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SWITCHING RESTRICTIONS FOR STABILITY DESPITE SWITCHING DELAY: APPLICATION TO SWITCHED TRACKING TASKS IN PARKINSON'S DISEASE

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Abstract. Switched nonlinear systems with delay in the switching instant could be destabilized, despite stable dynamics in each mode, if the delay is long enough. We identify a restriction on the switching scheme to assure stability despite a finite delay in switching instant. The restriction partitions the state-space in a time-varying manner for a known switching delay, and converges to a steady-state partition that can be determined from the intersection of Lyapunov functions in each mode. We apply this technique to experimental data from a manual pursuit tracking task performed by 14 subjects with Parkinson's disease, and 10 control subjects. Each subject manually tracks a moving target through a joystick-controlled cursor, with sudden changes in the tracking dynamics. The tracking task can be modeled as a 3-mode switched system. By calculating the maximal time delay for each mode pair and for each subject, we obtain a measure of relative stability that can be compared across groups and across tasks. Using the derived stability measure, subjects with Parkinson's disease were shown to be relatively less stable than control subjects.

Keywords. Switched systems, switching delay, Parkinson's disease, nonlinear dynamics

1 Introduction

We consider stability of switched systems [1, 2, 3] with delay in the switching instant. While such systems are ubiquitous in engineered systems (such as those with communication delays or with humans in the loop) we focus here on application to a biological system – motor control in a switched manual pursuit tracking task in Parkinson's disease – for which stability despite switching delay provides a measure of robustness to sudden changes in task dynamics.

For switched systems with stable dynamics in each mode, existence of a common Lyapunov function proves stability under arbitrary switching [4, 5]. If a common Lyapunov function does not exist or cannot be found, stability can be assessed for a known switching signal [6, 7, 8] or assured by constraining switching to occur in certain regions of the state-space [9]. However, even if a switching signal is determined to result in stable behavior, a delay in the switching instant could result in performance degradation or even destabilization of the system.

In Parkinson's disease, a neurodegenerative disorder in which voluntary movement is impaired by a lack of dopamine in the brain, slowness in switching between multiple tasks (e.g., between reaching and balancing) may underlie the empirical observation that PD subjects have difficulty performing simultaneous movements. Indeed, the slowness in switching may be a contributing factor to the high prevalence of falls in Parkinson's disease as compared to the general population. In addition, cognitive inflexibility in Parkinson's disease [10, 11] may closely correspond to an inability to adapt to sudden change. When sudden changes occur in a dynamic context, as with manual pursuit tracking tasks, delay in motor response to sudden changes may be related to stability despite delay in switching. In this paper, we investigate the effect of delay on switching in Parkinson's disease through the use of a manual pursuit tracking task whose dynamics have sudden changes – that is, a tracking task that has hybrid dynamics.

A set of three motor tasks were designed to evaluate the ability of subjects to respond to sudden and unexpected changes in tracking dynamics. When switching between multiple tracking tasks, the subject must stabilize their error dynamics. In recent work [12], mode detection algorithms were used to determine that the length of time required for a subject to detect a sudden change in task dynamics (based solely on their tracking performance) was longer in subjects with Parkinson's disease as compared to normal subjects. In this paper, we take a slightly different approach. We instead evaluate the stability of each subjects' tracking dynamics, and determine an upper bound on the maximum delay in switching instant that each subject could tolerate without destabilization.

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We interpret this delay as an indirect indicator of relative stability, such that longer delays correlate to a higher degree of stability.

Previous work in switched systems with bounded delay has focused on controller synthesis for delay differential equations, in which linear state-space equations depend on a time-delayed state [13, 14, 15, 16]. Work has also been done on stabilizing autonomous systems in which mode changes are detected with a time-varying delay [17, 18, 19]. In contrast, we focus here on autonomous systems for which a delay in the state measurement affects only the switching instant. Closely related to our work are the results presented in [20], in which switched systems with delay in the switching instant are stabilized by imposing a minimum dwell-time condition to prevent instability that could arise from switching too quickly between modes. These techniques are an extension of standard dwell-time methods [21, 22] that ensure stability by forcing a system to remain in a given mode for a minimum, finite duration.

We consider switched systems with stable dynamics in each mode, but for which no common Lyapunov function can be found to prove stability under arbitrary switching. We exploit restrictions for stability under state-constrained switching to determine, for a known delay in switching instant, the set of states for which switching will not violate restrictions for stable switching. By bounding the change in the Lyapunov functions of the current mode and the next mode over the duration of the switching delay, we introduce restrictions on the switching scheme. These restrictions result in a time-varying partition in the state-space for a given mode pair and bounded delay. By restricting the class of allowable switching signals to those which satisfy the time-varying state-space constraint, stability is assured, despite possible worst case evolution during the time-delay.

Our main contributions are 1) a method of synthesizing state constraints that guarantee global uniform asymptotic stability in a nonlinear switched system despite a switching delay, and 2) application of this technique to evaluate stability in a switched manual pursuit tracking task as a means of assessing motor performance in Parkinson's disease.

2 Problem formulation

Consider a switched nonlinear system

$$\dot{x} = f_{\sigma(t)}(t, x) \tag{1}$$

with $x \in \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \to \mathcal{P} \subset \mathbb{N}$ a piecewise constant switching signal, and $\mathcal{F} \triangleq \{f_p : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}^n \ : \ p \in \mathcal{P}\}$ a family of functions indexed by p that are piecewise continuous in t and locally Lipschitz in x on $\mathbb{R}_+ \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^n$ a domain containing the origin. We assume the origin is an equilibrium point for each $f_p \in \mathcal{F}$ without loss of generality, and consider local stability of the equilibrium point. Lastly, we use the notation $\|\cdot\|$ to indicate the $p-$norm of a vector in $\mathbb{R}^n$.

Definition 2.1 (Modified from [23])**.** The equilibrium point $x^* = 0$ of (1) is *stable under* Σ^*, a set of piecewise constant switching signals, if $\forall \epsilon > 0, \exists \delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq t_0 \geq 0 \tag{2}$$

for all $\sigma \in \Sigma^*$.

