Towards Switched Control Under Input and State Constraints

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***Abstract*— Engineered systems which are safety-critical, high-risk, or expensive must often meet multiple objectives. In addition to guarantees of safe and reliable operation, other goals, such as stability, tracking, or optimality are also important. Indeed, a system which is guaranteed to be "safe" may not be implementable in practice unless it is also guaranteed to meet other performance objectives. In previous work, the problem of control design to meet multiple objectives was addressed through *reachability analysis*. For a feedback linearizable system subject to bounded control input and nonlinear state constraints, we synthesized a single controller guaranteed to 1) stabilize the nonlinear system despite input saturation, and 2) prevent violation of the state constraints.**

However, a single controller may be restrictive. Switching between multiple controllers may increase the region of operation for the system as well as improve its overall performance. We focus here on recent developments to synthesize a switched multi-objective controller through reachability analysis. The result provides a mathematical guarantee that for all states within the computed reachable set, there exists a switched control law that will simultaneously satisfy two separate goals: envelope protection (no violation of state constraints), and stabilization despite saturation. This is accomplished through a reachability calculation, in which the state is extended to incorporate input parameters for the switched system.

Keywords: **hybrid systems, constrained control, switched systems, saturation, reachability, optimal switching, minimal time-to-reach, feedback linearization.**

I. INTRODUCTION

New methods and tools in reachability analysis can provide an alternative framework for control design in safety-critical systems such as civil jet aircraft. These methods provide a mathematical guarantee of the modeled system's behavior, in the presence of state and input constraints. Physical constraints arising from aerodynamic envelope protection, such as the speed at which an aircraft can stall, can be incorporated as constraints on the continuous state-space. Actuator saturation can be incorporated as an input constraint resulting in bounded control authority. When other goals, such as trajectory tracking, are incorporated into this analysis, multiple objectives can be simultaneously served through a single computational synthesis. This paper explores the synthesis of stable, switched control laws that will not saturate and will not violate state constraints, in order to increase the controllable range of the state-space.

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We show how to synthesize, in a single computation, switched controllers which simultaneously satisfy two separate objectives: 1) envelope protection, and 2) stabilization under input saturation. While in previous work [1] single-mode controllers were computed, in general, choosing a single controller may be restrictive. Switching between multiple controllers may significantly increase the region of operation for the system as well as improve its overall performance. In this paper, we choose a set of stabilizing controllers, each parameterized by a continuous variable, and compute how to optimally switch between them to reach the origin in minimal time. As in [1], the reachability computation is used as a controller synthesis to aid in the design of multi-objective controllers. The main contribution of this paper is in extending these techniques to now synthesize a hybrid controller – in addition to the continuous component, computed as in the previous work, a discrete switching scheme is also simultaneously synthesized.

To address the problem of envelope protection, we define safety as the ability to remain within a set of constraints in the continuous state-space, despite bounded control authority. We can compute, through standard reachability analysis and controller synthesis, the subset of those states in which we can guarantee the state of the system can always remain: this is the *invariant set*, which determines the "safe" region of operation [2]. States outside of this set comprise the *reachable set*, those states which can "reach" constraint violation. This technique, computationally based on a Hamilton-Jacobi partial differential equation, also synthesizes a set-valued control law which enforces safety by preventing the state of the system from entering the reachable set [3], [4]. An alternative approach, using viability theory [5] and numerical algorithms [6] has been developed to compute viability kernels and capture basins for continuous and hybrid systems [7], [8]. These are computationally based on a minimum-time-to-reach formulation [9].

