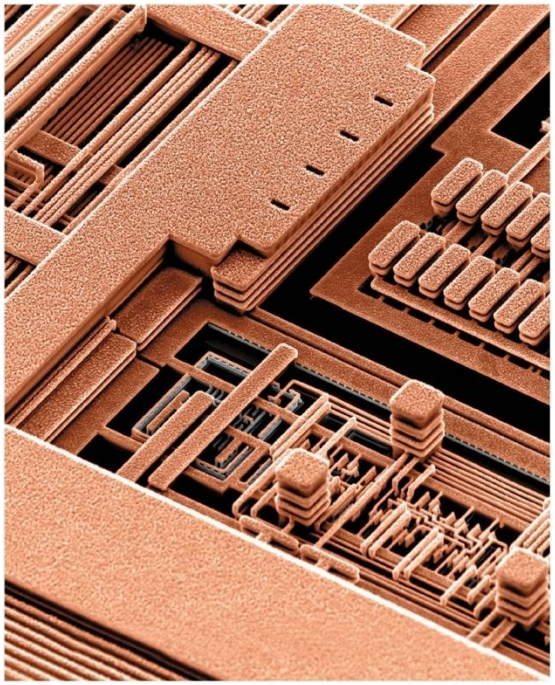
- Calculate the equivalent capacitance of the following network:
  a) when the switch is open
  b) when the switch is closed



Board Notes

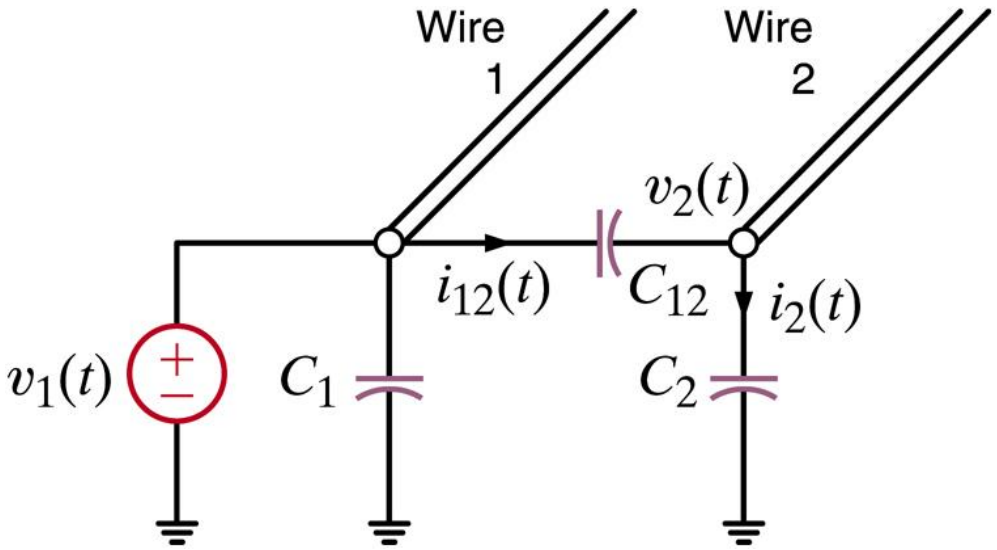
Application Example

- In integrated circuits, wires carrying high-speed signals are closely spaced as shown by the following micrograph. As a result, a signal on one conductor can “mysteriously” appear on a different conductor. This phenomenon is called crosstalk. Let us examine this condition and propose some methods for reducing it.



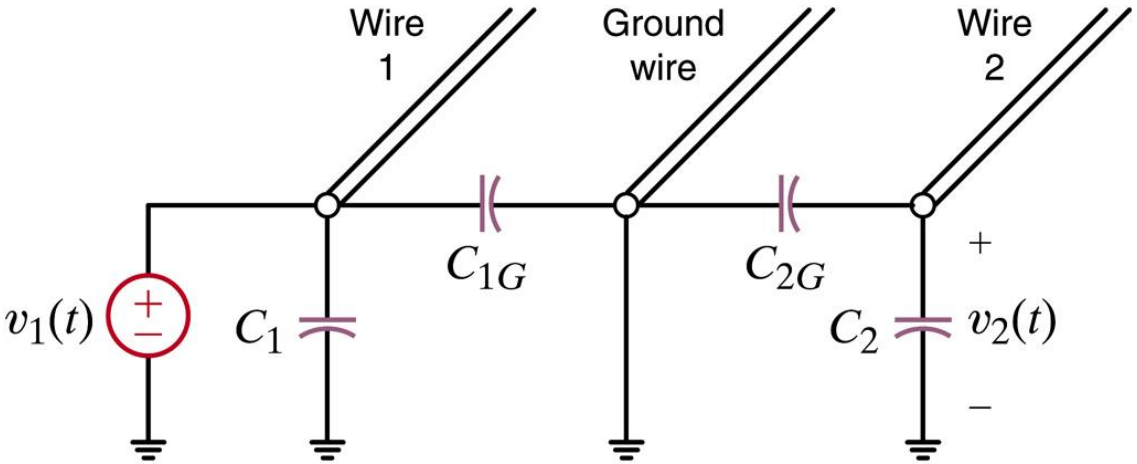
Application Example

- Simple model for investigating crosstalk:



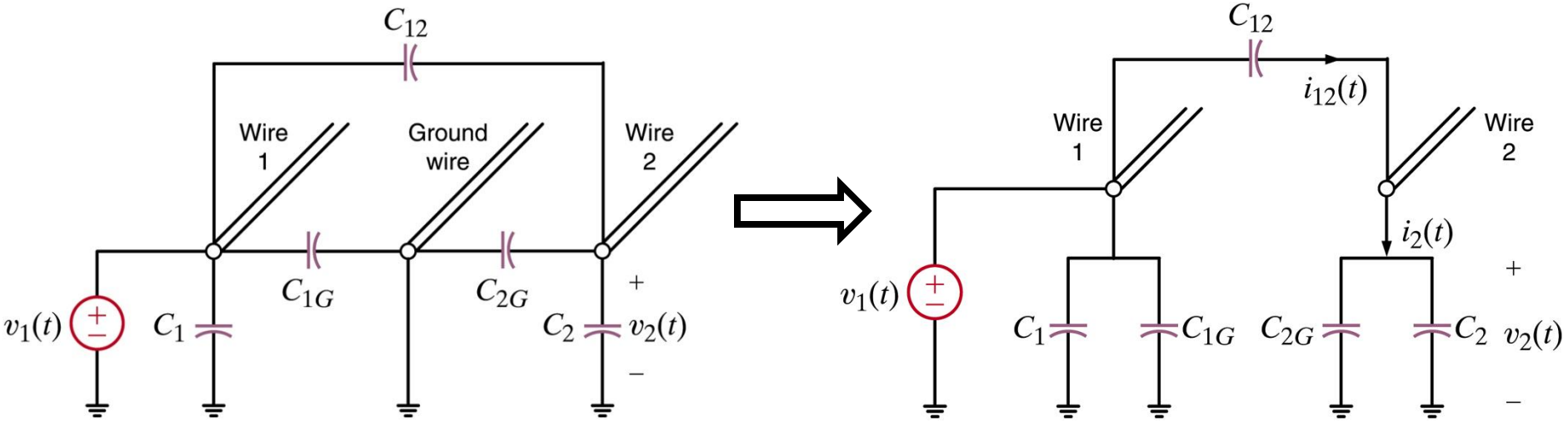
Application Example

- Use of ground wire to reduce crosstalk (simple, not too realistic model! Why?)