Lemma 2.1 (Modified from [23])**.** The equilibrium point $x^* = 0$ for (1) is

- *uniformly stable (US) under* Σ^* if and only if there exists a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

  $$\|x(t)\| \leq \alpha(\|x(t_0)\|),\ \forall t \geq t_0 \geq 0,\ \forall \|x(t_0)\| < c \tag{3}$$

  for all $\sigma \in \Sigma^*$.

- *uniformly asymptotically stable (UAS) under* Σ^* if and only if there exists a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

  $$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0),\ \forall t \geq t_0 \geq 0,\ \forall \|x(t_0)\| < c \tag{4}$$

  for all $\sigma \in \Sigma^*$.

The results of Lemma 2.1 hold globally for $c = \infty$.

Remark 2.1. If $\Sigma^* = \{p\}$, (i.e. $\sigma(t) \equiv p$), Definition 2.1 and Lemma 2.1 are equivalent to standard definitions of stability for a nonlinear system.

Assume that for (1), $\dot{x} = f_p(t, x), p \in \mathcal{P}$ has a *stable* equilibrium point $x^* = 0$. Define a piecewise continuous Lyapunov function

$$V(t, x) = V_{\sigma(t)}(t, x) \tag{5}$$

which satisfies conditions for stability in each mode p with a continuously differentiable Lyapunov function $V_p(t, x) : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}$ that is positive definite, and for which $\frac{d}{dt}V_p(t, x)$ is negative semidefinite, for all $t \geq 0$ [23].

Lastly, assume that $\delta(\epsilon, t_0)$ from Definition 2.1 is invertible, such that for any $\delta, t_0 \in \mathbb{R}_+$, one can compute

$$\epsilon = \epsilon(\delta, t_0) \tag{6}$$

satisfying (2).

Definition 2.2. Let Σ^{s} be the set of piecewise constant switching signals $\sigma : \mathbb{R}_+ \to \mathcal{P}$ such that (1) is stable.

From [7, 4], a sufficient condition for stability under a switching signal $\sigma \in \Sigma^s$ is that the piecewise continuous Lyapunov function (5) be non-increasing, that is, at each switching instant τ,

$$V_p(\tau, x(\tau^-)) - V_q(\tau, x(\tau)) \geq 0 \tag{7}$$

with $\sigma(\tau^-) = p$ and $\sigma(\tau) = q$.

Problem 1. Characterize a class of switching signals $\Sigma^{\mathrm{s}}_T \subseteq \Sigma^{\mathrm{s}}$ such that the piecewise continuous Lyapunov function (5) is non-increasing as in (7) for the system (1), despite a switching delay of duration T.

3 Stability under state constrained switching

We first develop state partitions that ensure (5) is strictly decreasing in the case in which there is no switching delay. We then examine the effect of the delay on those partitions, and introduce a "delay buffer" to account for the effect of the switching delay by bounding the possible changes in the Lyapunov function $V_p(t,x)$ of the current mode $p \in \mathcal{P}$ and the Lyapunov function $V_q(t,x)$ of the next mode $q \in \mathcal{P}, q \neq p$, over the duration of the time delay.

3.1 Restrictions on switching without delay

First consider the case when $T = 0$. To assure that the time derivatives of the two Lyapunov functions $V_p(t,x)$ and $V_q(t,x)$ are bounded, assume the following:

Assumption 3.1. There exists a class $\mathcal{K}$ function $\alpha_p(\|x\|)$ such that

$$-\alpha_p(\|x\|) \leq \frac{\partial V_p}{\partial t} + \frac{\partial V_p}{\partial x} f_p(t,x) \leq 0 \tag{8}$$

Assumption 3.2. There exists a class $\mathcal{K}$ function $\alpha_{p,q}(\|x\|)$ and a real constant $b_{p,q} \in \{-1, 1\}$ such that

$$\frac{\partial V_q}{\partial t} + \frac{\partial V_q}{\partial x} f_p(t,x) \leq b_{p,q} \alpha_{p,q}(\|x\|) \tag{9}$$

Lyapunov functions that are Lipschitz continuous in x will satisfy the restrictions (8), (9) for autonomous systems.

Hence to solve Problem 1 for the case in which $T = 0$, define the delay-free switching restriction

$$\bar{\mathcal{S}}(p,q,\tau) \triangleq \{x \in \mathbb{R}^n : V_p(\tau,x) - V_q(\tau,x) > 0\} \tag{10}$$

Theorem 3.1 (From [7]). Let Σ_0^{s} be the set of piecewise constant switching signals $\sigma : \mathbb{R}_+ \to \mathcal{P}$ such that $x(\tau) \in \mathcal{S}^{\mathrm{s}}(\sigma(\tau^-), \sigma(\tau), \tau)$ at every switching instant τ. Then (1) is stable under Σ_0^{s}.

Proof. Any switching signal $\sigma \in \Sigma_0^{\mathrm{s}}$ that generates trajectories that satisfy (10) at all switching instants will also satisfy (7), hence $\Sigma_0^{\mathrm{s}} \subseteq \Sigma^{\mathrm{s}}$. □

3.2 Restrictions on switching with delay

Theorem 3.2. Let Σ_T^s be the set of piecewise constant switching signals such that, for each switching instant τ, $x(\tau - T) \in \mathcal{S}^{\mathrm{s}}(\sigma(\tau^-), \sigma(\tau), \tau)$, with

$$\begin{aligned}\mathcal{S}^{\mathrm{s}}(p,q,\tau) \triangleq \{&x \in \mathcal{D} : \\ &V_p(\tau - T, x) - V_q(\tau - T, x) \geq \gamma^{\mathrm{s}}(p,q,\tau)\}\end{aligned} \tag{11}$$

the set of states for which switching from mode p to mode q is allowed for a time-varying delay buffer

$$\begin{aligned}\gamma^{\mathrm{s}}(p,q,\tau) = T \cdot [&\alpha_p(\Lambda(p,q,\tau)) \\ &+ \max(0, b_{p,q}) \cdot \alpha_{p,q}(\Lambda(p,q,\tau))]\end{aligned} \tag{12}$$

with $\Lambda(p,q,\tau) = \epsilon(\|x(\tau - T)\|, \tau - T)$, $\epsilon(\cdot,\cdot)$ as defined in (6), and $\alpha_p(\cdot)$, $\alpha_{p,q}(\cdot)$, and $b_{p,q}$ that satisfy (8) and (9). Then (1) is stable under Σ_T^{s}.

