THE UNIVERSITY OF BRITISH COLUMBIA

Schur-Based Decomposition for Reachability Analysis of Linear Time-Invariant Systems

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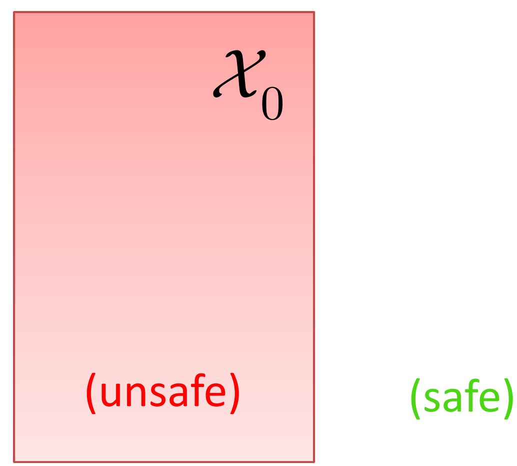
Introduction and Motivation

- Reachability analysis
- The "curse of dimensionality"
- Linear dynamics $\dot{x} = Ax + Bu,\ u \in \mathcal{U}$
- Efficient techniques (ellipsoidal, zonotopes)
- Safety control, non-convexity, arbitrary shapes

Introduction and Motivation

- Reachability analysis $\dot{x} = f(x,u,d), u \in \mathcal{U}, d \in \mathcal{D}$



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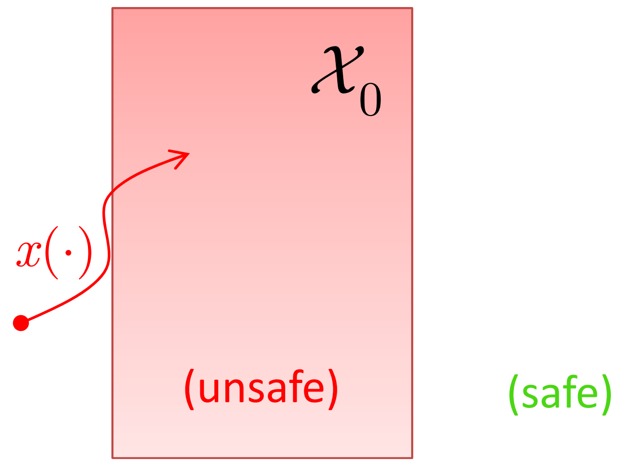

$$t \in [-\tau, 0], \tau > 0$$

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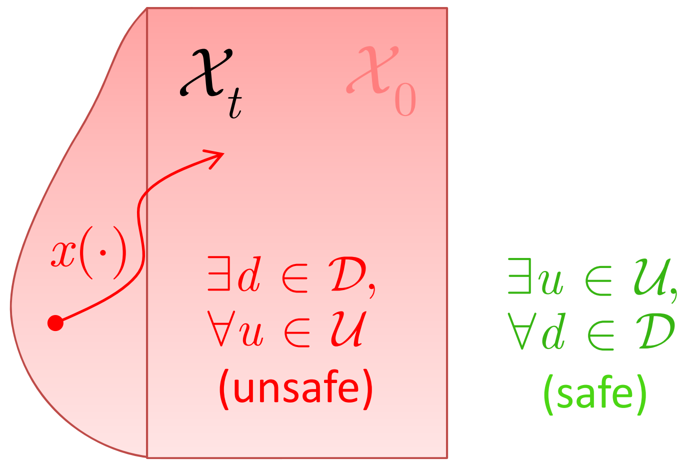

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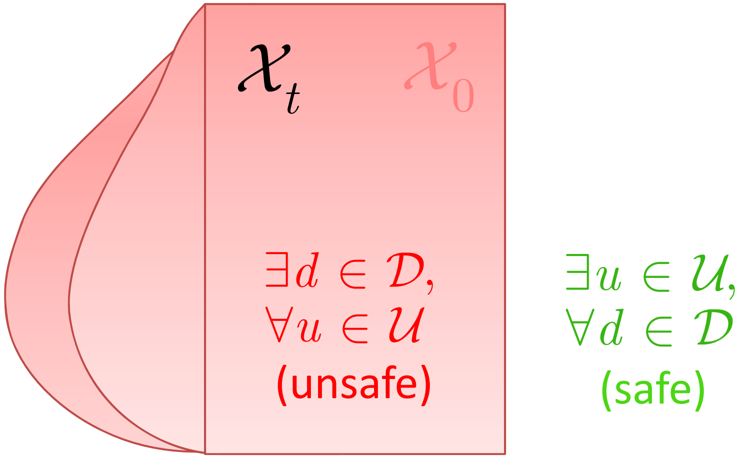