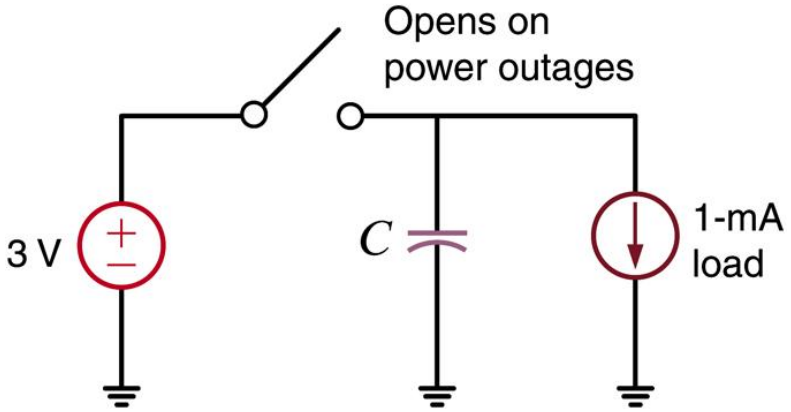
Application Example

- A more accurate model:



Design Example

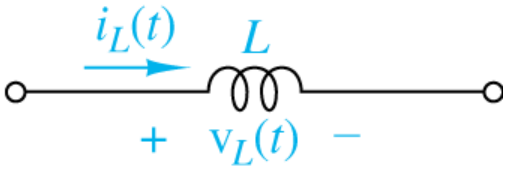
- We have all undoubtedly experienced a loss of electrical power in our office or our home. When this happens, even for a second, we typically find that we have to reset all of our digital alarm clocks. Let's assume that such a clock's internal digital hardware requires a current of 1 mA at a typical voltage level of 3.0 V, but the hardware will function properly down to 2.4 V. Under these assumptions, we wish to design a circuit that will “hold” the voltage level for a short duration, for example, 1 second.



Board Notes

Inductors

- An inductor is typically a coil of conducting wire.
- Inductor stores energy in its magnetic field.



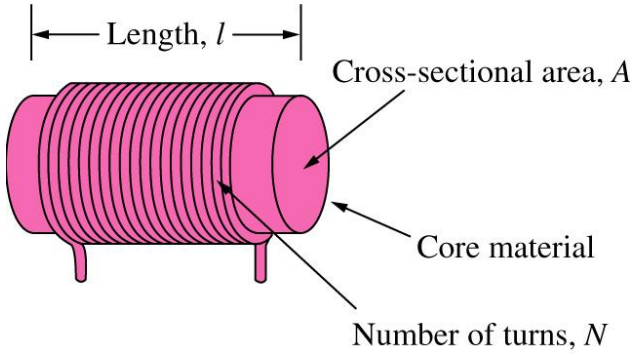
- If current passes through an inductor the voltage across the inductor is directly proportional to the time rate of change of the current:

$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

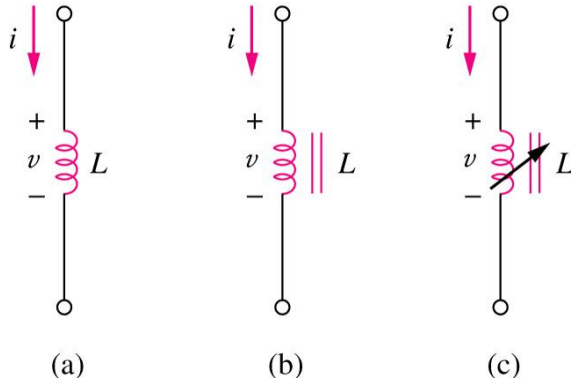
The constant of proportionality is the inductance of the inductor.

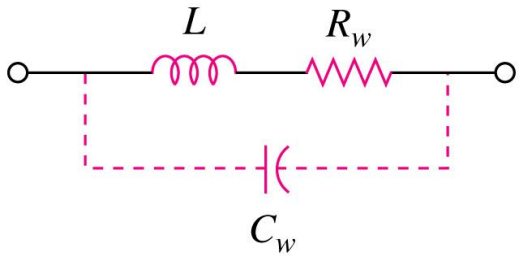
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)\cdot d\tau \quad \text{or} \quad i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(\tau)\cdot d\tau + i_L(t_0)$$

Inductors



$$L = \frac{N^2 \mu A}{l}$$





Model for a non-ideal inductor

Inductors

- In honor of Joseph Henry (1797-1878), an American physicist, the unit of inductance is named Henry (H).

- Note that:
  - An inductor acts like a short circuit to DC current.
  - Inductor impede instantaneous changes of its current.
- Instantaneous power delivered to the inductor is:

$$p_L(t) = v_L(t) i_L(t) = L \frac{di_L(t)}{dt} i_L(t)$$

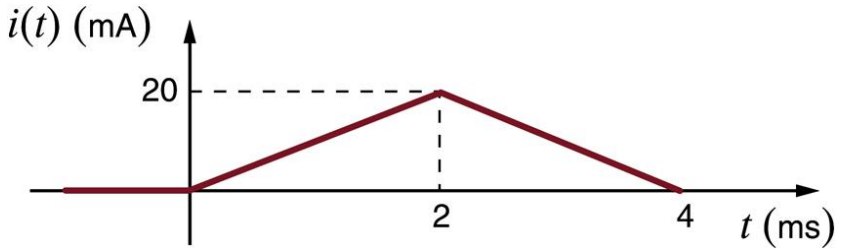
The total stored energy is:

$$W_L(t) = \int_{-\infty}^{t} p_L(\tau) \cdot d\tau = \int_{-\infty}^{t} L i_L(\tau) di_L(\tau) = \frac{1}{2} L i_L^2(t)$$

(Have we assumed anything in writing the above equation?!)

Example

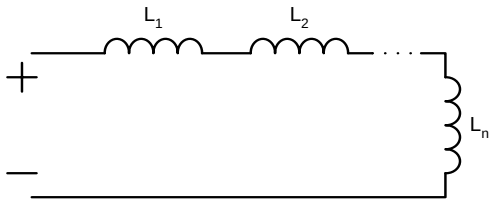
- The current in a 10-mH inductor has the following waveform. Find the voltage of the inductor.



