The Continual Reachability Set and its Computation Using Maximal Reachability Techniques

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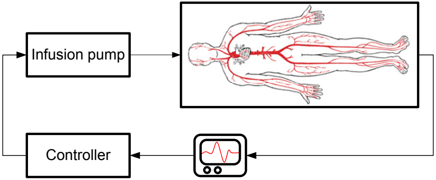
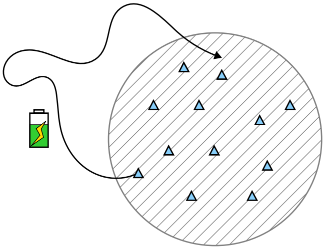
Motivation: Control of Anesthesia

- Control of depth of anesthesia
  - [Simanski, et al. 07; Ionescu, et al. 08; Syafiie, et al. 09; Dumont, et al. 09; Oliveira, et al. 09; Mendonca, et al. 09]
- Goal: closed-loop drug delivery system
  - Currently bolus-based open-loop system, 80+ patients via clinical trials
- Key element for obtaining regulatory certificate (FDA/Health Canada) are guarantees of safety and performance



Motivation: Other Application Domains

- Fleet of environmental monitoring motes with limited power source
- Must be dispersed using bounded input authority
- Alert depletion of battery **at least** t_a time units in advance
- Objectives:
  - Return to the base upon low-battery alert
  - Spend maximum possible time outside
  - Roam over as large of an area outside of the base as possible



Introduction

$$\begin{cases} \dot{x} = f(x,u), & x(0) = x_0 \\ u(t) \in \mathcal{U} & \text{(input constraint)} \\ \mathcal{K} \subseteq \mathcal{X} & \text{(target set/state constraint)} \end{cases}$$

- Reachability analysis
  - [Aubin, et al. 11; Kurzhanski and Varaiya 00; Tomlin, et al. 03; Lygeros 04]
- Typically used to guarantee safety
  - [Lygeros, et al. 98; Mitchell, et al. 05; Bayen, et al. 07]
- Safety constraints may be relaxed in favor of improved performance
  - **Continual reachability set** to explore that option

Outline

What is the Continual Reach Set?

Connection with Other Reachability Constructs

Properties and Implications

Approximation Techniques

Example: Control of Anesthesia

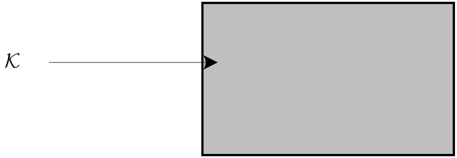
The Continual Reach Set

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) := \Big\{x_0 \in \mathcal{X} \mid \forall t \in [0,\tau], \\ \exists u(\cdot) \in \mathscr{U}_{[0,t]},\ \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\Big\}$$

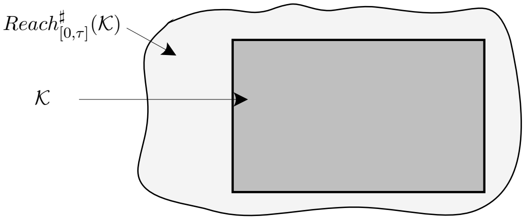
- Set of states that can reach $\mathcal{K}$ at any time within the finite horizon
- For any desired time there exists at least one input policy that can steer the system to the target
- Additional flexibility to a supervisory controller: a trade-off between the desired time-to-reach the target and the input effort

Backward Constructs and Their Connections

Target/Constraint Set



Maximal Reach Tube



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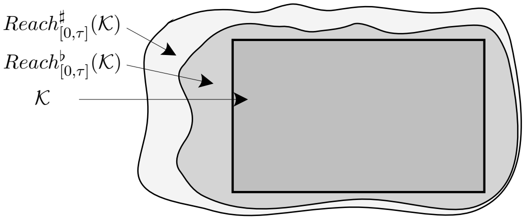
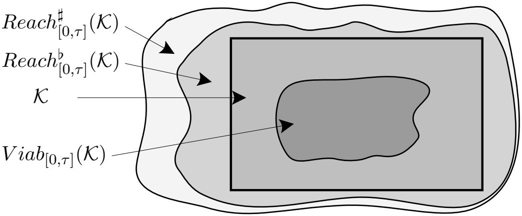


Backward Constructs and Their Connections

Viability Kernel



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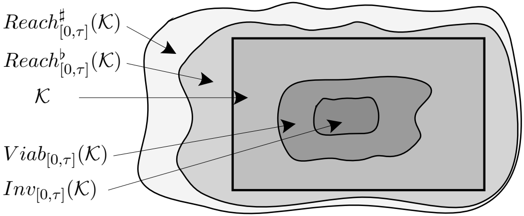
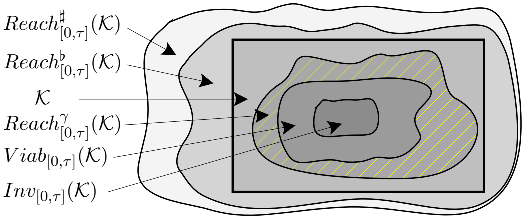
Invariance Kernel