Proof. We show that $\Sigma_T^{\mathrm{s}} \subseteq \Sigma^{\mathrm{s}}$ by finding a lower bound on the left-hand side of (7) based only on information available when the switch is triggered, i.e. $x(\tau - T)$, and partitioning the state space such that (7) holds at each switching instant, despite a switching delay T.

The Lypaunov function in mode $p \in \mathcal{P}$

$$\begin{aligned} V_p(\tau, x(\tau)) &= V_p(\tau - T, x(\tau - T)) \\ &\quad + \int_{\tau-T}^{\tau} \left(\tfrac{\partial V_p}{\partial t} + \tfrac{\partial V_p}{\partial x} f_p(t,x)\right) dt \\ &\geq V_p(\tau - T, x(\tau - T)) \\ &\quad - \int_{\tau-T}^{\tau} \alpha_p(\|x(t)\|) dt \end{aligned} \tag{13}$$

is bounded below by applying (8). Since $\|x(t)\|$ is bounded over $[\tau - T, \tau)$, applying (2), (6), produces a lower bound on $V_p(\tau, x(\tau))$ given $x(\tau - T)$.

$$\begin{aligned}V_p(\tau, x(\tau)) \geq V_p(&\tau - T, x(\tau - T)) \\ &- T \cdot \alpha_p(\epsilon(\|x(\tau - T)\|, \tau - T))\end{aligned} \tag{14}$$

Similarly, $V_q(\tau, x(\tau))$ is bounded above.

$$\begin{aligned} V_q(\tau, x(\tau)) &= V_q(\tau - T, x(\tau - T)) \\ &\quad + \int_{\tau-T}^{\tau} \left(\tfrac{\partial V_q}{\partial t} + \tfrac{\partial V_q}{\partial x} f_p(t,x)\right) dt \\ &\leq V_q(\tau - T, x(\tau - T)) \\ &\quad + \int_{\tau-T}^{\tau} b_{p,q}\alpha_{p,q}(\|x(t)\|) dt \end{aligned} \tag{15}$$

If $b_{p,q} = 1$, the integral term is positive, and the upper bound for $\|x(t)\|$ given by (2), (6) further bounds (13), and obtain a result similar to (14). However, if $b_{p,q} = -1$, a lower bound for $\|x(t)\|$ is required to further bound (13). In general, such a lower bound is unavailable, but can be conservatively approximated as 0.

$$\begin{aligned}V_q(\tau, x(\tau)) &\leq V_q(\tau - T, x(\tau - T)) \\ &+ \max(0, b_{p,q}) \cdot T \cdot \alpha_{p,q}(\epsilon(\|x(\tau - T)\|, \tau - T)\end{aligned} \tag{16}$$

Combining (14), (16) with (7),

$$V_p(\tau - T, x(\tau - T)) - V_q(\tau - T, x(\tau - T)) \geq \gamma^{\mathrm{s}}(p,q,\tau) \tag{17}$$

with γ^{s} given as in (12). Letting $\mathcal{S}^{\mathrm{s}}(p,q,\tau)$ be the subset of $\mathcal{D}$ where (17) holds, we obtain (11). Thus, for any piecewise constant switching signal $\sigma \in \Sigma_T^{\mathrm{s}}$, we have $\sigma \in \Sigma^{\mathrm{s}}$, thus $\Sigma_T^{\mathrm{s}} \subseteq \Sigma^{\mathrm{s}}$. □

The sets $\mathcal{S}^{\mathrm{s}}(p,q,\tau)$ thus partition the state space into regions where switching from mode p to mode q ensures

(5) is non-increasing, guaranteeing the stability of (1), despite a switching delay T. Computing the delay buffer γ^{s} given a delayed measurement $x(\tau-T)$ is trivial once functions α_p, $\alpha_{p,q}$ and constant $b_{p,q}$ have been determined, and can be easily be performed online, resulting in a computationally efficient manner of verifying whether a desired switch between two modes is allowable.

Corollary 3.1. Let Σ^{us} be the set of piecewise constant switching signals $\sigma : \mathbb{R}_+ \to \mathcal{P}$ such that (1) is uniformly stable. Let Σ_T^{us} be the set of piecewise constant switching signals such that, for each switching instant τ, $x(\tau - T) \in \mathcal{S}^{\mathrm{us}}(\sigma(\tau^-), \sigma(\tau), \tau)$, with

$$\mathcal{S}^{\mathrm{us}}(p,q,\tau) \triangleq \{x \in \mathcal{D} : \\ V_p(\tau - T, x) - V_q(\tau - T, x) \geq \gamma^{\mathrm{us}}(p,q,\tau)\} \quad (18)$$

the set of states for which switching from mode p to mode q is allowed for a time-varying delay buffer

$$\gamma^{\mathrm{us}}(p,q,\tau) = T \cdot [\alpha_p(\Lambda^{\mathrm{us}}(p,q,\tau)) \\ + \max(0, b_{p,q}) \cdot \alpha_{p,q}(\Lambda^{\mathrm{us}}(p,q,\tau))] \quad (19)$$

with $\Lambda^{\mathrm{us}}(p,q,\tau) = \bar{\alpha}_p(\|x(\tau-T)\|)$, $\bar{\alpha}_p(\cdot)$ that satisfies (3) and $\alpha_p(\cdot)$, $\alpha_{p,q}(\cdot)$, and $b_{p,q}$ that satisfy (8) and (9).

