Safety-Preserving Control of High-Dimensional Continuous-Time Uncertain Linear Systems

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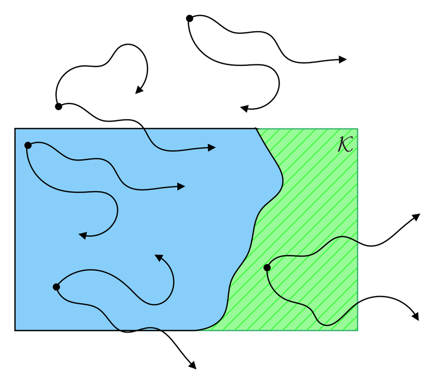
Motivation

"Scalability" is our main contribution.

- Presence of hard input and state constraints
- Particularly important with unknown (but bounded) disturbances/uncertainties
- Goal: Design a scalable permissive feedback controller that maintains feasibility/safety [Aubin 91; Blanchini 99; Lygeros, et al. 00; Tomlin, et al. 00; Morari 00; ...]

Background



- $\dot{x} = f(x, u, v)$, $u(t) \in \mathcal{U}$, $v(t) \in \mathcal{V}$, $x(t) \in \mathcal{K}$, $\forall t \in \mathbb{T} \triangleq [0, \tau]$
- Discriminating kernel

$$\mathrm{Disc}_{\mathbb{T}}(\mathcal{K}, \mathcal{U}, \mathcal{V}) \triangleq \{x_0 \in \mathcal{K} \mid \forall v(\cdot) \in \mathscr{V}_{\mathbb{T}}, \; \exists u(\cdot) \in \mathscr{U}_{\mathbb{T}}, \forall t \in \mathbb{T}, \; x(t) \in \mathcal{K}\}$$

  - ▹ Feedback strategy for control: $u(\cdot) = \hat{u}(x, \cdot) \in \mathscr{U}_{\mathbb{T}}$
  - ▹ Non-anticipative strategy for disturbance: $v(\cdot) = \varrho[u](\cdot) \in \mathscr{V}_{\mathbb{T}}$
  - ▹ When $\mathcal{K}$ is deemed *safe*, $u(\cdot)$ is *safety-preserving*

Conventional Approaches

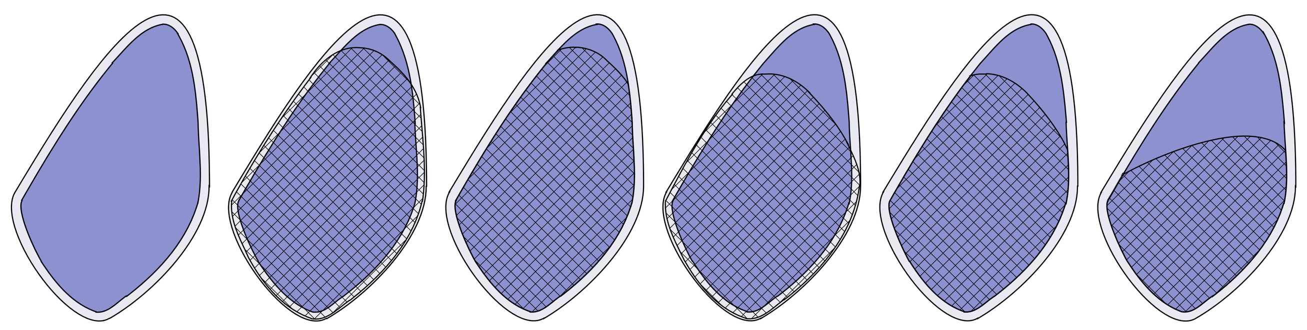
- Synthesize safety-preserving controllers based on the shape of the kernel
  - ▹ Contingent cones and proximal normals (per Nagumo's theorem)
  - ▹ Terminal constraint set for Receding Horizon Control
- Numerical solutions to compute the kernel
  - ▹ Eulerian methods, e.g. level-set techniques, Saint-Pierre's algorithm: grid-based
  - ▹ Recursive methods for DT LTI systems with polytopic constraints, e.g. Blanchini's algorithm: explosion of vertices
  - ▹ A control Lyapunov function sublevel-set for CT systems: possibly too conservative

Discriminating Kernel Approximation (Offline)

Approximate $\mathrm{Disc}_{\mathbb{T}}(\mathcal{K}, \mathcal{U}, \mathcal{V})$ via a nested sequence of sets robustly reachable in small sub-intervals of $\mathbb{T}$: $\mathrm{Reach}_t(\mathcal{K}, \mathcal{U}, \mathcal{V}) \triangleq \{x_0 \in \mathcal{X} \mid \forall v(\cdot) \in \mathscr{V}_{[0,t]}, \exists u(\cdot) \in \mathscr{U}_{[0,t]}, x(t) \in \mathcal{K}\}$

$$K_{|P|}(P) = \mathcal{K}_{\downarrow}(P) \triangleq \{x \in \mathcal{K} \mid \mathrm{dist}(x, \mathcal{K}^c) \geq M \|P\|\},$$
$$K_{k-1}(P) = \mathcal{K}_{\downarrow}(P) \cap \mathrm{Reach}_{t_k - t_{k-1}}(K_k(P), \mathcal{U}, \mathcal{V})$$

