THE UNIVERSITY OF BRITISH COLUMBIA

# Overapproximating the Reachable Sets of LTI Systems Through a Similarity Transformation

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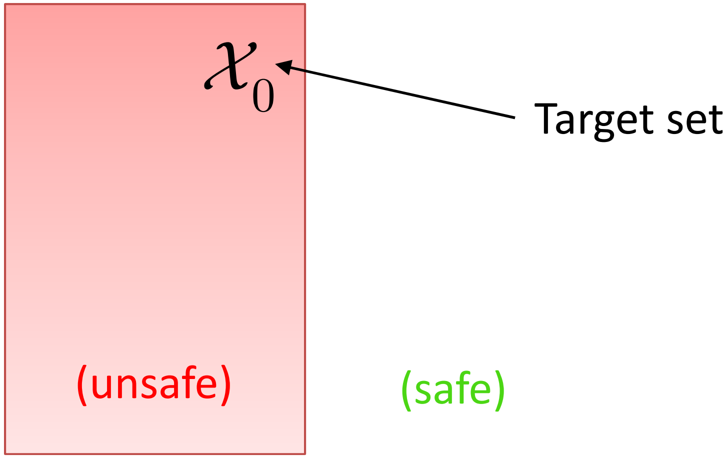
# Introduction and Motivation

- Reachability analysis
- The "curse of dimensionality"
- Linear dynamics $\dot{x} = Ax + Bu, \ u \in \mathcal{U}$
- Efficient techniques
- Safety control, non-convexity

# Introduction and Motivation

- Reachability analysis $\dot{x} = f(x, u, d), u \in \mathcal{U}, d \in \mathcal{D}$



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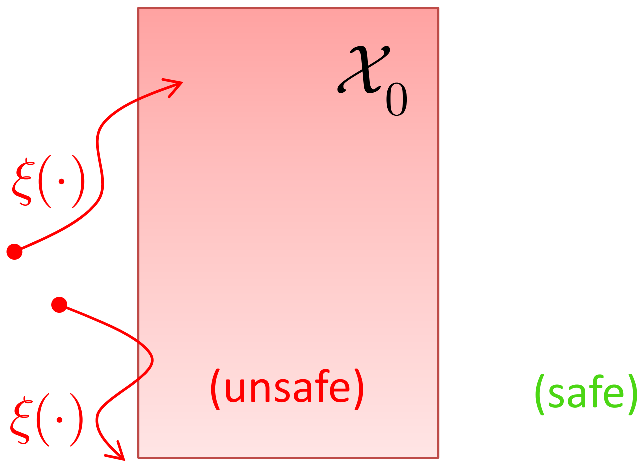