Proof. Similar to Theorem 3.2: When bounding equations (13) and (15), $\bar{\alpha}_p(\|x(\tau - T)\|)$ provides an upper bound for $\|x(t)\|$ over $[\tau - T, \tau)$ instead of (2), (6), thus $\Sigma_T^{\mathrm{us}} \subseteq \Sigma^{\mathrm{us}}$. □

Corollary 3.2. Let Σ^{uas} be the set of piecewise constant switching signals $\sigma : \mathbb{R}_+ \to \mathcal{P}$ such that (1) is UAS. Let Σ_T^{uas} be the set of piecewise constant switching signals such that, for each switching instant τ, $x(\tau - T) \in \mathcal{S}^{\mathrm{uas}}(\sigma(\tau^-), \sigma(\tau), \tau)$, with

$$\mathcal{S}^{\mathrm{uas}}(p,q,\tau) \triangleq \{x \in \mathcal{D} : \\ V_p(\tau - T, x) - V_q(\tau - T, x) \geq \gamma^{\mathrm{uas}}(p,q,\tau)\} \quad (20)$$

the set of states for which switching from mode p to mode q is allowed for a time-varying delay buffer

$$\gamma^{\mathrm{uas}}(p,q,\tau) = \int_{\tau-T}^{\tau} \alpha_p(\Lambda^{\mathrm{uas}}(p,q,\tau))dt \\ + \max(0, b_{p,q}) \cdot \int_{\tau-T}^{\tau} \alpha_{p,q}(\Lambda^{\mathrm{uas}}(p,q,\tau))dt \quad (21)$$

with $\Lambda^{\mathrm{uas}}(p,q,\tau) = \beta_p(\|x(\tau-T)\|, t-(\tau-T))$, bounding function $\beta_p(\cdot,\cdot)$ as defined in (4) for mode p, and with $\alpha_p(\cdot)$, $\alpha_{p,q}(\cdot)$, and $b_{p,q}$ that satisfy (8) and (9).

Proof. Similar to Theorem 3.2: When bounding equations (13) and (15), $\beta_p(\|x(\tau - T)\|, t - (\tau - T))$ provides an upper bound for $\|x(t)\|$ over $[\tau - T, \tau)$ instead of (2), (6), thus $\Sigma_T^{\mathrm{uas}} \subseteq \Sigma^{\mathrm{uas}}$. □

i	$V_i(x)$	$\alpha_i(y)$	$\alpha_{i,j}(y)$	$b_{i,j}$	$\beta_i(r,s)$
1	$x_1^2 + x_2^4$	$2y^2$	$2.5y^2$	1	$\left(\frac{4r^2}{r^2s+2}\right)^{\frac{1}{4}}$
2	$\frac{1}{2}(x_1^2 + x_2^2)$	$2y^2$	$3y^2$	1	$2\left(\frac{2r^2}{r^2s+2}\right)^{\frac{1}{2}}$

Table 1: Functions and constants necessary to apply Corollary 3.2 to Example 3.4.1.

3.3 Wait-time condition

Since the delay buffer γ^{uas} is time-varying, by waiting long enough before switching, the effect of the time delay on the state-based partitioning can be made arbitrarily small. Once this condition is satisfied, it is satisfied for all future times, and hence can be thought of as a *wait-time* condition.

Corollary 3.3. For $\sigma \in \Sigma_T^{\mathrm{uas}}$, as $t \to \infty$, the time-varying partition $\mathcal{S}^{\mathrm{uas}}(p,q,\tau) \to \bar{\mathcal{S}}(p,q,\tau)$ converges to the delay-free partition in (10) for all $p, q \in \mathcal{P}$, guaranteeing uniform asymptotic stability of (1).

Proof. We fix a "next mode" q and study the evolution of $\mathcal{S}^{\mathrm{uas}}(\sigma(t), q, t)$ under a switching signal $\sigma \in \Sigma_T^{\mathrm{uas}}$. Define the functional $\gamma^{\mathrm{uas}}(\cdot, q, \cdot) : \mathcal{P} \times \mathbb{R}_+ \to \mathbb{R}$, evolving under a switching signal $\sigma \in \Sigma_T^{\mathrm{uas}}$.

$$\gamma^{\mathrm{uas}}(\sigma(t), q, t) = \int_{t-T}^{t} \alpha_{\sigma(t)}(\Lambda^{\mathrm{uas}}(\sigma(t), q, t))dr \\ + \max(0, b_{\sigma(t),q}) \cdot \int_{t-T}^{t} \alpha_{\sigma(t),q}(\Lambda^{\mathrm{uas}}(\sigma(t), q, t))dr \quad (22)$$

with $\Lambda^{\mathrm{uas}}(\sigma(t), q, t) = \beta_{\sigma(t)}(\|x(t-T)\|, r - (t-T))$. For all $\sigma \in \Sigma_T^{\mathrm{uas}}$, (1) is uniform asymptotically stable, and by Lemma 2.1, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ satisfying (4). Hence $\|x(t)\| \to 0$ as $t \to \infty$, implying that the integral terms in (22) asymptotically approach 0 as well. Thus the delay buffer $\gamma^{\mathrm{uas}}(\sigma(t), q, t) \to 0$ as $t \to \infty$ for all $\sigma \in \Sigma_T^{\mathrm{uas}}$. Letting the final active mode of σ be p, the result follows. □

3.4 Examples

3.4.1 Autonomous nonlinear switched system

Consider a system (1) with $\mathcal{F} = \{f_1(x), f_2(x)\}$, $x = [x_1, x_2]^T \in \mathbb{R}^2$ and f_i

$$\begin{aligned} f_1(x) &= \begin{bmatrix} -x_1 + 2x_2^3 - 2x_2^4 \\ -x_1 - x_2 + x_1x_2 \end{bmatrix} \\ f_2(x) &= \begin{bmatrix} -x_2 - x_1^3 \\ x_1 - 2x_2^3 \end{bmatrix} \end{aligned} \quad (23)$$

restricted to $\mathcal{D} := \{x \in \mathbb{R}^2 \ : \ ||x||_2^2 \leq 1\}$, with a switching delay $T_D = .01$s.