Series and Parallel Inductors

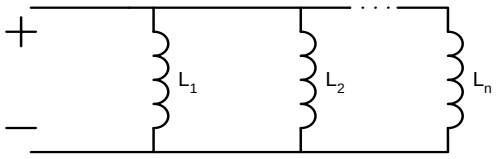
- The equivalent inductance of series-connected inductors is the sum of the individual inductances. Why?

$$L_{eq} = L_1 + L_2 + \cdots + L_n$$



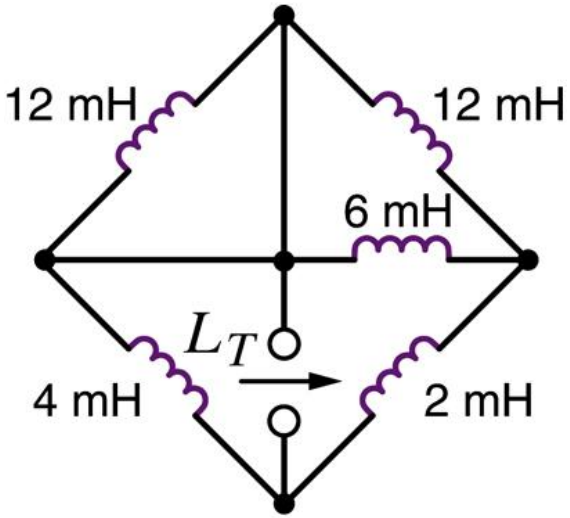
- The equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances. Why?

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$



Example

- Find the equivalent inductance (L_T) of the following network:



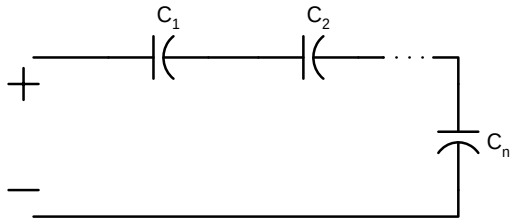
First-Order Circuits

- Applying KVL and/or KCL to purely resistive circuits results in algebraic equations.
- Applying these laws to RC and RL circuits results in differential equations.
- In general, differential equations are a bit more difficult to solve compared to algebraic equations!
- If there is only one C or just one L in the circuit the resulting differential equation is of the first order (and it is linear).
- A circuit that is characterized by a first-order differential equation is called a first-order circuit.

What Do We Mean By Equivalent Capacitor?

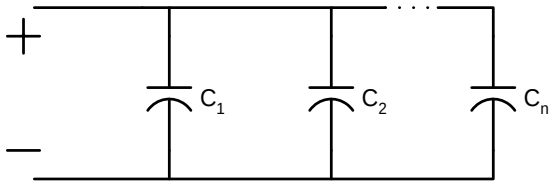
- The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances. Why?

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$



- The equivalent capacitance of parallel capacitors is the sum of the individual capacitances. Why?

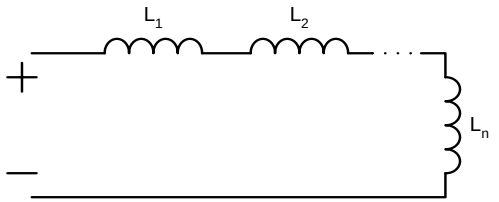
$$C_{eq} = C_1 + C_2 + \cdots + C_n$$



What Do We Mean by Equivalent Inductor?

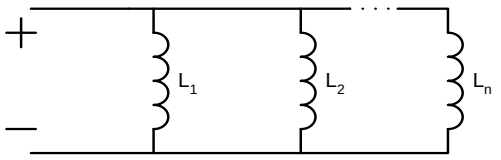
- The equivalent inductance of series-connected inductors is the sum of the individual inductances. Why?

$$L_{eq} = L_1 + L_2 + \cdots + L_n$$



- The equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances. Why?

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$



First-Order Circuits

- So in an RC circuit if we have more than one capacitor, however, we can combine the capacitors (series and/or parallel combination) and represent them with one equivalent capacitor, we still have a first-order circuit.

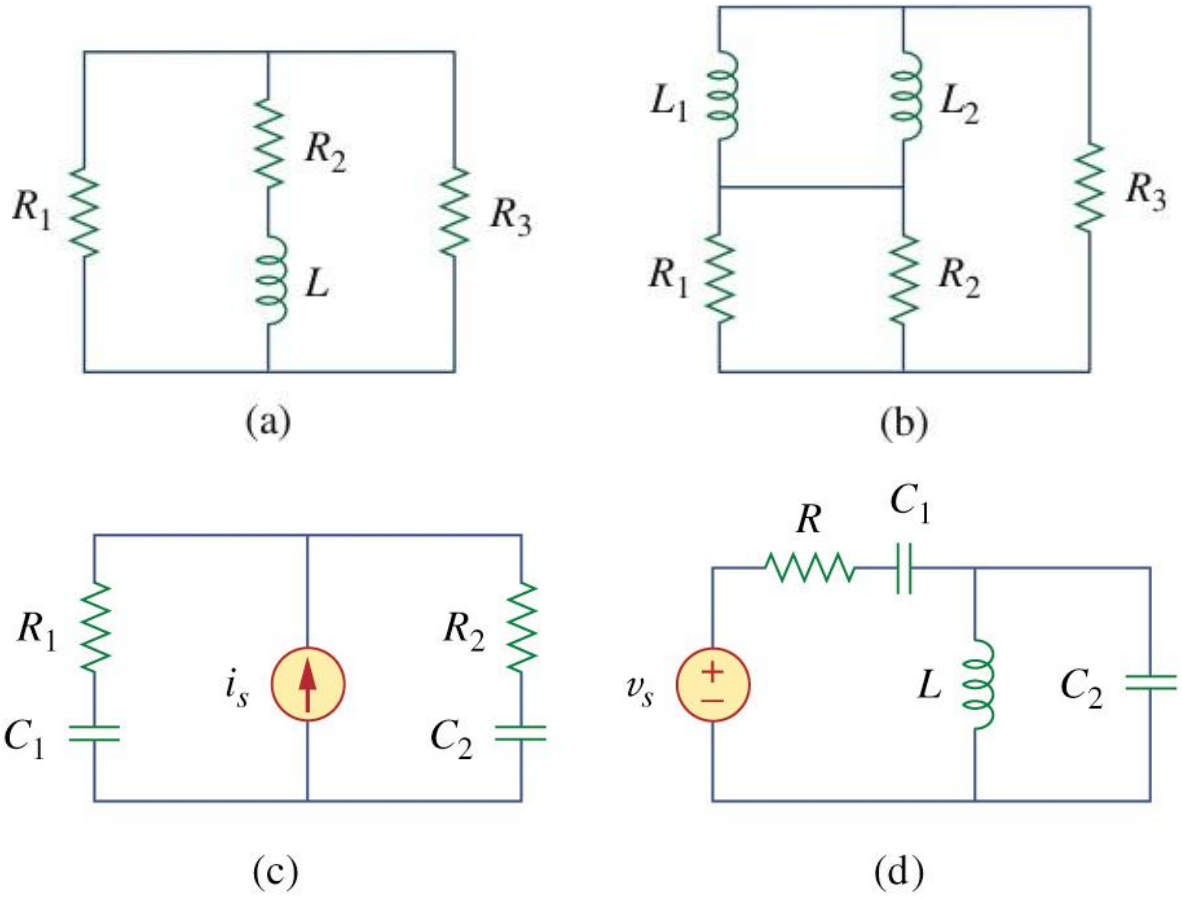
- The same is true for RL circuits, that is if we can combine all the inductors and represent them with one equivalent circuit then we still have a first-order circuit

- In such circuits we can find the Thevenin (or Norton) equivalent circuit seen by the equivalent capacitor (or Inductor) and then solve the circuit.