Backward Constructs and Their Connections

Continual Reach Set



Backward Constructs and Their Connections (cont'd)

$$Reach^{\flat}_{[0,\tau]}(\mathcal{K}^c) = (Viab_{[0,\tau]}(\mathcal{K}))^c$$
$$Reach^{\sharp}_{[0,\tau]}(\mathcal{K}^c) = (Inv_{[0,\tau]}(\mathcal{K}))^c$$

[Lygeros 04]

$$Reach^{\flat}_{[0,\tau]}(\mathcal{K}) \supseteq \bigcup_{t\in[0,\tau]} Reach^{\flat}_{t}(\mathcal{K})$$
$$Reach^{\sharp}_{[0,\tau]}(\mathcal{K}) = \bigcup_{t\in[0,\tau]} Reach^{\sharp}_{t}(\mathcal{K})$$

[Mitchell 07; Girard, et al. 06; Le Guernic 10; Kurzhanski and Varaiya 00; Kurzhanskiy and Varaiya 07; Stursberg and Krogh 03]

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[This paper]

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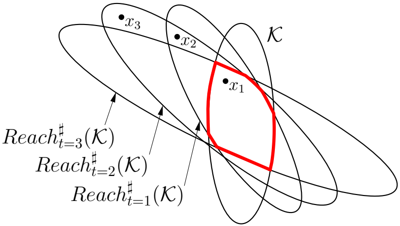
[This paper]

Some Properties

- For every $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ and some $t \in [0, \tau]$,

$$d(\xi_{x_0,0,u(\cdot)}(\hat{t}), \mathcal{K}) \leq d_{H_1}\left(Reach^{\sharp}_{t-\hat{t}}(\mathcal{K}), \mathcal{K}\right) \quad \forall \hat{t} \in [0, t]$$

  for any $u(\cdot) \in \mathscr{U}_{[0,t]}$ such that $\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}$
- States inside maximal reach tube but outside continual reach set can only reach target at specific times

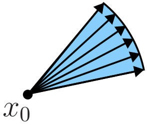


Implications

- What it means for performance
- More flexibility since viability control laws are a subset
- Performance + safety; a mixed scheme

$$Reach^{\gamma}_{[0,\tau]}(Viab_{[0,\tau]}(\mathcal{K})) = Viab_{[0,\tau]}(\mathcal{K})$$

(the original viability control laws are still a subset)



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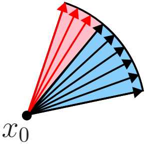


Approximation

- Maximal reachability techniques that offer under-approximation
- Approximation via the ellipsoidal techniques for linear systems

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \supseteq$$
$$\left\{ x \in \mathcal{X} \,\middle|\, \sup_{t\in[0,\tau]} \min_{\ell_\tau \in \mathcal{V}} \left\langle (x - x^*_t),\, (X^-_{\ell,t})^{-1}(x - x^*_t) \right\rangle \le 1 \right\}$$

- Implemented by Ellipsoidal Toolbox [Kurzhanskiy and Varaiya 06]

Example: Control of Anesthesia

- Discrete-time Laguerre models (6D); patient's response to rocuronium
- Target set: therapeutic bounds on output (pseudo-occupancy level), i.e. desired clinical effect
- Input constraint: actuator bounds (hard bounds on drug infusion rate)

Example: Control of Anesthesia (cont'd)

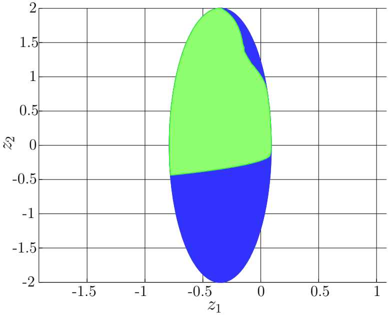
Issues to take into account:

1. The target is in the output space as opposed to the state space
2. The output signal should track a reference

Reformulate the problem by:

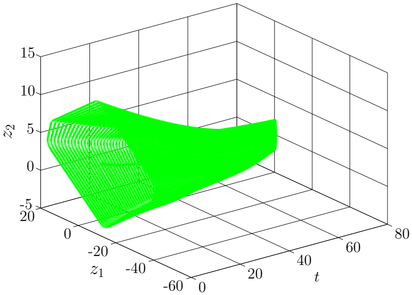
1. Projecting the bounds onto the state space
2. Making the control action regulatory

Example: Control of Anesthesia (cont'd)



A projection of the continual reach set computed using Ellipsoidal Toolbox.
(patient #80. 60 min surgery.)

Example: Control of Anesthesia (cont'd)



Projection of the maximal reach tube computed for 80 time steps.