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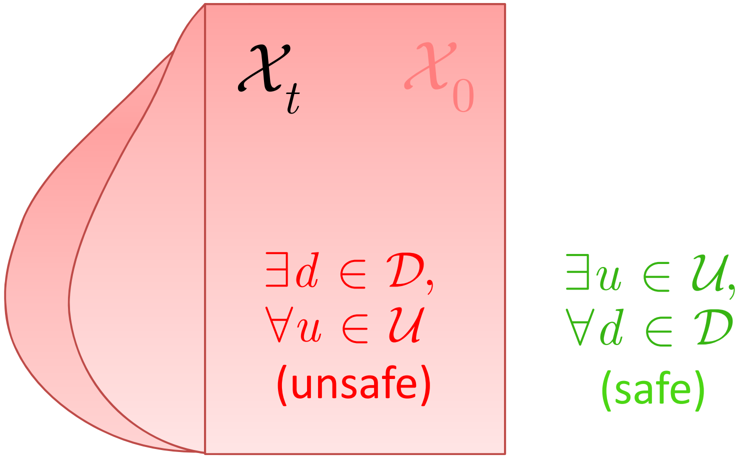

$$t \in [-\tau, 0], \tau > 0$$

$$\mathcal{X}_t = \{x(t) \in \mathbb{R}^n \mid \exists d \in \mathcal{D}, \exists x(\cdot), \forall u \in \mathcal{U}, x(0) \in \mathcal{X}_0\}$$

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Kurzhanskiy, Varaiya [2007]
Goncharova, Ovseevich [2007]
Asarin, Dang, Frehse, Girard, Le Guernic, Maler [2006]
Nasri, Lefebvre, Gueguen [2006]
Habets, Collins, Van Schuppen [2006]
Mitchell, Bayen, Tomlin [2005]
Yazarel, Pappas [2004]
Krogh, Stursberg [2003]
Nenninger, Frehse, Krebs [2002]
Botchkarev, Tripakis [2000]
Vidal, Schaffert, Lygeros, Sastry [2000]
Kurzhanski, Varaiya [2000]
Greenstreet, Mitchell [1999]
Chutinan, Krogh [1998]

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Introduction and Motivation

- Reachability analysis
- The "curse of dimensionality"
    - Set representation
        - Kurzhanskiy, Varaiya [2006]; Girard, Guernic, Maler [2006]; Krogh, Stursberg [2003],…
    - Model reduction, approximation, Hybridization, Projection, Structure decomposition
        - Han, Krogh [2004, 05]; Girard, Pappas [2007]; Asarin, Dang [2004]; Mitchell, Tomlin [2003]; Stipanovic, Hwang, Tomlin [2003],…
    - Combination
        - Han, Krogh [2006]

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  - Kurzhanskiy, Varaiya [2007]; Girard, Guernic, Maler [2006]; …
- Safety control, non-convexity, arbitrary shapes
  - Level-Set Methods (Mitchell, Bayen, Tomlin [2005])
  - Polytope-based MPT (Kvasnica, Grieder, Baotic, Morari [2004])
  - Parallelotope (Kostousova [2001])

Outline

- Introduction and motivation
- Structure decomposition
- Reachability in lower dimensions
- Examples

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- Structure decomposition
- Reachability in lower dimensions
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Structure Decomposition

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \xrightarrow{z = T^{-1}x} \dot{z} = \begin{pmatrix} \hat{A}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{A}_{22} \end{pmatrix} z + \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix} u$$

Structure Decomposition

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \hat{A}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{A}_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix} u$$

Structure Decomposition

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \xrightarrow{\quad T \quad} \dot{z} = \begin{pmatrix} \hat{A}_{11} & \mathbf{0} \\ \mathbf{0} & \hat{A}_{22} \end{pmatrix} z + \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix} u$$