- Let’s start with the circuits that have no source!

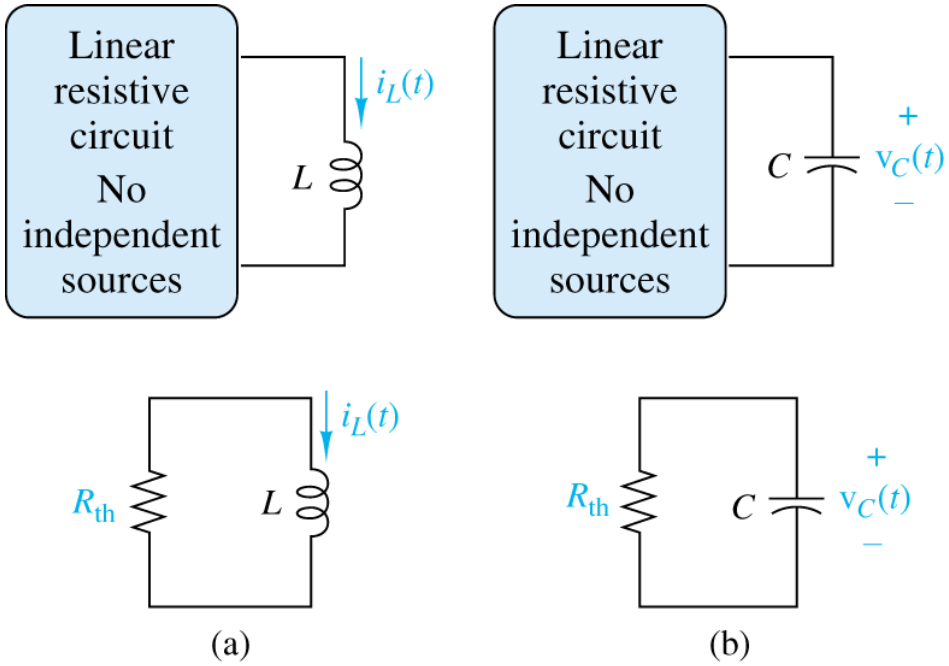
Example

- Which one of the following circuits is a first-order circuit?

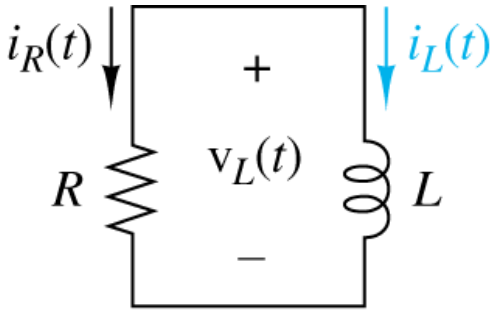


Source-Free or Zero-Input First-Order Circuit

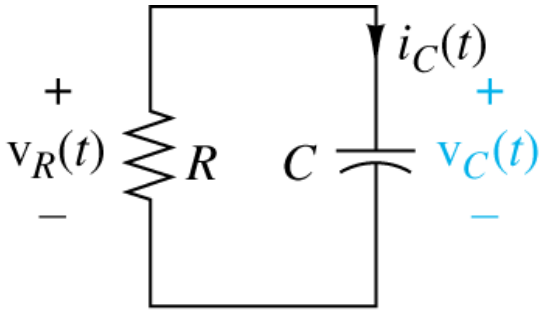
- Recall that in general if there is only one (equivalent) inductor or capacitor in the circuit one can model the circuit seen by the inductor or capacitor by its Thevenin equivalent circuit.
- In the case of source-free circuit (no independent source in the circuit) the Thevenin equivalent circuit will be a resistor.



Source-Free or Zero-Input First-Order Circuit



(a)



(b)

$$i_R(t) = -i_L(t)$$

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = \frac{L}{R}\frac{di_L(t)}{dt}$$

$$\frac{L}{R}\frac{di_L(t)}{dt} = -i_L(t) \Rightarrow \frac{di_L(t)}{dt} = -\frac{R}{L}i_L(t)$$

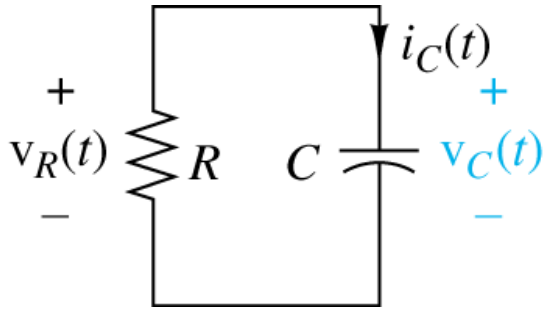
$$i_C(t) = -i_R(t)$$

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_C(t)}{R}$$

$$C\frac{dv_C(t)}{dt} = -\frac{v_R(t)}{R} \Rightarrow \frac{dv_C(t)}{dt} = -\frac{1}{RC}v_C(t)$$

Source-Free First-Order RC Circuit

- Let's assume that we know the charge or equivalently the voltage across the capacitor at time 0. That is: $v_C(0) = V_0$



- Recall: $$i_C(t) + i_R(t) = 0 \Rightarrow C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0 \Rightarrow \frac{dv_C(t)}{dt} = -\frac{1}{RC}v_C(t)$$

- Let's try to solve this differential equation. Because of the simple form of this equation we can re-arrange the term as:

$$\frac{dv_C(t)}{v_C(t)} = -\frac{1}{RC}dt$$

Source-Free First-Order RC Circuit

- Before going any further can you tell the units for RC from the equation:

$$\frac{dv_C(t)}{v_C(t)} = -\frac{1}{RC}dt$$

- Now let's go further! and integrate both sides of the equation from 0 to t:

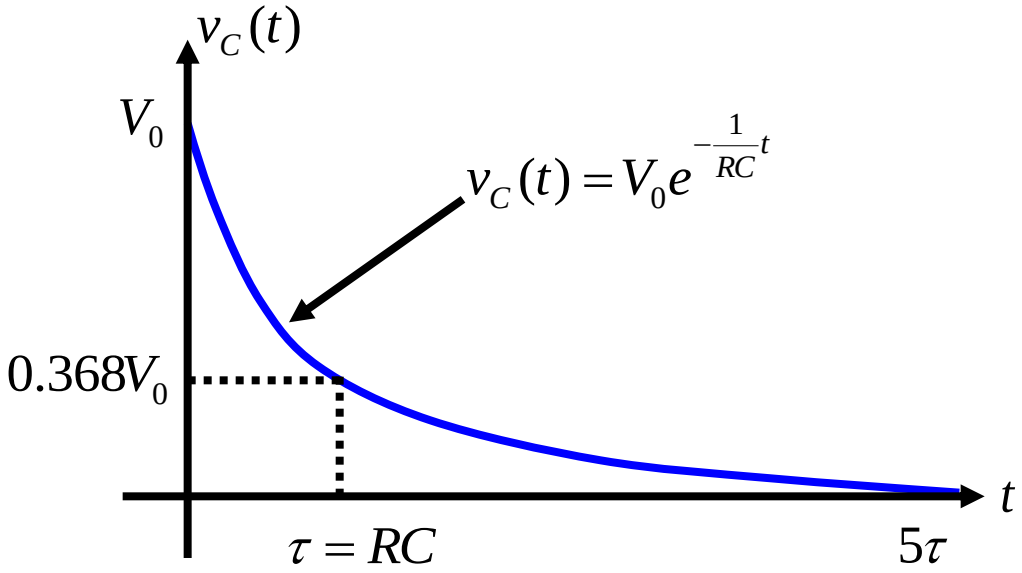
$$\int_{t'=0}^{t} \frac{dv_C(t')}{v_C(t')} = -\int_{t'=0}^{t} \frac{1}{RC}dt'$$