$$t \in [-\tau, 0], \tau > 0$$

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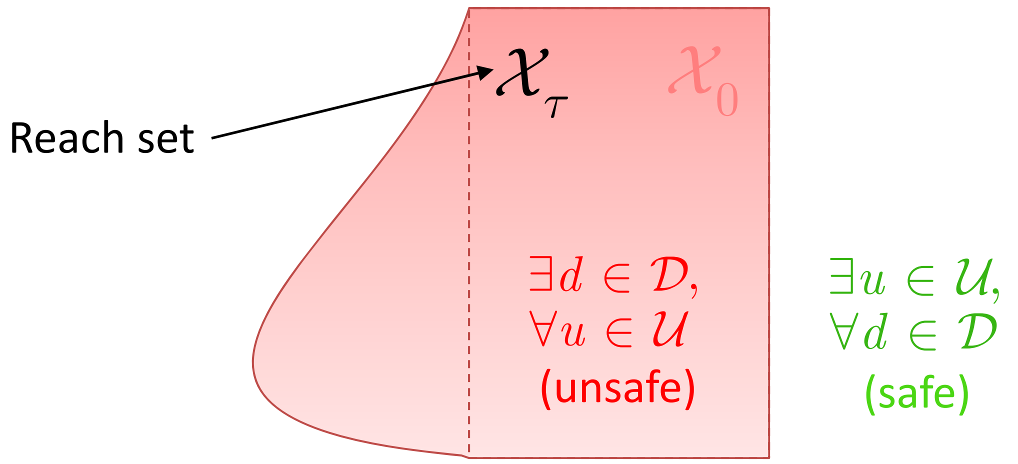
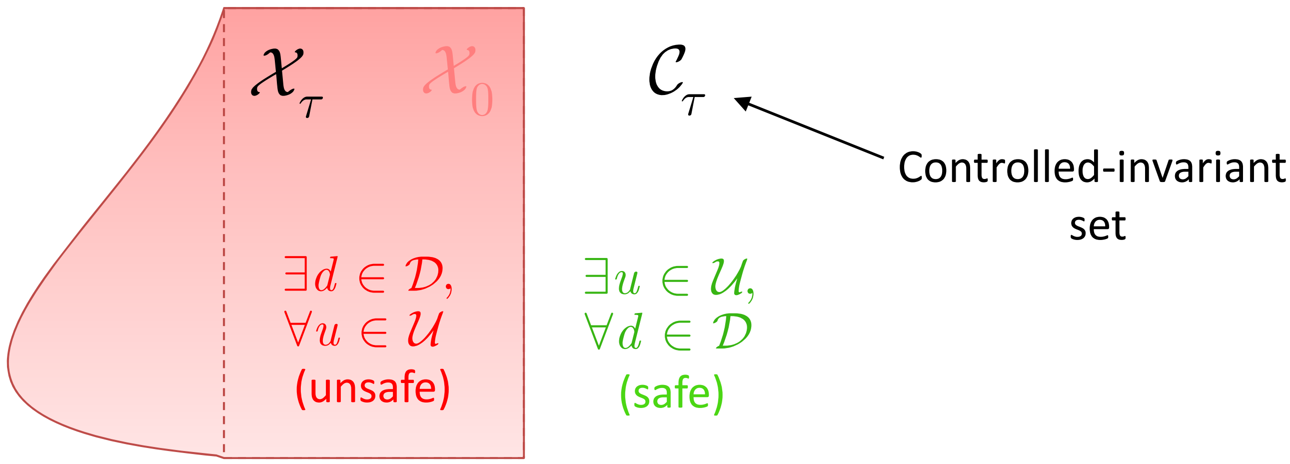



$$\mathcal{X}_\tau := \Big\{ x \in \mathbb{R}^n \mid \exists d(\cdot) \in \mathcal{D}, \exists s \in [-\tau, 0], \\ \xi_{x,-\tau,u(\cdot),d(\cdot)}(s) \in \mathcal{X}_0, \forall u(\cdot) \in \mathcal{U} \Big\}$$

# Introduction and Motivation

- Reachability analysis $\dot{x} = f(x, u, d),\, u \in \mathcal{U},\, d \in \mathcal{D}$




$$\mathcal{C}_\tau := \mathcal{X}_\tau^{\complement} = \Big\{ x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathcal{U}, \xi_{x,-\tau,u(\cdot),d(\cdot)}(s) \in \mathcal{X}_0^{\complement}, \forall s \in [-\tau, 0], \forall d(\cdot) \in \mathcal{D} \Big\}$$

# Introduction and Motivation

Reachability:

$$\mathcal{X}_\tau := \Big\{ x \in \mathbb{R}^n \mid \exists d(\cdot) \in \mathcal{D}, \exists s \in [-\tau, 0], \\ \xi_{x,-\tau,u(\cdot),d(\cdot)}(s) \in \mathcal{X}_0, \forall u(\cdot) \in \mathcal{U} \Big\}$$

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Attainability:

$$\mathcal{A}_\tau := \Big\{ x \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathcal{U}, \exists s \in [-\tau, 0], \\ \xi_{x,-\tau,u(\cdot),d(\cdot)}(s) \in \mathcal{X}_0, \forall d(\cdot) \in \mathcal{D} \Big\}$$