To address stabilization under saturation, we parameterize feedback linearizing control laws subject to bounded control input, such that the parameters reflect system performance goals (e.g. damping, overshoot). We formulate constraints that input saturation and stability place on the input parameters. Feedback linearization is a popular technique for differentially flat systems [10], [11], but can generate inputs with high-magnitude. Synthesizing non-saturating feedback linearizing control laws is a non-trivial problem [12], [13], [14] for stabilization [15] as well as for tracking [16]. Trajectory generation for differentially flat systems often involves saturation and rate constraints [17], [18]. Other common techniques to incorporate state and input constraints are model predictive control [19], [20], [21], and control Lyapunov functions [22], [23], however finding such functions is often difficult and done heuristically. For linear systems, quadratic Lyapunov functions can be synthesized [24], [25]. A variety of techniques have been investigated to control linear systems with constraints [26], [27], [28], [29].

In this paper, after formulating the switched controller synthesis problem, we provide a brief description of the reachability computation and the resultant invariant set. This method is demonstrated on the double integrator. Lastly directions for future work are discussed.

II. Problem Formulation

Consider the input-output full-state feedback linearizable system

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u, x \in \mathcal{X} \subseteq \mathbb{R}^n, u \in \mathbb{R} \\ y & = & h(x), y \in \mathbb{R} \end{array} \quad (1)$$

with bounded input $u \in \mathcal{U} = [u_{\min}, u_{\max}]$, and constraint set $\mathcal{C} \subset \mathbb{R}^n$ which encodes the set of states which satisfy the constraints on the system (e.g., speeds above the aircraft's stall speed). We express the state constraints through the inequality

$$\mathcal{C} = \{x \mid c(x) \geq 0\}. \quad (2)$$

Consider a switched feedback linearizing control law $u(x, \beta_q) : \mathbb{R} \times \mathbb{R}_+ \times Q \rightarrow \mathbb{R}$ to stabilize (1) around the equilibrium $x^* = 0$. The control law in mode $q \in Q$ is

$$u(x, \beta_q) = \frac{1}{L_g h} \left(-L_f^{n-1} h - \sum_{i=1}^{n} \beta_q[i] x^{(i-1)} \right) \quad (3)$$

with Lie derivatives $L_f h = \frac{\partial h}{\partial x} f(x)$, $L_f^2 h = \frac{\partial}{\partial x}(L_f h)$, $\cdots$, $L_f^{n-1} h = \frac{\partial}{\partial x}(L_f^{n-2} h)$, and $x^{(i)}$ the ith time derivative of x. Constant coefficients $\beta_q = \left[\begin{array}{cccc} \beta_q[1] & \beta_q[2] & \cdots & \beta_q[n] \end{array} \right]^T$ are chosen such that the polynomial in s

$$s^n + \sum_{i=1}^{n} \beta_q[i] s^{i-1} \quad (4)$$

is Hurwitz, assuring stability within each mode. Additionally, to assure stability despite arbitrary switching, the resulting switched closed-loop system

$$\dot{\xi} = A_q \xi \quad (5)$$

must have a common Lyapunov function.

To prevent saturation, the control $u(x, \beta_q)$ must remain within its allowable bounds $\mathcal{U}$ for all $x \in \mathcal{X}$, $q \in Q$.

$$u_{\min} \leq u(x, \beta_q) \leq u_{\max} \quad (6)$$

Define the reachable set as the set of states in $\mathcal{C}$ for which all values of a measurable function $u(\cdot)$ in $\mathcal{U}$ drive the system state out of the constraint set $\mathcal{C}$. We presume that the equilibrium $x^* = 0 \in \mathcal{C}$ is contained in the constraint set. We compute the reachable set and its complement, known as the invariant set, through Hamilton-Jacobi techniques.

We wish to satisfy two goals with a single controller: 1) envelope protection, and 2) stabilization under saturation. Further, we wish to determine the largest set of states from which this controller is guaranteed to fulfill these goals.

Statement of Problem 1: Given the dynamical system (1), with state constraints (2), and with a switched feedback linearizing control law $u(x, \beta_q)$ (3) parameterized by a constant vector $\beta \in \mathbb{R}^{nm}$, determine 1) the invariant set $\mathcal{W}$, which is the largest set of states x for a switched non-saturating controller $u(x, \beta_q)$ that will reach the origin without violating the state constraints $x \in \mathcal{C}$, 2) β_q such that the feedback linearizing control law in each mode $q \in Q$ is both non-saturating (6) and stable (4), and 3) the optimal state-based switching scheme required to minimize the time to reach a small-radius ball $\mathcal{B}_n$ around the equilibrium point.