$$\ln v_C(t')\Big|_{t'=0}^{t} = -\frac{1}{RC}t'\Big|_{t'=0}^{t}$$

$$\ln \frac{v_C(t)}{V_0} = -\frac{1}{RC}t \Rightarrow v_C(t) = V_0 e^{-\frac{1}{RC}t}$$

Source-Free First-Order RC Circuit

- The voltage response of an source-free first-order RC circuit is an exponential decay from its initial voltage value:



- The time that is required for the response to decay by a factor of 1/e (36.8% or by engineering approximation! 37%) of its initial value is called *time constant* of the circuit and is typically denoted by τ.

Source-Free First-Order RC Circuit

- Philosophical question: When there is no source in the circuit, how come we have such a response? What is the response due to?

- In general, the response of a source-free circuit which is due to the initial energy stored in the storage elements (in this case C) and not due to external sources is called *natural response.*

- In first order RC (and RL) circuits natural response is a decaying exponential.

- To find the natural response of a first-order RC circuit we need two pieces of information:
  - Initial voltage across the capacitor
  - The time constant τ=RC

Source-Free First-Order RC Circuit

- Time constant of the circuit gives us an indication of how rapidly the response decays, in other words how fast is the response.

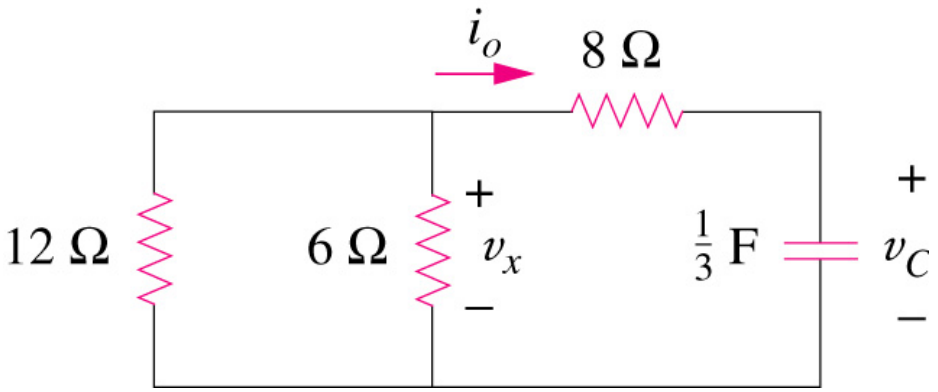
- Let's calculate the natural response $v_C(t) = V_0 e^{-\frac{1}{RC}t}$ for times equal different multiples of the time constant:

t	$v_C(t)$
τ	$0.3679V_0$
2τ	$0.1353V_0$
3τ	$0.0498V_0$
4τ	$0.0183V_0$
5τ	$0.0067V_0$

- For all practical purposes it is typically assumed that the response reaches its final value after 5τ.

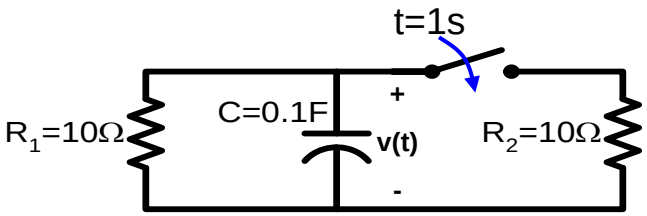
Example

- Assuming $v_C(0)=30V$, determine v_C and v_x, and i_o for t≥0



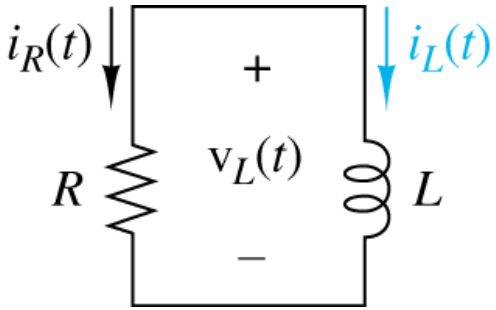
Source-Free First-Order RC Example

- In the following circuit, find the voltage across the capacitor for t≥0. Assume that v(0)=10V.



Source-Free First-Order RL Circuit

- Let's assume that we know the initial current in the inductor at time 0. That is: $i_L(0) = I_0$



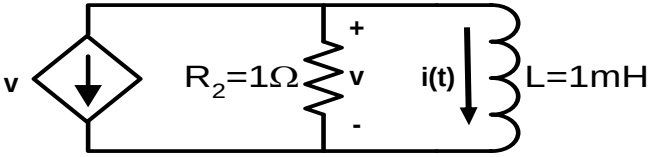
- After a bit of equation writing! we have:

$$\frac{di_L(t)}{dt} = -\frac{R}{L} i_L(t) \Rightarrow i_L(t) = I_0 e^{-\frac{R}{L}t}$$

- What is the time constant of this circuit?
- To find the natural response of a first-order RL circuit we need two pieces of information:
  - Initial current through the inductor
  - The time constant τ=L/R

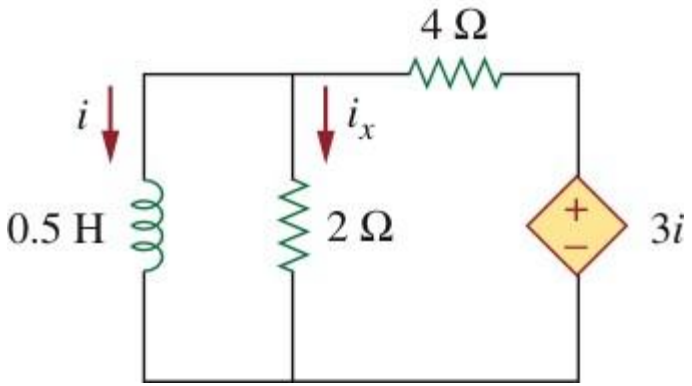
Source-Free First-Order RL Example

- In the following circuit, find the current through the inductor for t≥0. Assume that i(0)=1A.