Table 1 describes the Lyapunov functions, constants, class $\mathcal{K}$ functions and class $\mathcal{KL}$ functions needed to apply

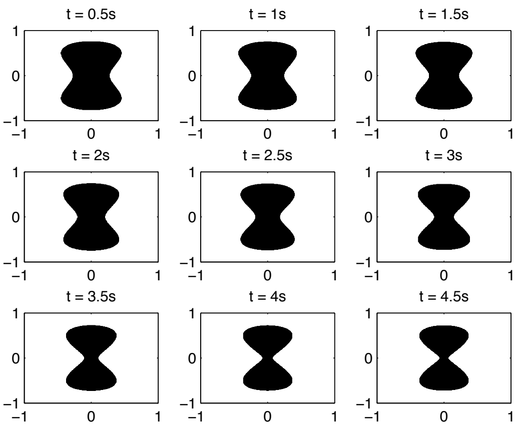

Figure 1: Snapshots of the partition $\mathcal{S}^{\mathrm{uas}}(1,2,t)$ (white) evolving over time under the switching signal $\sigma(t) \equiv 1$. Notice that the black region (states from which a switch from mode 1 to mode 2 is disallowed) shrinks over time, since $\mathcal{S}^{\mathrm{uas}}(1,2,t) \to \bar{\mathcal{S}}(1,2)$ as $t \to \infty$ (10).

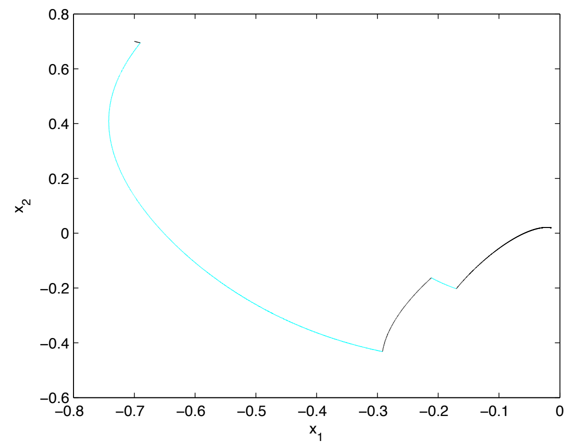

Figure 2: Example 3.4.1. Phase-plane trajectory generated by a two mode system (23) under switching signal $\sigma \in \Sigma^{\mathrm{uas}}_{T_D}$, with portions of the trajectory evolving according to $\dot{x} = f_1(x)$ plotted in black (dark), and those evolving according to $\dot{x} = f_2(x)$ plotted in cyan (light).

Corollary 3.2. The functions $\alpha_i(\cdot)$, $\alpha_{i,j}(\cdot)$ and constants $b_{i,j}$ are all obtained by exploiting the equivalence of norms over $\mathbb{R}^n$ and the fact that $\|x^r\| < \|x^s\|$ for all $r > s$ and $\|x\| < 1$. The functions $\beta_i(\cdot,\cdot)$ are solved as in Theorem 4.9, Lemma 4.4 and Appendix C.5 of [23]. The resulting second-order scalar ODE has an analytic solution.

Figure 1 shows snapshots of $\mathcal{S}^{\mathrm{uas}}(1,2,t)$ (white) evolving over time under the switching signal $\sigma(t) \equiv 1$. Initially, $\mathcal{S}^{\mathrm{uas}}(1,2,t)$ is not very large (recall that the domain is the unit circle), but as the system evolves, the buffer delay $\gamma^{\mathrm{uas}}(1,2,t)$ decreases, and its effect becomes less important. The set $\mathcal{S}^{\mathrm{uas}}(1,2,t) \to \bar{\mathcal{S}}(1,2)$ converges to the delay-free partition in (10). A sample phase-plane trajectory is presented in Figure 2, with portions of the trajectory evolving according to $\dot{x} = f_1(x)$ plotted in black (dark), and those evolving according to $\dot{x} = f_2(x)$ plotted in cyan (light). A switch occurs as soon as the trajectory enters a region of the state-space in which switching is allowed. The system spends a relatively long time in mode 2 initially because $\mathcal{S}^{\mathrm{uas}}(1,2,t)$ is relatively small (Figure 1). In the last snapshot of Figure 1, $\mathcal{S}^{\mathrm{uas}}(1,2,t)$ occupies approximately half of the unit circle. Hence, as the effect of the time delay lessens as γ^{uas} decreases, switching between modes is enabled and occurs more frequently.

3.4.2 Time-varying linear switched system

Consider a system (1) with $\mathcal{F} = \{f_1(t,x), f_2(t,x)\}$,

$$f_i(x) = \begin{bmatrix} -x_1 - g_i(t)x_2 \\ x_1 - x_2 \end{bmatrix}, \tag{24}$$

with $x = [x_1, x_2]^T \in \mathbb{R}^2$, $g_i : \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{aligned} g_1(t) &= \tfrac{3}{1+t} \\ g_2(t) &= \tfrac{e^t}{(1+e^t)} \end{aligned} \tag{25}$$

i	$\underline{k}_i$	$\overline{k}_i$	$\alpha_i(y)$	$\alpha_{i,j}(y)$	$b_{i,j}$	$\beta_i(r,s)$
1	3	3	$7y^2$	$2y^2$	1	$2e^{-\frac{3}{8}s}r$
2	1	0	$3y^2$	$5y^2$	-1	$\sqrt{2}e^{-\frac{3}{4}s}r$

Table 2: Functions and constants to apply Corollary 3.2 to Example 3.4.2 with $V_i(x) = x_1^2 + (1 + g_i(t))x_2^2$ in each mode i.

and switching delay $T_D = .01$s.

Since each continuously differentiable $g_i(t)$ satisfies

$$\begin{aligned} 0 \le g_i(t) \le \underline{k}_i \\ -\bar{k}_i \le \dot{g}(t) \le g(t) \end{aligned} \tag{26}$$

for some real $\underline{k}_i, \overline{k}_i \ge 0$, it is possible to construct the necessary Lyapunov functions, class $\mathcal{K}$ functions, and class $\mathcal{KL}$ functions. These functions and constants were solved (Table 2) in a similar manner as those in the previous example, except that equivalence of norms was not necessary as all terms were second-order. Figure 2 shows a sample phase-plane trajectory with portions of the trajectory evolving according to $\dot{x} = f_1(x)$ plotted in black (dark), and those evolving according to $\dot{x} = f_2(x)$ plotted in cyan (light). A switch occurs as soon as the trajectory enters a region of the state space in which switching is allowed. We see that in this example, as opposed to Example 3.4.1, switching occurs much less frequently. This highlights the effect of both the system dynamics and Lyapunov function structures on delay buffer.