III. Method

Assume that there are m controllers. We first append the parameter vector $\beta = [\beta_1, \cdots, \beta_m] \in \mathbb{R}^{nm}$ to the state such that $\tilde{x} = [x, \beta] \in \mathbb{R}^{n(m+1)}$. The extended dynamics

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x) \left(-L_f^{n-1} h - \frac{1}{L_g h} \sum_{i=1}^{n} \beta_q[i] x^{(i-1)} \right) \\ \dot{\beta} & = & 0 \end{array} \quad (7)$$

ensure that β remains constant in the reachability computation.

Note that special mode structures can allow simplification of the switched stability constraints. For example, for modes with first- and second-order continuous dynamics, the requirements for stability (4) within each mode simplify to

$$\beta_q[i] > 0 \quad (8)$$

while for higher-order systems, (4) can be represented as a set of inequalities in $\beta_q[i]$. For switched systems with exactly two modes and stable dynamics within each mode, sufficient conditions for exponential stability under arbitrary switching can be reduced to checking whether the product of the two matrices contain any negative real eigenvalues.

$$\mathrm{eig}(A_1 A_2) \notin \mathbb{R}_- \quad (9)$$

This can be determined prior to the calculation [30]. Depending on the switched system structure, other techniques to find a common Lyapunov function can also be used [31].

Through reachability analysis and controller synthesis we can determine the *backwards reachable set* $\overline{\mathcal{W}(t)}$ and its complement, the *controlled invariant set* $\mathcal{W}(t)$. Given a dynamically evolving system (7) and a constraint set $\tilde{\mathcal{C}}$, we define the backwards reachable set as the set of all states which will exit the constraint set $\tilde{\mathcal{C}}$ in the time $[0, t]$. The controlled invariant set is simply the complement of this result. To calculate the backwards reachable set, define a continuous function $J_0 : \mathcal{X} \times \mathbb{R}^{mn} \rightarrow \mathbb{R}$ such that

$$\tilde{\mathcal{C}} = \{x \in \mathcal{X}, \beta \in \mathbb{R}^{mn} \mid J_0(x, \beta) \geq 0\}. \quad (10)$$

The backwards reachable set $\overline{\mathcal{W}(t)}$ can be found by solving the terminal value Hamilton-Jacobi (HJ) partial differential equation (PDE) [32], [2], [3]

$$\frac{\partial J(\tilde{x},t)}{\partial t} + \min\left[0, H\left(\tilde{x}, \frac{\partial J(\tilde{x},t)}{\partial \tilde{x}}\right)\right] = 0 \qquad \text{for } t < 0;$$
$$J(\tilde{x},0) = J_0(\tilde{x}) \text{ for } t = 0; \tag{11}$$

As shown in [3], the implicit representation of the backwards reachable set is $\overline{\mathcal{W}(t)} = \{x \in \mathcal{X}, \beta \in \mathbb{R}^{mn} | J(\tilde{x}, -t) \leq 0\}$. If (11) converges as $t \to -\infty$, then $J(\tilde{x}, -t) \to J(\tilde{x})$ and the reachable set converges to a fixed point $\overline{\mathcal{W}(t)} \to \overline{\mathcal{W}}$.