Decomposing Transformation

$$\begin{pmatrix} B_{11} & \cdots & B_{1p} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$$\begin{pmatrix} B_{21} & \cdots & B_{2p} \end{pmatrix}$$

Decomposing Transformation

$$\begin{pmatrix} A_{11} & A_{12} & \big| & B_1 \\ A_{21} & A_{22} & \big| & B_2 \end{pmatrix}_x$$

$$\Big\downarrow U$$

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \big| & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \big| & \tilde{B}_2 \end{pmatrix}_y$$

Decomposing Transformation

$$\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X-X\tilde{A}_{22}+\tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z
\end{tikzcd}$$

Decomposing Transformation

$$\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & & \left(\begin{array}{cc|c} \tilde{A}_{11} & \mathbf{0} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[rr, "W=\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}"'] & & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \arrow[u]
\end{tikzcd}$$

Decomposing Transformation

$$\begin{pmatrix} A_{11} & A_{12} & \bigg| & B_1 \\ A_{21} & A_{22} & \bigg| & B_2 \end{pmatrix}_x \xrightarrow{\;T = UW\;} \begin{pmatrix} \tilde{A}_{11} & \mathbf{0} & \bigg| & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \bigg| & \hat{B}_2 \end{pmatrix}_z$$

Decomposing Transformation

decoupled subsystems

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \hat{B}_1 u_1$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Decomposing Transformation

$$
\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z
\end{tikzcd}
$$

Decomposing Transformation

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$$\Big\downarrow U$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \xrightarrow[W=\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}]{} \left(\begin{array}{cc|c} \tilde{A}_{11} & \left\|\tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12}\right\|_\infty & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z$$

Decomposing Transformation

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$$\Big\downarrow U$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \xrightarrow[W=\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)]{} \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z$$

$$\boxed{X = \underset{Q \in \mathbb{R}^{k \times (n-k)}}{\arg\min} \left\| \tilde{A}_{11} Q - Q \tilde{A}_{22} + \tilde{A}_{12} \right\|_\infty}$$

Decomposing Transformation

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \left(\begin{array}{cc|c} \tilde{A}_{11} & A_c & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \arrow[u]
\end{tikzcd}$$

Decomposing Transformation

$$\begin{tikzcd}[ampersand replacement=\&]
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] \& \left(\begin{array}{cc|c} \tilde{A}_{11} & A_c & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}"'] \& \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \arrow[u]
\end{tikzcd}$$

Decomposing Transformation

unidirectionally
weakly-coupled
subsystems

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \hat{B}_1 u_1 + A_c z_2$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Decomposing Transformation

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \hat{B}_1 u_1 + \overbrace{A_c z_2}^{\text{disturbance}}$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Decomposing Transformation
(disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \hat{B}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{B}_{22} & \hat{B}_{23} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau$$

Decomposing Transformation (disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \hat{B}_{11} & \mathbf{0} & \hat{B}_{13} \\ \mathbf{0} & \hat{B}_{22} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau$$

Decomposing Transformation (disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \mathbf{0} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau$$

Decomposing Transformation
(disjoint control input)

partitioning of input across candidate subsystems must be mutually exclusive and exhaustive

$$u_i \in \mathcal{U}_i \subset \mathbb{R}^{p_i}, \quad p = \sum_{i=1}^{N} p_i$$

Decomposing Transformation
(non-disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \mathbf{0} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau$$

Decomposing Transformation
(non-disjoint control input)

$$
\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}
$$

$$
+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau
$$

Decomposing Transformation
(non-disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-\tau) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \\ u_3(\tau) \end{pmatrix} d\tau$$

Not enough for the dynamics to be decoupled; Inputs must be disjoint too!