P: partition of $\mathbb{T}$
M: uniform bound on f



- Guaranteed under-approximation, & applicable to general systems and sets
- Arbitrary precision $\mathrm{Disc}_{\mathbb{T}}(\mathring{\mathcal{K}}, \mathcal{U}, \mathcal{V}) \subseteq \bigcup_{P \in \mathscr{P}(\mathbb{T})} K_0(P) \subseteq \mathrm{Disc}_{\mathbb{T}}(\mathcal{K}, \mathcal{U}, \mathcal{V})$

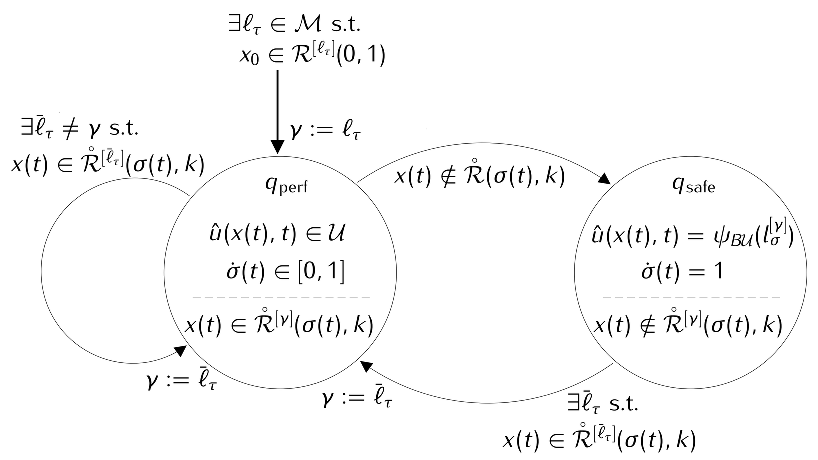
Piecewise Ellipsoidal Algorithm (Offline)

- Based on the ellipsoidal techniques for maximal reachability (LTI) [Kurzhanski and Valyi 96]
- For fixed terminal direction ℓ_τ and partition P, with $K^{[\ell_\tau]}_{|P|}(P) = K_{|P|}(P) = \mathcal{K}_{\downarrow}(P)$, do

$$K^{[\ell_\tau]}_{k-1} = \mathrm{maxvol}\left(K_{|P|}(P) \cap \mathrm{Reach}^{[\ell_\tau]}_{t_k - t_{k-1}}(K^{[\ell_\tau]}_k(P), \mathcal{U}, \mathcal{V})\right)$$

- Generates an **ellipsoidal** set $K^{[\ell_\tau]}_0(P)$ such that $\bigcup_{\ell_\tau \in \mathcal{M}} K^{[\ell_\tau]}_0(P) \subseteq \mathrm{Disc}_{\mathbb{T}}(\mathcal{K}, \mathcal{U}, \mathcal{V})$
- Finer partition = less accuracy loss (empirically)
- Computational complexity $\sim \mathcal{O}(n^3)$ (ellipsoidal reach & SDP)

Safety-Preserving Synthesis (Online)



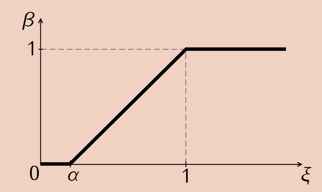
[Based on Kurzhanski and Valyi 96; Kurzhanskiy and Varaiya 08; Tomlin, et al. 00]

- Guarantees safety over at least $\mathbb{T}$ (pseudo-time $\sigma(t) = s + \int_s^t \dot{\sigma}(\lambda) d\lambda$ prolongs safety)
- Infinite-horizon control (via satisfaction of robust controlled-invariance conditions offline)
- Computational complexity $\sim \mathcal{O}(n^2)$ (inexpensive matrix multiplications)

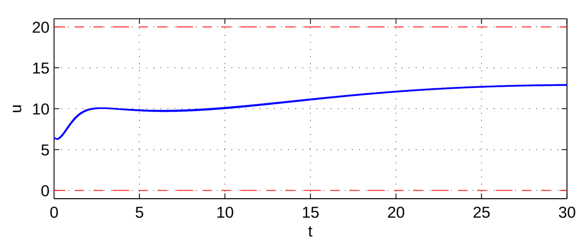
Smoothing Modification

- Safety and performance control laws may be conflicting: pathological behavior
- One solution: with $\phi^{[\gamma]}$ a measure of how deep inside ellipsoid γ,