We incorporate the state bounds, non-saturation constraints, and stability constraints into the initial cost function

$$J_0(x,\beta) = \max_{q\in Q}\left\{\min\left\{J_0^{\text{state}}(x,\beta_q), J_0^{\text{sat}-\max}(x,\beta_q),\right.\right.$$
$$\left.\left. J_0^{\text{sat}-\min}(x,\beta_q), J_0^{\text{stability}}(x,\beta_q)\right\}\right\} \tag{12}$$

for which we define the functions

$$\begin{array}{rcl} J_0^{\text{state}}(x,\beta_q) & = & c(x) \\ J_0^{\text{sat}-\max}(x,\beta_q) & = & u_{\max} - u(x,\beta_q) \\ J_0^{\text{sat}-\min}(x,\beta_q) & = & u(x,\beta_q) - u_{\min} \\ J_0^{\text{stability}}(x,\beta_q) & = & \beta_q[i] \end{array} \tag{13}$$

such that they are positive in those regions where the constraints are satisfied. We then define the Hamiltonian

$$H\left(\tilde{x}, \tfrac{\partial J}{\partial \tilde{x}}\right) = \max_{q\in Q_V(x)} \left(\tfrac{\partial J}{\partial x}\right)^T \left(f(x) + g(x)\left(-L_f^{n-1}h\right.\right.$$
$$\left.\left. - \tfrac{1}{L_g h}\sum_{i=1}^{n} \beta_q[i] x^{(i-1)}\right)\right) + \left(\tfrac{\partial J}{\partial \beta}\right)^T \cdot 0, \tag{14}$$

maximized over the set of "valid" modes at any given state $\tilde{x}$

$$Q_V(\tilde{x}) = \left\{q \in Q \mid [J_0^{\text{state}}(x,\beta_q), J_0^{\text{sat}-\max}(x,\beta_q),\right.$$
$$\left. J_0^{\text{sat}-\min}(x,\beta_q), J_0^{\text{stability}}(x,\beta_q)] > 0\right\}. \tag{15}$$

The control law used in (14) will stabilize (1), ensure envelope protection, and will not saturate.

The result of the reachability computation is the largest set of states x for a given input parameter β for which trajectories that begin in this set and are controlled through (3) will reach the origin in minimal time without violating any state constraints (2) or saturating the input (6).

The advantage of this framework is that the computed result inherently meets the required constraints for both switched stability and non-saturation, while maintaining invariance within the constraint set. As there are generally no analytic ways to synthesize such controllers, this computation provides an alternative to simply picking various β and various switching schemes and checking whether they fulfill the conditions for stability and non-saturation.

At first glance, solving (11), (14) involves a reachability calculation in $n(m+1)$ dimensions – no small feat due to the complexity of the calculation, $\mathcal{O}(d^{n(m+1)})$ with d grid points in $n(m+1)$ dimensions. However, by exploiting structure in the pole placement, we can reduce the computation to $\mathcal{O}(d^{n+1})$, with d grid points in $n+1$ dimensions. Consider a simple double integrator with two modes, $q \in \{1, 2\}$, $\beta \in \mathbb{R}^4_+$. If we assume both modes are parameterized by $\eta \in \mathbb{R}$, with $\beta_1 = [2\eta, \eta^2]$, $\beta_2 = [2\eta, 2\eta^2]$, the computation can proceed in $[x, \eta] \in \mathbb{R}^3$ and therefore is reduced from 6 to 3 dimensions.

IV. Example

To demonstrate this method, consider the system $\ddot{x} = u$, with state $x = [x_1, x_2] \in \mathcal{C} = \mathcal{X} = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$, input $u \in \mathcal{U} = [u_{\min}, u_{\max}]$, and output $h = x_2$. We design a switched feedback linearizing control law, $u(x, \beta_q) = -\beta_q^T x$, with $\beta_q \in \mathbb{R}^2_+$, such that the resultant switched closed-loop system is stable. The two modes are chosen such that the closed-loop eigenvalues in mode $q = 1$ occur co-located on the negative real line, and in mode $q = 2$ occur as a complex conjugate pair with damping ratio $\zeta = 1/\sqrt{2}$. By parameterizing β_1 and β_2 with the same value η, the dimension of the extended state $\tilde{x} = [x, \eta] \in \mathbb{R}^3$ is significantly reduced.

$$\dot{x} = A_q x, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -2\eta & -\eta^2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2\eta & -2\eta^2 \end{bmatrix} \tag{16}$$

With these two matrices, (9) holds for $\eta > 0$, assuring stability under arbitrary switching.