Decomposing Transformation

$$
\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z
\end{tikzcd}
$$

Decomposing Transformation

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$$\Big\downarrow U$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \xrightarrow[W=\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)]{} \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z$$

$$X = \underset{Q \in \mathbb{R}^{k\times(n-k)}}{\arg\min} \left\| \tilde{A}_{11}Q - Q\tilde{A}_{22} + \tilde{A}_{12} \right\|_\infty \quad \text{s.t.} \quad Q\hat{B}_2 = \hat{B}_1$$

Decomposing Transformation

$$
\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \left(\begin{array}{cc|c} \tilde{A}_{11} & A_c & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W=\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \arrow[u]
\end{tikzcd}
$$

Decomposing Transformation

$$
\begin{tikzcd}
\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \arrow[d, "U"] & \left(\begin{array}{cc|c} \tilde{A}_{11} & A_c & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \\
\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \arrow[r, "W = \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)"'] & \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{11}X - X\tilde{A}_{22} + \tilde{A}_{12} & \hat{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \hat{B}_2 \end{array}\right)_z \arrow[u]
\end{tikzcd}
$$

Decomposing Transformation

unidirectionally coupled subsystems

$$\dot{z}_1 = \tilde{A}_{11} z_1 + A_c z_2$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Decomposing Transformation

trivially-uncontrollable

unidirectionally
coupled
subsystems

$$\boxed{\dot{z}_1 = \tilde{A}_{11} z_1 + A_c z_2}$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Decomposing Transformation

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \overbrace{A_c z_2}^{\text{disturbance}}$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \hat{B}_2 u_2$$

Outline

- Introduction and motivation
- Structure decomposition
- Reachability in lower dimensions
- Examples

Reachability in Lower Dimensions

1: transform target set $\mathcal{Z}_0 \leftarrow T^{-1}\mathcal{X}_0$

2: project onto each subspace $\mathcal{Z}_0^i \leftarrow proj(\mathcal{Z}_0, i), \quad i = \{1,2\}$

3: for lower subsystem:

i) [isolated] $\mathcal{Z}_t^2 \leftarrow Reach(\mathcal{Z}_0^2)$

4: for upper subsystem:

i) compute upper-bound $||A_c z_2||_\infty \leq ||A_c||_\infty \cdot \xi$

ii) [perturbed] $\mathcal{Z}_t^1 \xleftarrow{\text{conserv.}} Reach(\mathcal{Z}_0^1)$

5: back-project, intersect, reverse-transform

$$\widehat{\mathcal{X}}_t := T\Big((\mathcal{Z}_t^1 \times \mathcal{R}_2) \cap (\mathcal{Z}_t^2 \times \mathcal{R}_1)\Big) \supseteq \mathcal{X}_t$$

Outline

- Introduction and motivation
- Structure decomposition
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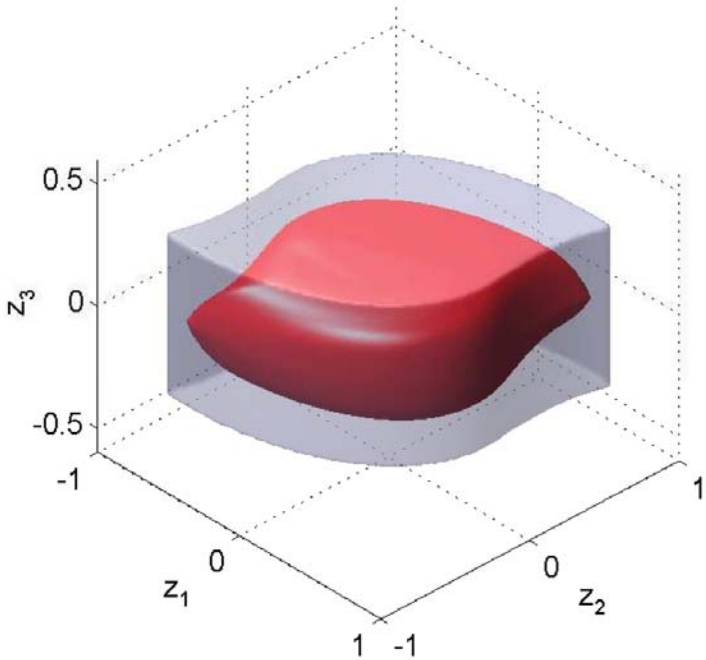
- Examples