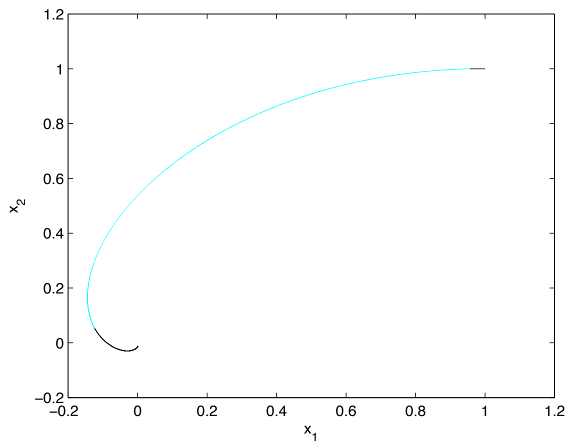

Figure 3: Example 3.4.2. Phase-plane trajectory of (24) under a switching signal $\sigma \in \Sigma_{T_D}^{\text{uas}}$. Portions of the trajectory evolving according to $\dot{x} = f_1(x)$ plotted in black (dark), and those evolving according to $\dot{x} = f_2(x)$ plotted in cyan (light).

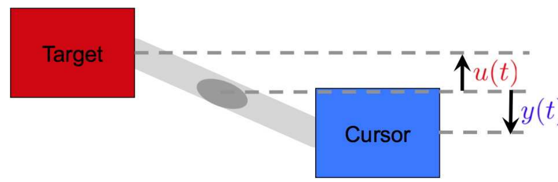

Figure 4: Experimental setup. The target trajectory is $u(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$. Users are instructed to keep the cursor $y(t)$ level with the target. The error $u(t) - y(t)$ is scaled by 0.3 in 'Better' mode, by 2.0 in 'Worse' mode, and unchanged in 'Normal' mode.

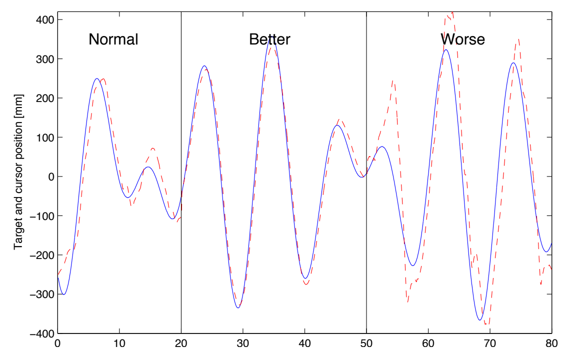

Figure 5: Switched manual pursuit tracking task for a normal subject. The desired task is blue (solid), the actual tracking performance is red (dashed). Task modes Better, Normal, Worse, correspond to scaling of the subject's displayed error by 0.3, 1.0, and 2.0, respectively, such that their cursor position appears to be better than expected, as expected, or worse than expected.

4 Delayed switching in Parkinson's Disease

4.1 Experiment Description

Fourteen PD subjects (on and off L-dopa medication) with clinically diagnosed, mild to moderate PD and ten healthy, age-matched subjects without active neurological disorders conducted a series of experiments at the Pacific Parkinson's Research Centre at the University of British Columbia at Vancouver, Canada. The study was approved by the Ethics Board at UBC, and all subjects first provided informed consent (a full description of the experimental setup can be found in [12]). Subjects were asked to perform a tracking task by using a joystick in response to visual stimuli displayed on a computer screen, as shown in Figure 4. A horizontal "glass rod" connecting two boxes (each 60mm × 45mm) was shown on the display, where the box on the left (Target) oscillated in the vertical direction at a linear combination of two constant frequencies (ω_1 and ω_2), thus giving it a smooth but fairly complex appearing motion. Subjects were instructed to move the box on the right (Cursor) by using the joystick so that the glass rod remained horizontal at all times. PD subjects performed the task once after an overnight withdrawal (minimum of 12 hours since their last dose of L-dopa, minimum of 18 hours since the last dose of dopamine agonists) of their anti-Parkinson drugs and again one hour after admission of L-dopa.

Part 1: Single tracking task. Subjects were first trained on three separate tracking tasks. Over a single 90-second interval, a sequence of three separate tracking tasks was performed, with a short delay (5-10 seconds) between each task to mark its end. In each task, the visual feedback of the actual tracking errors was either amplified, attenuated or unaltered (but did not switch between the three options). In the 'Normal' task, the vertical distance between the target and cursor displayed on the monitor reflected the true error generated by the subject. In the 'Better' task, this distance was artificially reduced on the computer screen to 30% of the true error. In practice, the attenuation essentially made the tracking error better than expected. Finally, in the 'Worse' task, the the distance between the target and the cursor was artificially doubled, making the tracking error worse than expected. Subjects performed eight sets of the 90-second intervals (e.g., a total of 8×3 tasks).

Part 2: Switched tracking task. The same sequences of three different tasks was again performed over a single 90-second interval, but *without* a delay between tasks, as shown in Figure 5. Subjects were not provided with any additional signal that might indicate that the task had changed, although two unenunciated mode switches occurred in every 90-second interval. With a 10-second pause at the start of each interval, the first task lasted 20 seconds, and the remaining two tasks each lasted 30 seconds. This pattern was repeated eight times, re-

	Normal	Parkinson's off med	Parkinson's on med
Better	$(0.54, 1.09) \pm (0.29, 0.15)$	$(0.53, 1.51) \pm (0.29, 0.34)$	$(0.46, 1.81) \pm (0.26, 0.40)$
Normal	$(0.67, 0.91) \pm (0.35, 0.24)$	$(0.58, 1.26) \pm (0.22, 0.43)$	$(0.55, 1.56) \pm (0.12, 0.49)$
Worse	$(0.53, 0.84) \pm (0.20, 0.17)$	$(0.43, 1.26) \pm (0.12, 0.28)$	$(0.41, 1.59) \pm (0.11, 0.33)$

Table 3: Mean and standard deviations of (ζ, ω_n) for combinations of subject groups and tasks.