The saturation constraints (6), state constraints, and stability constraints are formulated in each mode as

$$\begin{array}{l} J_0^{\text{sat}}(x,\beta_q) = \min\{u_{\max} - \beta_q^T x, \beta_q^T x - u_{\min}\} \\ J_0^{\text{state}}(x,\beta_q) = \min\{\overline{x}_1 - x_1, x_1 - \underline{x}_1, \overline{x}_2 - x_2, x_2 - \underline{x}_2\} \\ J_0^{\text{stability}}(x,\beta_q) = \eta. \end{array} \tag{17}$$

For the reachability computation, we combine the above three functions into one initial cost function

$$J_0(x,\beta_q) = \max_{q\in Q}\left\{\min\{J_0^{\text{sat}}(x,\beta_q), J_0^{\text{state}}(x), J_0^{\text{stability}}(\beta_q)\}\right\} \tag{18}$$

to maximize the area of the state-space in which at least one non-saturating, stabilizing controller is feasible.

Figure 1 represents the invariant set in $[x_1, x_2]$ for four different values of the control parameter η. Under any switching scheme, the closed-loop system will be stable. Notice that each value of η results in a different switching scheme, as indicated by the color of each grid cell. The dark (red, blue) colored regions are the set of initial conditions for which the state will be driven to the equilibrium in minimal time without saturating the input or violating the state constraints, presuming the switching scheme indicated is implemented exactly as shown.

V. Conclusion and Future Work

We presented a method to determine, through a Hamilton-Jacobi reachability computation, the set of states in safety-critical systems which will reach the desired equilibrium without saturating the input or violating the state constraints. Thus both envelope protection and stabilization under saturation are simultaneously achieved. This involves a reachability analysis on an extended state space which

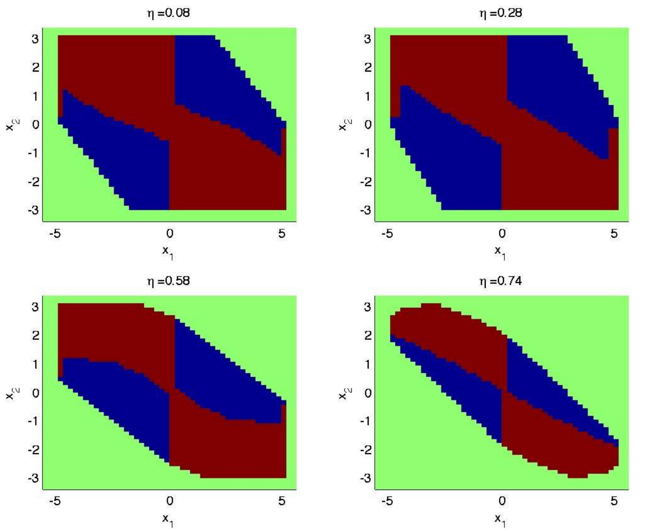

Fig. 1. Invariant set $\mathcal{W}_{\beta_q}$ plotted in x for $\eta \in \{0.08, 0.28, 0.58, 0.74\}$. Dark (blue) indicates $q = 1$, medium (red) indicates $q = 2$, and light (green) indicates states in the reachable set.

incorporates a parameter from the feedback linearizing input. By incorporating the input saturation, stability, and state constraints simultaneously in the initial cost function, the resultant invariant set will be the largest set of states, given bounded input, which will stabilize the system and always remain within a given constraint set.

The method and two examples presented contribute to the difficult problem of determining stabilizing controllers for safety-critical systems under nonlinear state and input constraints. Future work includes 1) determining solutions for the computed switching surfaces, 2) switched control synthesis for open-loop hybrid systems with different constraint sets and control bounds in each mode, and 3) identification of unstable dynamics which can result in an optimal and stable closed-loop switched system, 4) exploitation of the η dynamics to reduce computational effort.

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