Example

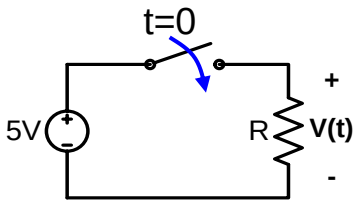
- In the following circuit, assuming $i(0)=10\text{A}$, calculate $i(t)$ and $i_x(t)$.



Unit Step Function

- Step function is a very useful function to model the signals in the circuits that have switches.

- Example: In the following circuit, find the voltage across the resistor R for $-\infty<t<\infty$.



Unit Step Function

- To model abrupt changes in a voltage or current one can use a unit step function.

- The unit step function is defined as follows:

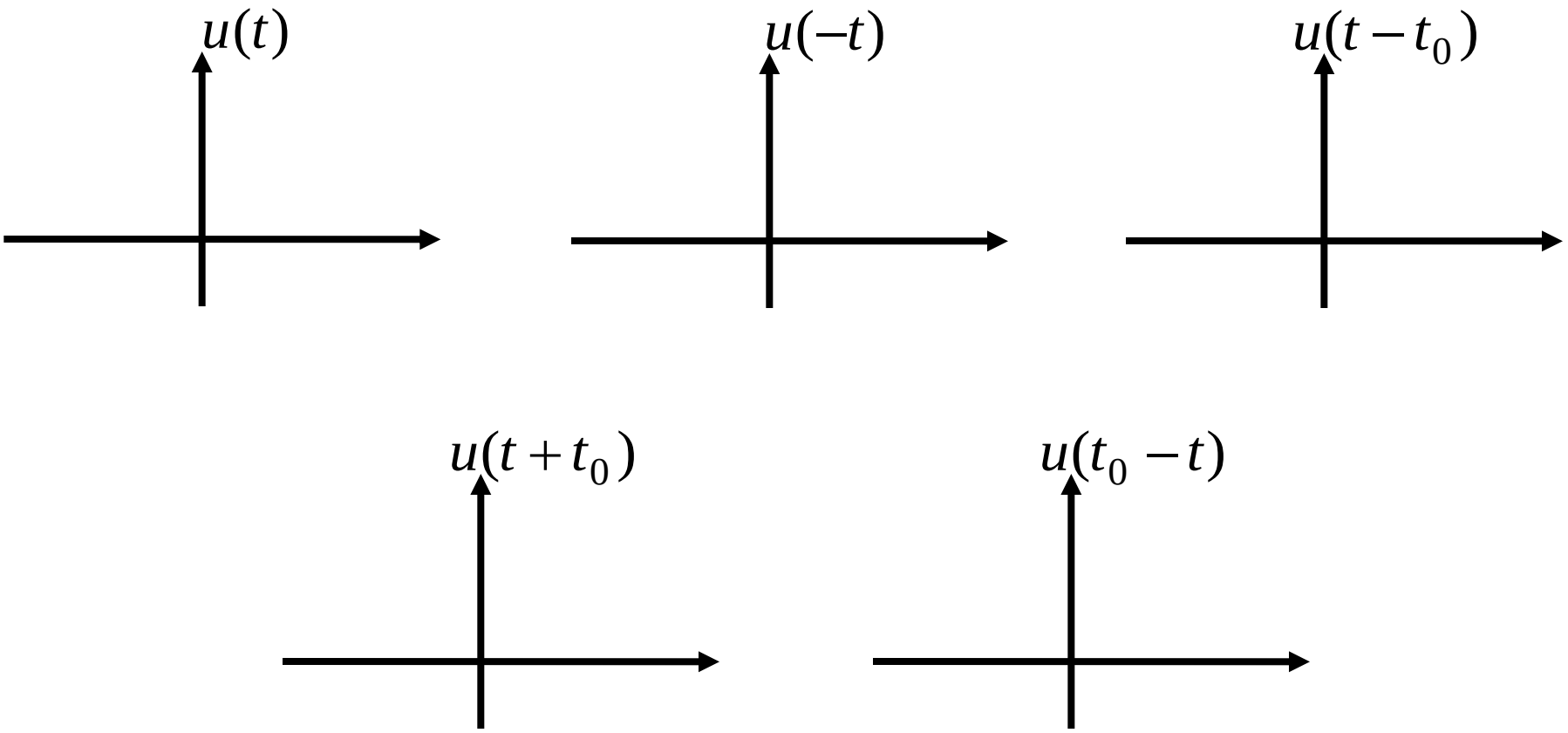
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Use the step function to express the voltage across the resistor in the previous example:

v(t)=

Unit Step Function Examples

- Assuming t_0 is a given positive time, plot the following functions:



Unit Step Function Examples

- Write the functions on the previous slide in mathematical terms, e.g.,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Recall: Differential Equations

- In general, the differential equation that model a first-order RC or RL circuit with a source that is switched in at $t = t_0$, is of the form:

$$\frac{dx(t)}{dt} + \alpha x(t) = f(t - t_0)u(t - t_0), \qquad x(t_0) = x_0$$

  valid for $t \geq t_0$, where $x(t)$ is the voltage or current of interest and x_0 is the initial condition at time t_0 and $f(t)$ is a function of the source (or force function).

- For notation simplicity and without loss of generality, let's assume $t_0 = 0$, then, the equation can be written as:

$$\frac{dx(t)}{dt} + \alpha x(t) = f(t), \qquad x(0) = x_0$$

- Note that this is a special type of differential equations! **(What is so special about it?)**

Recall: Differential Equations

- Many techniques for solving this type of differential equations exist.
- The fundumental theorem of differential equations states that if $x_p(t)$ is a solution of

$$\frac{dx(t)}{dt} + \alpha x(t) = f(t)$$

and $x_h(t)$ is a solution to the homogeneous equation

$$\frac{dx(t)}{dt} + \alpha x(t) = 0$$

then $x(t) = K_1 x_h(t) + x_p(t)$, where K_1 is a constant is a solution to the original differential equation.

- $x_p(t)$ is called the **particular** solution or **forced** response.
- $x_h(t)$ is called the **homogeneous** solution or **natural** response (also called complementary solution).

Recall: Differential Equations

- If we only have DC sources in the circuit, then

$$f(t) = F$$

where F is a constant.

- Can you find $x_p(t)$ and $x_h(t)$?

Recall: Differential Equations

- In general, the solution to: $\frac{dx(t)}{dt} + \alpha x(t) = F$

  is of the form of:

$$x(t) = K_1 e^{-\alpha t} + K_2$$

- α is called the natural frequency of the circuit! or $\tau = 1/\alpha$ is called the time constant of the circuit. Recall, that the first term in the above expression is called ***natural response*** (is due to stored energy or initial condition) and the second term is called ***forced response*** (is due to independent sources).

- How do we find K_1 and K_2?

DC or Step-Response of First-Order Circuits

- When a DC source in an RC or RL circuit is suddenly applied (for example by turning on a switch), the voltage or current source can be modeled using the source and a switch (using a step function!).

- The response of the circuit to such a sudden change (when the excitation is a step function) is called the “step response” of the circuit.

- In general the DC or step response of a first-order circuit satisfies a differential equation of the following form (assuming that the step is applied at $t = 0$):

$$\frac{dx(t)}{dt} + ax(t) = Fu(t), \qquad x(0^+) = x_0$$

Do you know what do we mean by 0^+ and why we are using it ?