Examples

Arbitrary system (3D)

$$A = \begin{bmatrix} -0.5672 & -0.7588 & -0.6282 \\ +3.1364 & -1.1705 & +2.3247 \\ +1.8134 & -1.7689 & -2.6930 \end{bmatrix}, \quad B = \begin{bmatrix} +0.0731 & -0.1639 \\ -0.7377 & -0.3578 \\ +0.1470 & +0.2410 \end{bmatrix}$$



Produced using the Level-Set Toolbox

Examples
Arbitrary system (3D)

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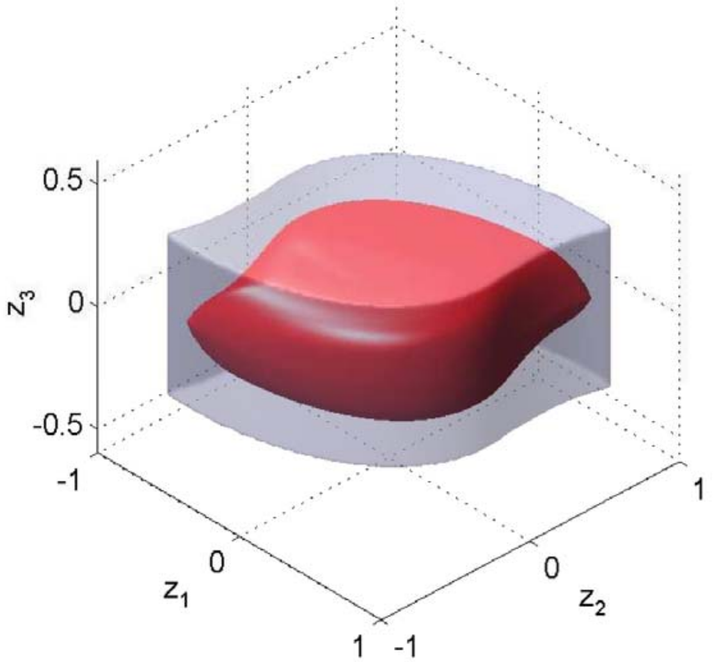


Full-order: 5823.73

Schur-based: 22.87

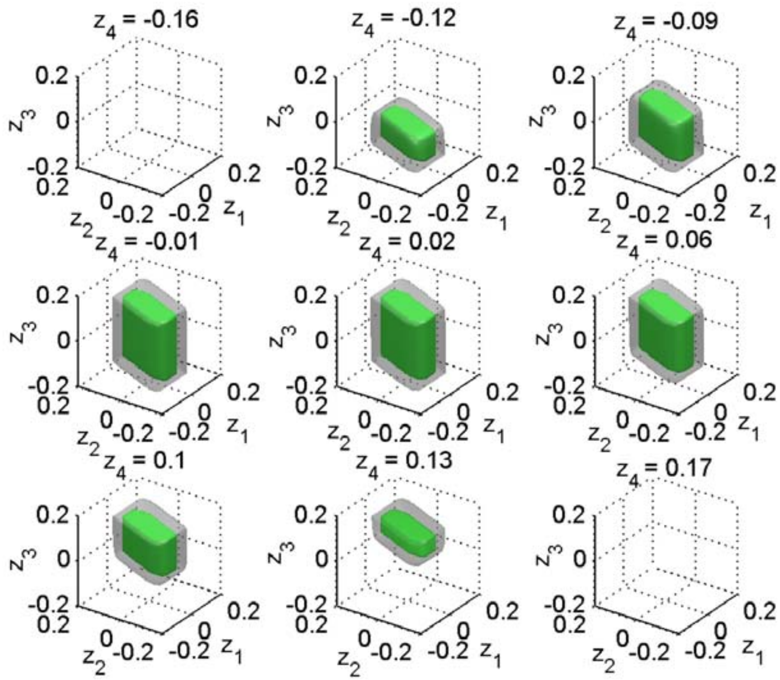
Produced using the Level-Set Toolbox

Examples

Longitudinal aircraft dynamics (4D)

$$A = \begin{bmatrix} -0.0030 & +0.0390 & 0 & -0.3220 \\ -0.0650 & -0.3190 & +7.7400 & 0 \\ +0.0200 & -0.1010 & -0.4290 & 0 \\ 0 & 0 & +1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} +0.0100 \\ -0.1800 \\ -1.1600 \\ 0 \end{bmatrix}$$

Source: A. Bryson, Control of Spacecraft and Aircraft. Princeton Univ. Press, 1994.