Mode	A_p	B_p	D_p
Better	$\begin{bmatrix} 1.998 & 1 \\ -0.999 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.002 \\ -0.003 \end{bmatrix}$	0.006
Normal	$\begin{bmatrix} 1.994 & 1 \\ -0.994 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.005 \\ -0.006 \end{bmatrix}$	0.004
Worse	$\begin{bmatrix} 1.917 & 1 \\ -0.925 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.039 \\ -0.031 \end{bmatrix}$	0.035

Table 4: System matrices for a typical normal subject.

sulting in a total of 8×2 mode switches. A total of 4 sequences were each tested twice: 'Normal-Better-Worse', 'Worse-Normal-Better', 'Better-Worse-Normal', and 'Better-Normal-Worse'.

In prior work [12, 24], black-box system identification algorithms were used to determine second-order LTI dynamical system models of each subjects' tracking performance from Part 1 (e.g., for each of the three tasks), with input as the Target position and output as the Cursor position. Using standard methods [25] implemented in Matlab's system identification toolbox [26], for each subject $\times$ task, four sets of data were used to create a second-order model, and the remaining four sets of data were used to validate the model. The resulting models for Parkinson's subjects on and off medication are qualitatively similar, although as described in [12], damping ratio and natural frequency vary depending on group and task, respectively. As compared to control subjects, Parkinson's subjects off medication are more damped, and Parkinson's subjects on medication are less damped. For all groups, the natural frequency increased with task difficulty, with highest values for the 'Worse' task. Mean and standard deviations for damping ratio and natural frequency are shown in Table 3; full description of the analysis and results can be found in [12].

We then model each subject performing the switched tracking task (Part 2) as a 3-mode switched linear system

$$\begin{aligned} \dot{x} &= A_p x + B_p u \\ y &= C_p x + D_p u \end{aligned} \tag{27}$$

with modes $\mathcal{P} = \{\text{Better, Normal, Worse}\}$ and state matrices (A_p, B_p, C_p, D_p) identified numerically for each subject $\times$ task with $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, $y \in \mathbb{R}$. A subject that adapts to the sudden changes in tracking dynamics will in essence switch modes. From the subjects' perspective, switches could occur at any time, so the switching scheme is assumed to be arbitrary for the purpose of modeling.

System matrices for a typical normal subject are shown in Table 4, with $C_p = [1, 0]$ in all modes $p \in \mathcal{P}$.

Subject type	Subjects with no CQLF	Total subjects
Normal	5	10
Parkinson's off med	9	14
Parkinson's on med	9	14

Table 5: Subjects without a common quadratic Lyapunov function (CQLF). The higher rate of subjects with Parkinson's disease whose dynamics are not stable under arbitrary switching coincides with clinical observations that Parkinson's disease reduces robustness to uncertainty.

4.2 Arbitrary Switching

To determine whether the identified models for each subject were stable under arbitrary switching, we apply converse theorem for arbitrary switching [9]. For each subject, solving the LMI

$$\sum_{p \in \mathcal{P}} (A_p^T R_p + R_p A_p) > 0 \tag{28}$$

for positive definite matrices R_p disproves the existence of a common quadratic Lyapunov function. As expected (Table 5), proportionally fewer subjects with Parkinson's disease had switched systems that were stable under arbitrary switching as compared to normal subjects. This is consistent with clinical expectations that normal subjects are more robust to uncertainty than subjects with Parkinson's disease. For the subjects with common quadratic Lyapunov functions, delays in switching will never destabilize the system.

4.3 Restrictions on Switching

Since the dynamics of all subjects are stable in each mode $p \in \{\text{Better, Normal, Worse}\}$, Lyapunov functions exist for all subjects $\times$ tasks. A stable switching scheme $\sigma \in \Sigma^{\mathrm{s}}$ was identified for each subject for whom no common quadratic Lyapunov function could be found (Table 5), based on normalized quadratic Lyapunov functions calculated for each mode with $Q_p = -I$.

$$V_p(x(t)) = x(t)^T P_p x(t) / \|P_p\|, \; P_p = P_p^T > 0, P_p \in \mathbb{R}^{n \times n} \tag{29}$$

Normalized Lyapunov functions (29) were calculated for all 5 Normal + 9 PD on medication + 9 PD off medication subjects $\times$ tasks. Each subject was confirmed to have non-empty intersections of Lyapunov functions at a

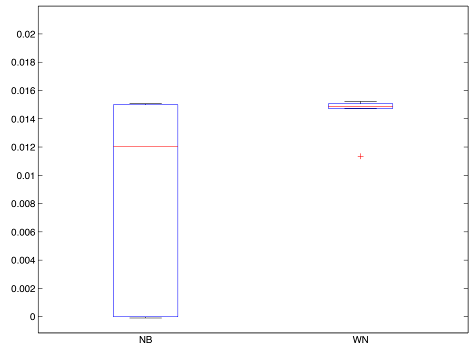

Figure 6: The differences in time delays $\Delta_{\mathrm{BN}} = T_{\max}(\text{Normal, Better}) - T_{\max}(\text{Better, Worse})$ and $\Delta_{\mathrm{NW}} = T_{\max}(\text{Worse, Normal}) - T_{\max}(\text{Better, Worse})$ for subjects with Parkinson's disease off medication, are close to statistical significance, with $p = 0.0869$.

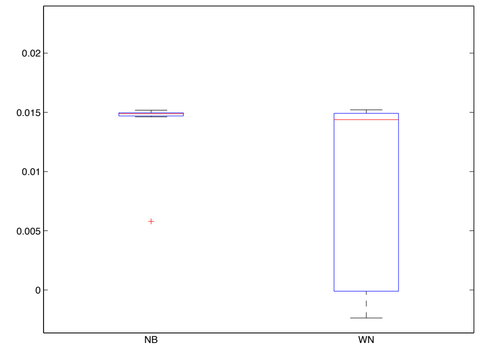

Figure 7: The differences in time delays $\Delta_{\mathrm{BN}} = T_{\max}(\text{Normal, Better}) - T_{\max}(\text{Better, Worse})$ and $\Delta_{\mathrm{NW}} = T_{\max}(\text{Worse, Normal}) - T_{\max}(\text{Better, Worse})$ for subjects with Parkinson's disease on medication, are close to statistical significance, with $p = 0.0878$.

constant energy level for each of the 3 mode pairs (Better, Worse), (Normal, Better), (Worse, Normal). Hence the existence of a stable switching scheme for each subject without a common quadratic Lyapunov function is assured.