# Introduction and Motivation

- Reachability analysis

Reachability / attainability

Girard, Le Guernic [2008, 2010]
Kurzhanskiy, Varaiya [2007]
Goncharova, Ovseevich [2007]
Asarin, Dang, Frehse, Girard, Le Guernic, Maler [2006]
Nasri, Lefebvre, Gueguen [2006]
Habets, Collins, Van Schuppen [2006]
Mitchell, Bayen, Tomlin [2005]
Yazarel, Pappas [2004]
Krogh, Stursberg [2003]
Nenninger, Frehse, Krebs [2002]
Botchkarev, Tripakis [2000]
Vidal, Schaffert, Lygeros, Sastry [2000]
Kurzhanski, Varaiya [2000]
Greenstreet, Mitchell [1999]
Chutinan, Krogh [1998]
⋮

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- Linear dynamics $\dot{x} = Ax + Bu, \ u \in \mathcal{U}$
- Efficient techniques
- Safety control, non-convexity

# Introduction and Motivation

- Reachability analysis
- The “curse of dimensionality”
  - Set representation
    - Kurzhanski, Varaiya [2006]; Girard, Guernic, Maler [2006]; Krogh, Stursberg [2003],…
  - Model reduction, approximation, Hybridization, Projection, Structure decomposition
    - Han, Krogh [2004, 05]; Girard, Pappas [2007]; Asarin, Dang [2004]; Mitchell, Tomlin [2003]; Stipanovic, Hwang, Tomlin [2003]; Kaynama, Oishi [2009]; …
  - Combination
    - Han, Krogh [2006]

# Introduction and Motivation

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- The "curse of dimensionality"
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# Introduction and Motivation

- Reachability analysis
- The "curse of dimensionality"
- Linear dynamics $\dot{x} = Ax + Bu, \ u \in \mathcal{U}$
- Efficient techniques
  - Attainability: Ellipsoidal, zonotopes, support functions, …
    Kurzhanskiy, Varaiya [2007]; Girard, Le Guernic, Maler [2006]; Girard, Le Guernic [2008, 2010]; …
- Safety control, non-convexity

# Introduction and Motivation

- Reachability analysis
- The "curse of dimensionality"
- Linear dynamics $\dot{x} = Ax + Bu, \; u \in \mathcal{U}$
- Efficient techniques (Attainability)
- Safety control, non-convexity
  - Level-set methods (Mitchell, Bayen, Tomlin [2005])
  - Polytopic MPT (Kvasnica, Grieder, Baotic, Morari [2004])
  - Viability algorithms (Cardaliaguet, Quincampoix, Saint-Pierre [2004]; Gao, Lygeros, Quincampoix [2006])

# Outline

- Introduction and motivation
- Structure decomposition
- Reachability in lower dimensions
- Example

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- Structure decomposition
- Reachability in lower dimensions
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# Structure Decomposition

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n$$

$$\longrightarrow$$

$$\dot{z} = \tilde{f}(z, u) \qquad z \in \mathbb{R}^n$$

$$\dot{z}_1 = g(z_1, u) \qquad z_1 \in \mathbb{R}^k$$

$$\dot{z}_2 = h(z_2, u) \qquad z_2 \in \mathbb{R}^{(n-k)}$$

# Structure Decomposition

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

$$B_1 = \begin{pmatrix} B_{11} & \dots & B_{1p} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} B_{21} & \dots & B_{2p} \end{pmatrix}$$

# Structure Decomposition

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \xrightarrow{\quad T \quad} \dot{z} = \begin{pmatrix} \tilde{A}_{11} & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$

# Structure Decomposition
## (disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-r) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-r) \end{pmatrix} \begin{pmatrix} \tilde{B}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{B}_{22} & \tilde{B}_{23} \end{pmatrix} \begin{pmatrix} u_1(r) \\ u_2(r) \\ u_3(r) \end{pmatrix} dr$$

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# Structure Decomposition
## (non-disjoint control input)