Example: Control of Anesthesia (cont'd)

- A guarantee of performance; desired clinical effect can be reached at arbitrary times
- Minimize total administered drug, or achieve a desired depth of anesthesia arbitrarily fast
- Optimal infusion rate to keep within the target clinical effect may not be physiologically ideal (discontinuous/bang-bang)
- May choose to temporarily relax the state constraint in exchange for a better-suited (less aggressive, mildly varying) infusion rate
- Physiologically more optimized to meet the operating conditions and patient's ability to handle drug (patient-oriented design)

Conclusions and Future Work

- Continual reach set to guarantee performance
- Approximation based on available maximal reachability techniques
- Additional degree of freedom to a supervisory controller
- Facilitate a physiologically more relevant control of anesthesia

- Synthesizing continual reachability control laws
- Accounting for model uncertainty
- Implementation of a safety- + performance-based controller

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Backward Constructs

Maximal Reachability Set:

$$Reach_t^{\sharp}(\mathcal{K}) := \left\{ x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \right\}$$

Maximal Reachability Tube:

$$Reach_{[0,\tau]}^{\sharp}(\mathcal{K}) := \left\{ x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \exists t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \right\}$$

Minimal Reachability Set:

$$Reach_t^{\flat}(\mathcal{K}) := \left\{ x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \right\}$$

Minimal Reachability Tube:

$$Reach_{[0,\tau]}^{\flat}(\mathcal{K}) := \left\{ x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \exists t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \right\}$$

Backward Constructs (cont'd)

Invariance Kernel:

$$Inv_{[0,\tau]}(\mathcal{K}) := \left\{x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \forall t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\right\}$$

Viability Kernel:

$$Viab_{[0,\tau]}(\mathcal{K}) := \left\{x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \forall t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\right\}$$

Continual Reachability Set:

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) := \left\{x_0 \in \mathcal{X} \mid \forall t \in [0,\tau],\, \exists u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\right\}$$

Approximation via the Ellipsoidal Techniques

$$
\begin{aligned}
Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq {} & Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \\
& = \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau \in \mathcal{L}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right)
\end{aligned}
$$

$$
\begin{aligned}
Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq {} & Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \\
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\end{aligned}
$$

$$
Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \supseteq
$$

$$
\left\{ x \in \mathcal{X} \;\middle|\; \sup_{t\in[0,\tau]} \min_{\ell_\tau \in \mathcal{V}} \langle (x - x^*_t), (X^-_{\ell,t})^{-1}(x - x^*_t) \rangle \leq 1 \right\}
$$

Approximation via the Ellipsoidal Techniques

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) = \bigcap_{t\in[0,\tau]}\left(\bigcup_{\ell_\tau\in\mathcal{L}} \mathcal{E}(x^*(t), X^-_\ell(t))\right)$$

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \supseteq \bigcap_{t\in[0,\tau]}\left(\bigcup_{\ell_\tau\in\mathcal{V}} \mathcal{E}(x^*(t), X^-_\ell(t))\right)$$

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Approximation via the Ellipsoidal Techniques

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$$
$$= \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau \in \mathcal{L}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right)$$

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Example: Control of Anesthesia

$$\begin{cases} x(t+1) = Ax(t) + Bu(t),\ y(t) = Cx(t),\ t \in \mathbb{Z}^+ \\ \mathcal{K}_0 := [0.1, 1] \\ \mathcal{U}_0 := [0, 0.8] \end{cases}$$

Example: Control of Anesthesia (cont'd)

Reformulate the problem by:

1. Projecting the bounds onto the state space
2. Making the control action regulatory

Example: Control of Anesthesia (cont'd)

Reformulate the problem by:

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Example: Control of Anesthesia (cont'd)

$$\begin{bmatrix} & C & \\ \mathbf{0}_{5\times 1} & & I_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix} =: \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_6 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ w_2 \\ \vdots \\ w_6 \\ y^* \end{bmatrix} = \begin{bmatrix} y - y^* \\ w_2 \\ \vdots \\ w_6 \\ y^* \end{bmatrix} =: \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_6 \\ z_7 \end{bmatrix}.$$

Example: Control of Anesthesia (cont'd)

Reformulate the problem by:

1. Projecting the bounds onto the state space
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Example: Control of Anesthesia (cont'd)

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} A - I & B \\ C & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ y^* \end{bmatrix}$$

Example: Control of Anesthesia (cont'd)

$$
\begin{cases}
z(t+1) = \widetilde{A}z(t) + \widetilde{B}u(t), \; y(t) = \widetilde{C}z(t), \; t \in \mathbb{Z}^+ \\
\mathcal{K} := (\mathcal{K}_0 - y^*) \times \mathbb{R}^6 \\
\mathcal{U} := \mathcal{U}_0 - u_{ss}
\end{cases}
$$

Other Application Domains (revisited)

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- Must be dispersed using bounded input authority
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- Objectives:
  - Return to the base upon low-battery alert
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- Solution: $Reach^{\gamma}_{[t_a,\tau]}(\mathcal{K})$