DC or Step-Response of First-Order Circuits

- Using $\alpha = \frac{1}{\tau}$ the solution to $\frac{dx(t)}{dt} + ax(t) = Fu(t), \qquad x(0^+) = x_0$ is of the form:

$$x(t) = K_1 e^{-\frac{t}{\tau}} + F\tau$$

- Note that:

$$x(\infty) = \lim_{t\to\infty} x(t) = F\tau$$

$$K_1 = x(0^+) - x(\infty)$$

- Thus, the response of a first-order circuit has the following form:

$$x(t) = \left[x(0^+) - x(\infty)\right]e^{-\frac{t}{\tau}} + x(\infty)$$

The step response of any voltage or current in a first-order circuit has the above form.

DC or Step-Response of First-Order Circuits

- If the step is applied at $t = t_0$ (or the switch changes its position at $t = t_0$), given the initial condition at $t = t_0^+$ then the general form of the solution is of the form:

$$x(t) = \left[x(t_0^+) - x(\infty)\right]e^{-\frac{(t-t_0)}{\tau}} + x(\infty)$$

DC or Step-Response of First-Order Circuits

- For example, in a first-order LR circuit the step response of the current through the inductor is of the form:

$$i_L(t) = \left[i_L(t_0^+) - i_L(\infty) \right] e^{-\frac{(t-t_0)}{L/R}} + i_L(\infty)$$

and in a first-order RC circuit the step response of the voltage across the capacitor is of the form:

$$v_C(t) = \left[v_C(t_0^+) - v_C(\infty) \right] e^{-\frac{(t-t_0)}{RC}} + v_C(\infty)$$

- These equations are very useful! and in general for a step response of any first-order circuit we have:

$$\text{any voltage or current} = (\text{Initial value} - \text{Final value}) e^{-\frac{\text{elapsed time}}{\text{time constant}}} + \text{Final value}$$

- The initial value can be found using the initial condition of the circuit.

DC or Step-Response of First-Order Circuits

- The complete response can be divided into two portions:

$$\text{Complete Response} = \underbrace{\text{Transient Response}}_{\substack{\text{temporary part} \\ \text{due to stored energy}}} + \underbrace{\text{Steady-State Response}}_{\substack{\text{permanent part} \\ \text{due to independent sources}}}$$

- The transient response is the circuit's temporary response that will die out with time.

- The steady-state response is the portion of the response that remains after the transient response has died out (behavior of the circuit a long time after the external excitation is applied).

DC or Step-Response of First-Order Circuits

- What are the transient and steady-state portions of the following response:

$$v_C(t) = \left[v_C(t_0^+) - v_C(\infty)\right] e^{-\frac{(t-t_0)}{RC}} + v_C(\infty)$$

- To find the complete response of a first-order circuit we need to find initial value, final value, and time constant of the circuit:
  - Initial value can be found using the initial condition.
  - Time constant can be found by finding the **Thevenin** equivalent resistance seen across the capacitor (or inductor)
  - How about the final value.

DC or Step-Response of First-Order Circuits

- Couple of interesting points (tricks) that are **only valid** for calculating final values of DC step-response:

  – A capacitor acts like a open circuit long time after the external excitation is applied. Can you intuitively justify this statement?

  – An inductor acts like a short circuit long time after the external excitation is applied. Why?

Steady-State Response

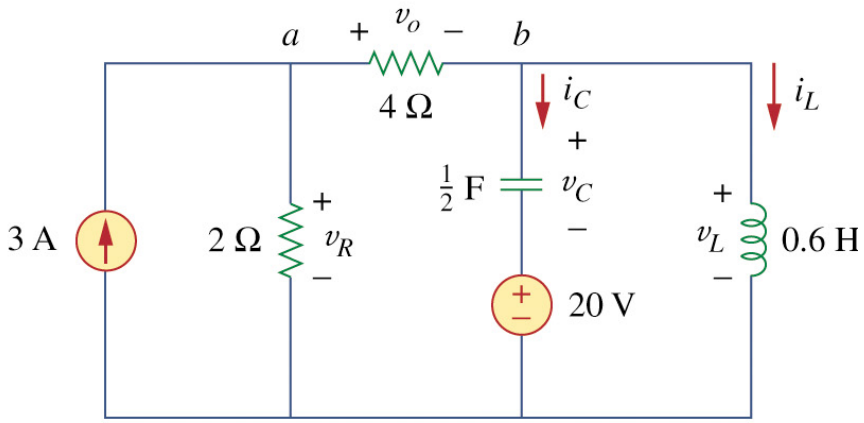
- Loosely speaking, the behaviour of the circuit a long time after an excitation is applied to the circuit is called steady-state response.

- For example, if in a circuit a switch is opened (or closed) the response of the circuit to this excitation long time after the switch is opened (or closed) is referred to as steady-state response.

- If we only have DC sources in the circuit, at steady state capacitors act like open circuit and inductors act like a short circuit.

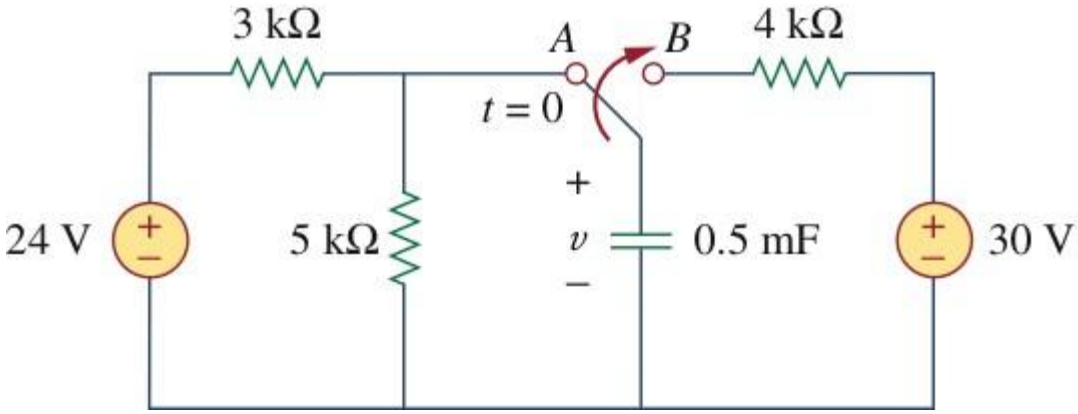
Example

- In the following circuit find the energy that is stored in the inductor and capacitor, when the circuit reaches steady state.