For a switched system (1) with LTI dynamics and quadratic Lyapunov functions (29) for each mode $p \in \mathcal{P}$, and $Q_p(q) \triangleq -(A_q^T P_p + P_p A_q)$ to track the evolution of the Lyapunov function in mode q while $\sigma(t) = p$, the partition $\mathcal{S}^{\mathrm{s}}$ (10) reduces to

$$\begin{aligned} \mathcal{S}(p,q) &\triangleq \left\{ x \in \mathbb{R}^n : \frac{x^T(P_p - P_q)x}{\|x\|^2} > \gamma(p,q) \right\} \\ \gamma(p,q) &= \Theta(p) \cdot \Lambda(p,q) \end{aligned} \tag{30}$$

with $\Theta(p) = \frac{c_p^2}{2\lambda_p}(1 - e^{2\lambda_p T})$, $\Lambda(p,q) = \lambda_{\max}(Q_p(p)) - \min(0, \lambda_{\min}(Q_q(p)))$, and $c_p, \lambda_p > 0$ constants for mode p such that

$$\|x(t)\| \leq c_p e^{-\lambda_p(t-(\tau-T))} \|x(\tau - T)\| \tag{31}$$

4.4 Robustness to delayed switching

For each subject, the amount of time required to violate (11) can be determined for a given mode pair.

$$x^T(P_p - Q_p - \gamma(p,q) \cdot I)x > 0 \tag{32}$$

Exploiting symmetry and realness for 2D matrices, the positive solution to the quadratic equation

$$0 = \gamma_{\max}(p,q)^2 - \mathrm{Trace}(P_p - Q_p) \cdot \gamma_{\max}(p,q) + |P_p - Q_p| \tag{33}$$

is the largest delay buffer $\gamma_{\max}(p,q)$, hence an upper bound on the maximal delay that will not destabilize the system is

$$T_{\max}(p,q) = -\frac{1}{2\lambda_p} \ln\left(1 - \frac{2\lambda_p}{c_p^2} \cdot \frac{\gamma_{\max}(p,q)}{\Lambda(p,q)}\right) \tag{34}$$

Since $\mathcal{S}(p,q)$ is determined conservatively (for all subjects), and the bounds on the evolution of the Lyapunov functions are not tight, we obtain an upper bound on the maximum delay, as opposed to the actual maximum delay. Let the switching scheme σ with delay T be denoted by $\sigma_T \in \Sigma_T^s$. Hence delays $T > T_{\max}(p,q)$ as in (34) could destabilize the system under switching scheme σ_T.

Within subject groups, delay $T_{\max}$ is statistically significant across tasks, with p-values of $p = 2.420 \times 10^{-3}$ for normal subjects, $p = 7.704 \times 10^{-5}$ for Parkinson's subjects off medication, and $p = 3.156 \times 10^{-5}$ for Parkinson's subjects on medication. In all groups, the mean value of $T_{\max}(\text{Better, Worse})$ is smaller than the mean values of $T_{\max}(\text{Normal, Better})$ and $T_{\max}(\text{Worse, Normal})$. This is consistent with the hypothesis that switching between tasks Better and Worse is the most difficult amongst task pairs. In addition, switching between Better and Worse is less "robust" to delay – that is, the system is most likely to become unstable due to a delay in a transition between modes whose dynamics are extremely different. In this case, the attenuation factor switches from its lowest value (0.3) to its highest value (2.0) and a long delay in adapting to the sudden change will significantly impair tracking performance.

Using a paired t-test, we examine the differences in delay across task pairs $\Delta_{\mathrm{BN}} = T_{\max}(\text{Worse, Normal}) - T_{\max}(\text{Better, Worse})$ and $\Delta_{\mathrm{NW}} = T_{\max}(\text{Normal, Better}) - T_{\max}(\text{Better, Worse})$ for each subject. While for normal subjects the relative delays are *not* significant ($p = 0.6280$), for subjects with Parkinson's disease, both on and off medication, the relative delays are close to statistical significance ($p = 0.0878$ and $p = 0.0869$, respectively). Further, in Parkinson's subjects on medication the mean values follow the trend $\overline{\Delta}_{\mathrm{WN}} = 0.01376 < \overline{\Delta}_{\mathrm{NB}} = 0.00890$ where in Parkinson's subjects off medication, the reverse trend holds: $\overline{\Delta}_{\mathrm{WN}} = 0.01443 > \overline{\Delta}_{\mathrm{NB}} = 0.00814$, as shown in Figures 6 and 7. This reversal indicates relatively increased difficulty for Parkinson subjects in managing the transitions involving the Worse mode after medication, and the transitions involving the Better mode before medication. While counterintuitive at first glance, this trend may be related to impaired response to error that is known to worsen after medication. For example, we found in earlier work on Part 1 of this same experiment that Parkinson's subjects on medication were more underdamped in their responses (lower damping ratios on average) than Parkinson's subjects off medication. The difficulty with the Worse mode after medication may be the same phenomenon, in which medication may apparently worsen some motor responses. This may also reflect altered reward processing in PD subjects that may be reversed with medication [27].

5 Conclusion and Future Work

We present conditions for stability for nonlinear switched systems despite a delay in switching instant. Exploiting sufficient conditions for switched stability, in addition to mild assumptions about the boundedness of Lyapunov functions in each mode, we determine those set of states that meet sufficient criteria for stability despite a constant time delay in switching instant. The criteria essentially creates a time-varying partition of the state-space. The effect of the time delay decreases the system evolves.

We apply these techniques to experimentally obtained switched LTI models of a manual pursuit tracking task performed by 10 normal subjects and 14 subjects with Parkinson's disease, on and off medication. We identify the maximum time delay that a given switching sequence could withstand without violating the sufficient conditions for stability of a switched system, and compare the result across groups and mode pairs. Across all subject groups, the Better $\rightarrow$ Worse task change is the least tolerant to switching delays, as expected. In addition, amongst subjects with Parkinson's disease, the relative difficulty of certain task changes reverses after medication. This may be related to clinically observed phenomena in which medication may make subjects hypersensitive to motor error and hence degrade some aspects of their motor performance.

Acknowledgements

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