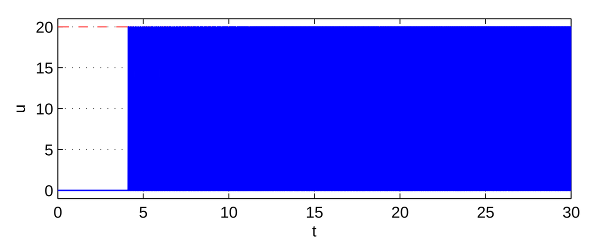
$$u(t) = \left(1 - \beta_\alpha(\phi^{[\gamma]}(x(t), \sigma(t)))\right) u_{\mathrm{perf}}(t) + \beta_\alpha(\phi^{[\gamma]}(x(t), \sigma(t))) u_{\mathrm{safe}}(t)$$



- Benefit: $u(t)$ is **continuous** across the automaton's transitions
  - ▹ When disturbance plays optimally (with inexact arithmetic)
    - → Unmodified policy: Chattering in HA and u
    - → Modified policy: HA less likely to chatter, u does not chatter
  - ▹ When disturbance allowed to play non-optimally
    - → Unmodified policy: Chattering in HA and u as fast as disturbance itself (Zeno!)
    - → Modified policy: Neither HA nor u chatters



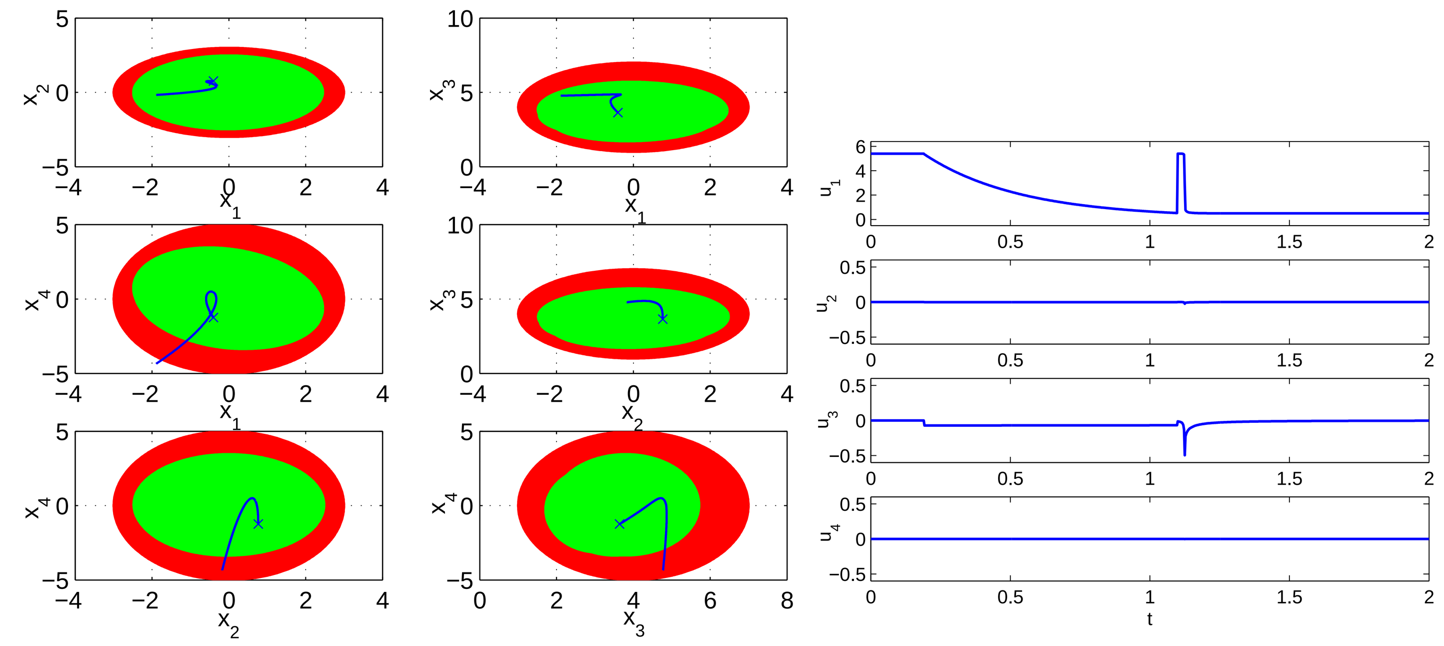
vs.