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t-t_0) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix}$$

$$+ \int_{t_0}^{t} \begin{pmatrix} \Phi_{11}(t-r) & \mathbf{0} \\ \mathbf{0} & \Phi_{22}(t-r) \end{pmatrix} \begin{pmatrix} \mathbf{0} & \tilde{B}_{12} & \tilde{B}_{13} \\ \tilde{B}_{21} & \tilde{B}_{22} & \mathbf{0} \end{pmatrix} \begin{pmatrix} u_1(r) \\ u_2(r) \\ u_3(r) \end{pmatrix} dr$$

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Not enough for the dynamics to be decoupled; Inputs must be disjoint too!

# Structure Decomposition

$$\dot{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \xrightarrow{\quad T \quad} \dot{z} = \begin{pmatrix} \tilde{A}_{11} & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

$T_1$
(Schur)

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$
(Schur)

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# Structure Decomposition

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$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$$

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# Structure Decomposition

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$T_1$
(Schur)

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$
(Schur)

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \|\tilde{A}_{12}\| & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

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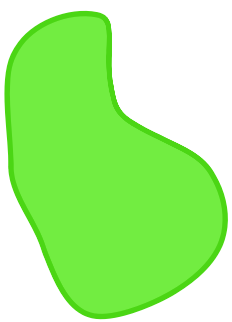
# Structure Decomposition



$\mathcal{Y}$

$$T_2 = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha = 1000$$

$$\mathcal{W} = \left\{ w \mid w = T_2^{-1} y, \ y \in \mathcal{Y} \right\}$$

$\mathcal{W}$

# Structure Decomposition

$$\begin{pmatrix} A_{11} & A_{12} & \big| & B_1 \\ A_{21} & A_{22} & \big| & B_2 \end{pmatrix}_x$$

$T_1$ (Schur)

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \big| & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \big| & \tilde{B}_2 \end{pmatrix}_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \big| & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \big| & \tilde{B}_2 \end{pmatrix}_w$$

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} \tilde{A}_{11} & \Pi & \big| & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \big| & \tilde{B}_2 \end{pmatrix}_z$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$
(Schur)

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

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# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \qquad \left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

$T_1$
(Schur)

$$\tilde{B}_2^\dagger := (\tilde{B}_2^T \tilde{B}_2)^{-1} \tilde{B}_2^T$$

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^\dagger \\ 0 & I \end{pmatrix}$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y \qquad \left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$ (Schur) ↓

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$ →

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix}$ ↑

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

$$\Pi := \alpha^{-1}(\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^{\dagger} - \tilde{B}_1\tilde{B}_2^{\dagger}\tilde{A}_{22} + \tilde{A}_{12})$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$

(Schur)

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^\dagger \\ 0 & I \end{pmatrix}$$

$$\alpha = \max\left\{1, \frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|}\right\} + \epsilon$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x$$

$T_1$ (Schur) ↓

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_y$$

$$T_2 := \begin{pmatrix} \alpha I & 0 \\ 0 & I \end{pmatrix} \longrightarrow$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \alpha^{-1}\tilde{A}_{12} & \alpha^{-1}\tilde{B}_1 \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_w$$

$$T_3 := \begin{pmatrix} I & \alpha^{-1}\tilde{B}_1\tilde{B}_2^{\dagger} \\ 0 & I \end{pmatrix} \uparrow$$

$$\alpha = \max\left\{1, \frac{\left\| \tilde{A}_{11}\tilde{B}_1\tilde{B}_2^{\dagger} - \tilde{B}_1\tilde{B}_2^{\dagger}\tilde{A}_{22} + \tilde{A}_{12} \right\|}{\left\| \tilde{A}_{12} \right\|}\right\} + \epsilon$$

$$\left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

# Structure Decomposition

But how large is $\alpha = \max\left\{1, \frac{\left\| \tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12} \right\|}{\left\| \tilde{A}_{12} \right\|}\right\} + \epsilon$ ?