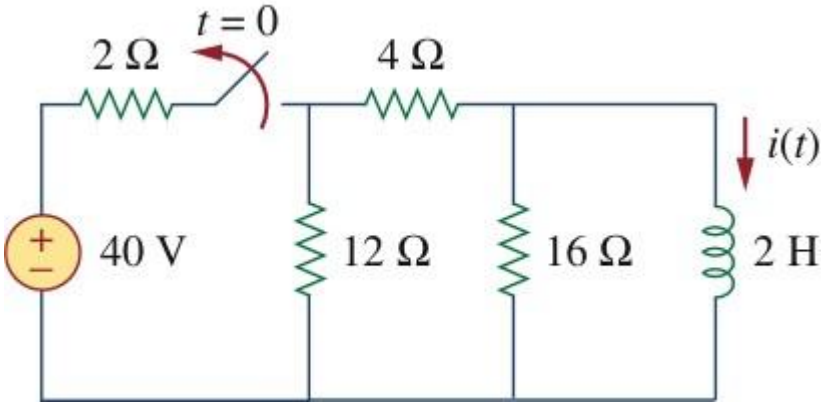
Example

- In the following circuit, the switch has been in position A for a long time and then at t=0, the switch moves to position B. Find the energy stored in the capacitor just before the switch moves. Also, what is the energy stored in the capacitor a long time after switch is moved to B, i.e., t=∞..



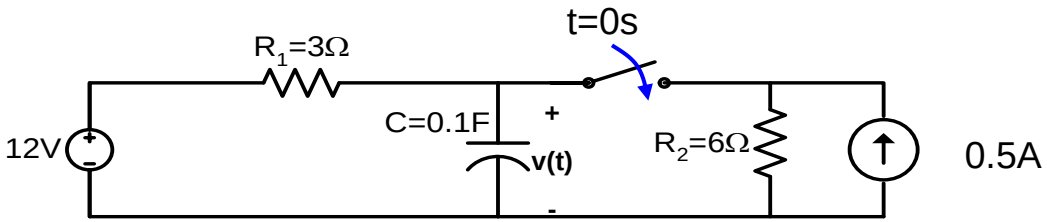
Example

- In the following circuit, the switch has been closed for a long time and at t=0 the switch is opened. What is the energy stored in the inductor just before the switch is opened? What is the energy stored in the inductor a long time after the switch is opened. i.e., t=∞.



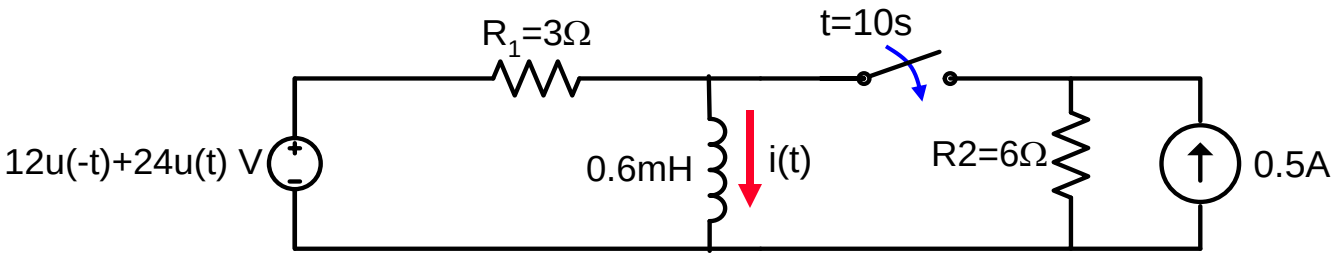
Example

- Find v(t) for t>0 in the following circuit. Assume the switch has been open for a long time before it is closed at t=0.



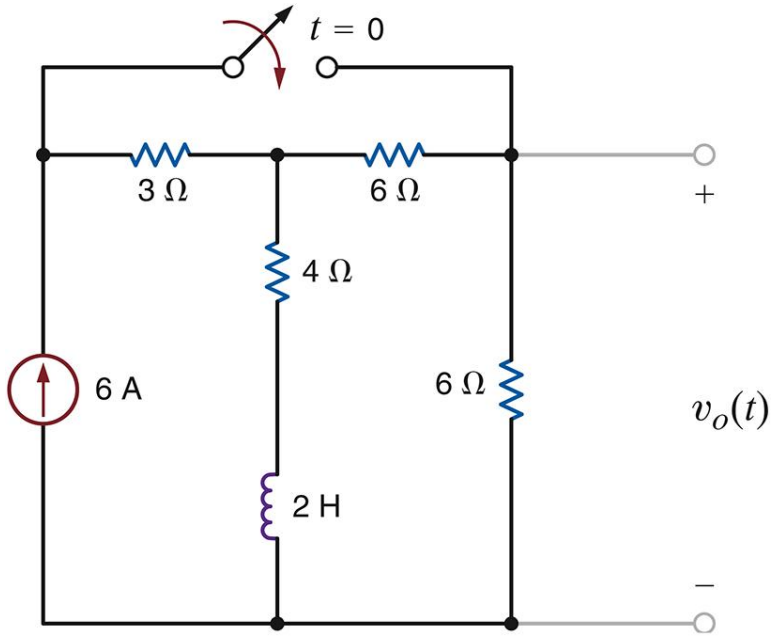
Example

- Find i(t) for t>0 in the following circuit. Assume the switch has been open for a long time before it is closed at t=0.



Example

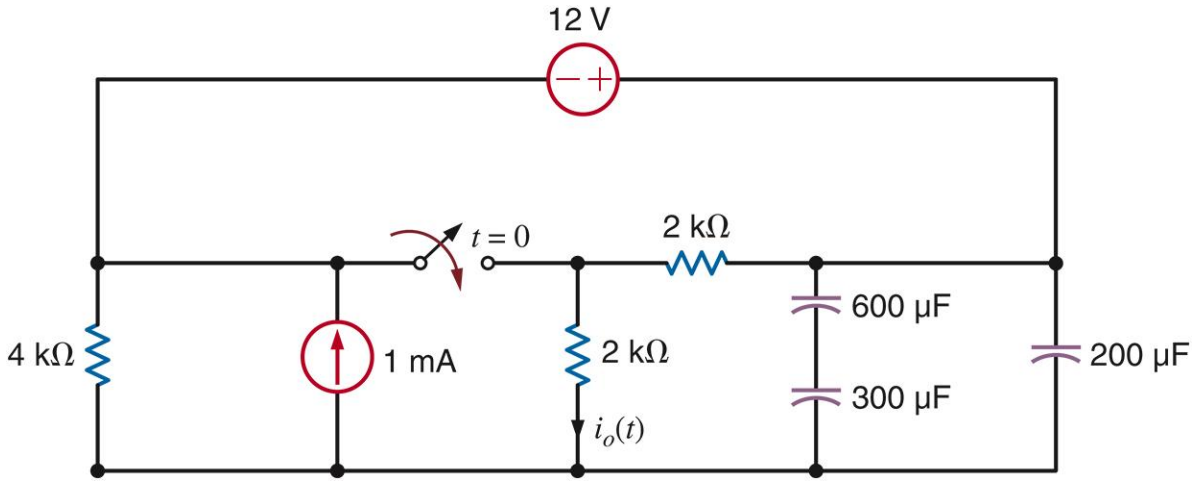
- In the following circuit, assume the switch has been open for a long time before being closed at time 0. Find $v_0(t)$ for $t>0$



Notes

Example

- In the following circuit, assume the switch has been open for a long time before being closed at time 0. Find $i_0(t)$ for $t>0$



Notes