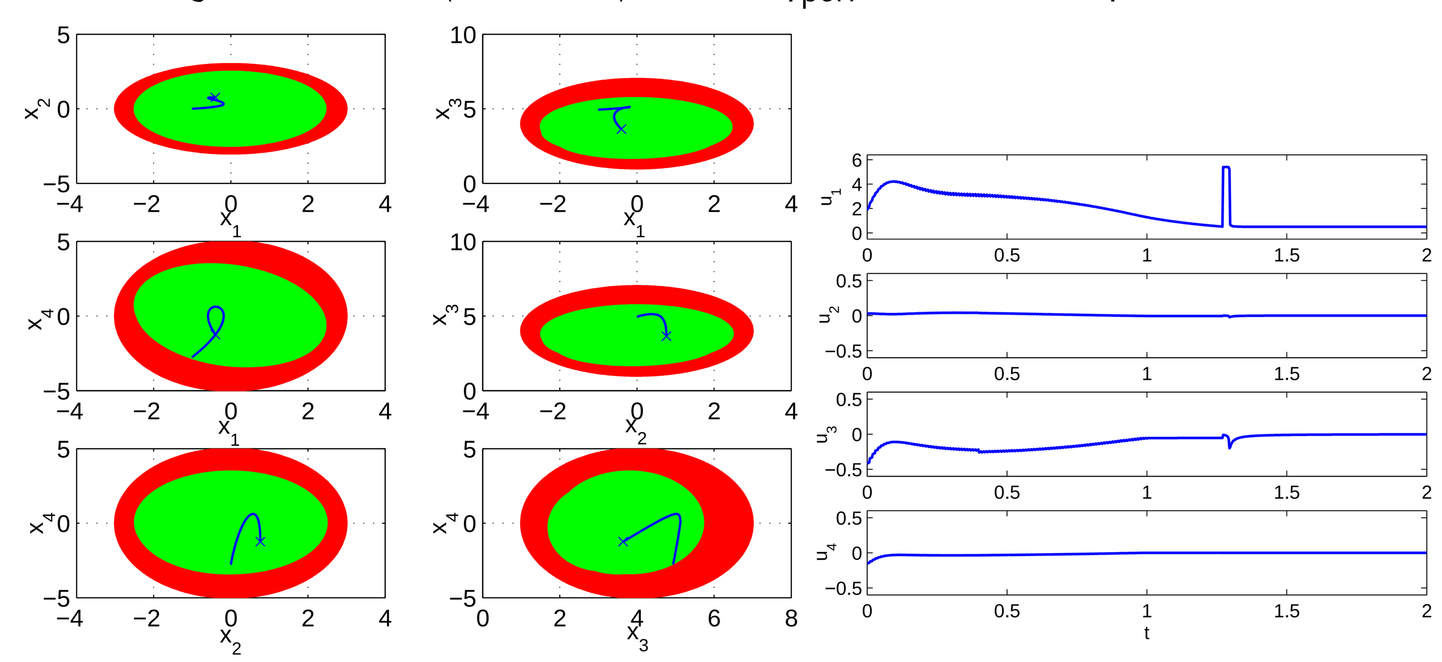
Application: 12D Quadrotor

- 12D model [Cowling, et al. 10]. Linearized about hover. $x = [x \; y \; z \; \dot{x} \; \dot{y} \; \dot{z} \; \phi \; \theta \; \psi \; \dot{\phi} \; \dot{\theta} \; \dot{\psi}]^T$
- Control $u = [u_1 \; u_2 \; u_3 \; u_4]^T \in [0.5, 5.4] \times [-0.5, 0.5]^3$
- Disturbance is wind: unknown but bounded ($v \sim$ uniform$(0, +0.1)$ in simulations)
- Physical constraints. Also, keep 1–7 m above ground for at least 2 s
- Performance: LQR (saturated) to move to $x_{ss} = [0 \; 0 \; 5 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0]^T$

Case 1: No safety controls; (Saturated) LQR; Failed at 1.8 s



Case 2: Safety control with (saturated) LQR in q_{perf}; Can extend upto 4.5 s



Further Readings

- S. Kaynama, I. M. Mitchell, M. Oishi, and G. A. Dumont, "Scalable safety-preserving robust control synthesis for continuous-time linear systems," submitted to *IEEE Transactions on Automatic Control*, Special Issue on Control of Cyber-Physical Systems (Feb 2013)
- J. Maidens, S. Kaynama, I. M. Mitchell, M. Oishi, and G. A. Dumont, "Lagrangian methods for approximating the viability kernel in high-dimensional systems," to appear in *Automatica* (accepted in Sept 2012 as Regular Paper)
- S. Kaynama, J. Maidens, M. Oishi, I. M. Mitchell, and G. A. Dumont, "Computing the viability kernel using maximal reachable sets," in *Proc. Hybrid Systems: Computation and Control*, Beijing, China, April 17–19, 2012, pp. 55–63