Produced using the Level-Set Toolbox

Examples

Longitudinal aircraft dynamics (4D)

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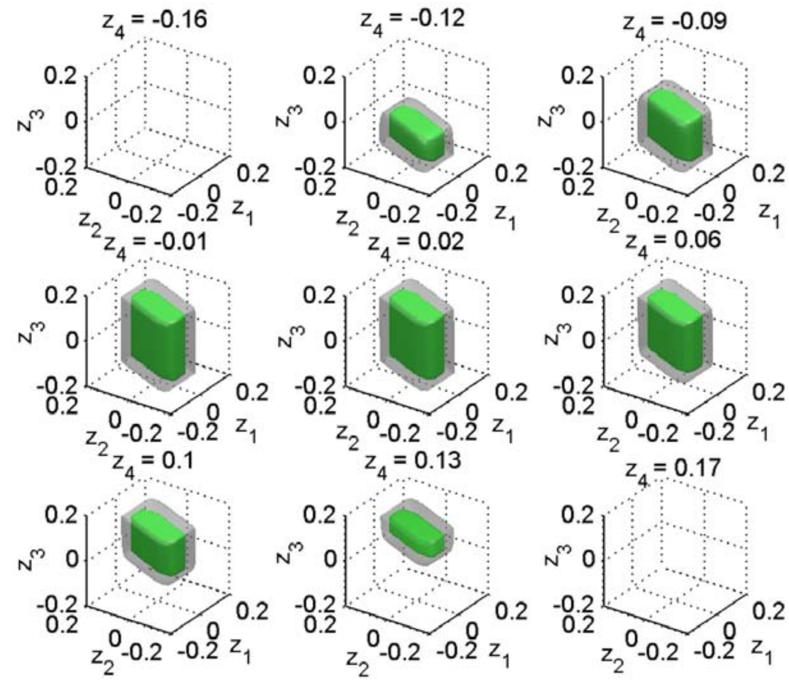


Full-order: 28546.80

Schur-based: 54.64

Produced using the Level-Set Toolbox

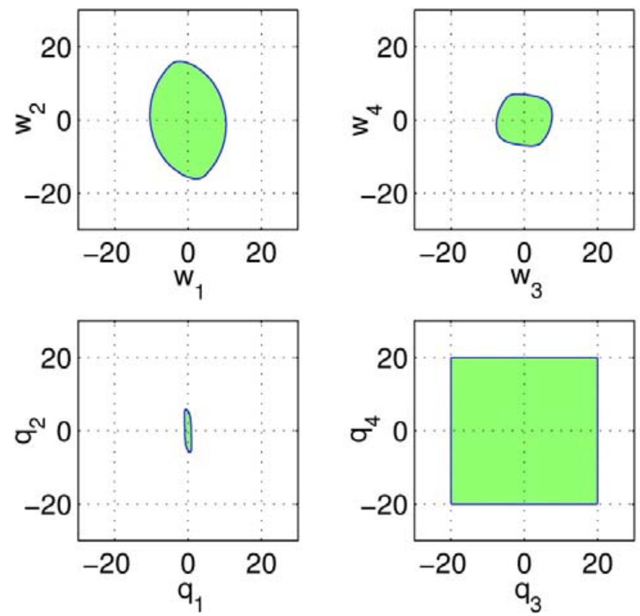
Examples

Distillation Column (8D)

$$A=\begin{bmatrix} -0.5774 & +3.0567 & +0.0073 & -0.8121 & +0.3034 & -0.3035 & +0.0072 & -0.1542 \\ -2.7290 & -0.7147 & -0.3430 & +1.5321 & +0.6643 & +0.2896 & -0.0013 & +0.0926 \\ 0 & 0 & -0.3891 & -0.9956 & +0.0182 & +0.0235 & +0.0049 & +0.0506 \\ 0 & 0 & +1.3640 & -1.3363 & -0.9037 & -0.4686 & -0.0009 & -0.1887 \\ 0 & 0 & 0 & 0 & -0.7357 & -0.2275 & -0.0082 & -0.0021 \\ 0 & 0 & 0 & 0 & 0 & -0.2259 & +0.0021 & -0.0457 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0052 & +0.0024 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0755 \end{bmatrix}$$

$$B=\begin{bmatrix} -0.0335 & -0.4534 & -0.8005 & +0.5497 & +1.2886 & +0.3132 & +0.7117 & +0.0599 \\ -0.1228 & -0.0711 & -0.2612 & -0.1344 & -0.0504 & -0.2249 & -0.6994 & -0.3014 \end{bmatrix}^T$$

Source: S. Skogestad and I. Postlethwaite, Multivariable Feedback Control; analysis and design. West Sussex, UK: John Wiley & Sons, 2007



Produced using the Level-Set Toolbox

Examples

Distillation Column (8D)

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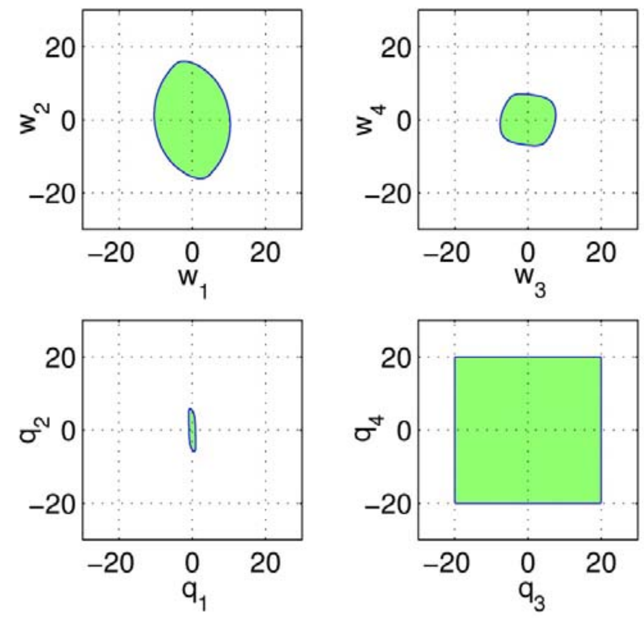

Full-order: NOT POSSIBLE!

Schur-based: 94.31

Produced using the Level-Set Toolbox

Summary

- Computations scale poorly with dimension
- Safety control, non-convex or arbitrarily shaped sets
- Appropriate coordinate transformation
- Significant computational advantage
- Easily extendible to hybrid systems
- Motivated by safety verification in anesthesia

THE UNIVERSITY OF BRITISH COLUMBIA

Schur-Based Decomposition for Reachability Analysis of Linear Time-Invariant Systems

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HJI PDE
(non-disjoint control input)

$$\mathcal{S}(A,B): A = \mathrm{diag}(A_{11}, A_{12}),\;\; B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},\;\; u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{U}$$

$$\mathcal{X}_0 = \{x \mid g(x) \le 0\}$$

$$- -$$

$$\frac{\partial \phi(x,t)}{\partial t} + \min\Big[0, H(x, \nabla_x \phi(x,t))\Big] = 0,$$

$$t \in [-\tau, 0], \qquad \phi(x,0) = g(x)$$

$$H(x,p) = \max_{u \in \mathcal{U}} \Big(p^T A x + p^T B u\Big)$$

HJI PDE
(non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \max_{u \in \mathcal{U}} \Big(p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 \\ + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)$$

HJI PDE
(non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \max_{u \in \mathcal{U}} \Big(p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 \\ + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)$$

HJI PDE

(non-disjoint control input)

$$
\begin{aligned}
H_{\mathcal{S}}(x,p) = \max_{u \in \mathcal{U}} \Big(& p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 \\
& + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)
\end{aligned}
$$

$$
H_{\mathcal{S}}(x,p) \neq \sum_{k=1}^{2} H_{\mathcal{S}_k}(x_k, p_k).
$$

HJI PDE
(non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \max_{u \in \mathcal{U}} \Big(p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + \cancel{p_1^T B_{11} u_1} + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + \cancel{p_1^T B_{12} u_2} \Big)$$

$$H_{\mathcal{S}}(x,p) = \sum_{k=1}^{2} H_{\mathcal{S}_k}(x_k, p_k).$$