# Structure Decomposition

But how large is $\alpha = \max\left\{1, \frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^{\dagger} - \tilde{B}_1\tilde{B}_2^{\dagger}\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|}\right\} + \epsilon$ ?

# Structure Decomposition

But how large is $\alpha = \max\left\{1, \frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|}\right\} + \epsilon$ ?

$$\frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|} \leq \|\Delta\| \cdot \left(\frac{\left\|\tilde{A}_{11}\right\| + \left\|\tilde{A}_{22}\right\|}{\left\|\tilde{A}_{12}\right\|}\right) + 1$$

$$\Delta := \tilde{B}_1\tilde{B}_2^\dagger - \operatorname*{arg\,min}_{Q \in \mathbb{R}^{k \times (n-k)}} \left\|\tilde{A}_{11}Q - Q\tilde{A}_{22} + \tilde{A}_{12}\right\|$$

# Structure Decomposition

But how large is $\alpha = \max\left\{1, \frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|}\right\} + \epsilon$ ?

$$\frac{\left\|\tilde{A}_{11}\tilde{B}_1\tilde{B}_2^\dagger - \tilde{B}_1\tilde{B}_2^\dagger\tilde{A}_{22} + \tilde{A}_{12}\right\|}{\left\|\tilde{A}_{12}\right\|} \le \left\|\Delta\right\| \cdot \left(\frac{\left\|\tilde{A}_{11}\right\| + \left\|\tilde{A}_{22}\right\|}{\left\|\tilde{A}_{12}\right\|}\right) + 1$$

$$\Delta := \tilde{B}_1\tilde{B}_2^\dagger - \underset{Q \in \mathbb{R}^{k\times(n-k)}}{\arg\min} \left\|\tilde{A}_{11}Q - Q\tilde{A}_{22} + \tilde{A}_{12}\right\|$$

$$\therefore \quad \lim_{\left\|\Delta\right\| \to 0} \alpha = 1 + \epsilon$$

# Structure Decomposition

$$\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{array}\right)_x \xrightarrow{T = T_1 T_2 T_3} \left(\begin{array}{cc|c} \tilde{A}_{11} & \Pi & \mathbf{0} \\ \mathbf{0} & \tilde{A}_{22} & \tilde{B}_2 \end{array}\right)_z$$

# Structure Decomposition

unidirectionally
weakly-coupled
subsystems

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \Pi z_2$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{B}_2 u_2$$

# Structure Decomposition

trivially-uncontrollable

unidirectionally weakly-coupled subsystems

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \Pi z_2$$

$$\dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{B}_2 u_2$$

# Structure Decomposition

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \overbrace{\Pi\, z_2}^{\text{disturbance}}$$

$$\dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{B}_2 u_2$$

# Outline

- Introduction and motivation
- Structure decomposition
- Reachability in lower dimensions
- Example

# Reachability in Lower Dimensions

1: transform target set $\mathcal{Z}_0 \leftarrow T^{-1}\mathcal{X}_0$

2: project onto each subspace $\mathcal{Z}_0^i \leftarrow \mathrm{Proj}(\mathcal{Z}_0, i), \quad i = 1,2$

3: for lower subsystem:

i) [isolated] $\mathcal{Z}_\tau^2 \leftarrow \mathrm{Reach}(\mathcal{Z}_0^2)$

4: for upper subsystem:

i) compute upper-bound $\|\Pi z_2\| \leq \|\Pi\|\zeta, \quad \zeta := \sup_{z_2 \in \mathcal{Z}_\tau^2} \|z_2\|$

ii) [perturbed] $\mathcal{Z}_\tau^1 \xleftarrow{\text{conserv.}} \mathrm{Reach}(\mathcal{Z}_0^1)$

5: back-project, intersect, reverse-transform

$$\widehat{\mathcal{X}}_\tau := T\Big((\mathcal{Z}_\tau^1 \times \mathcal{R}_2) \cap (\mathcal{Z}_\tau^2 \times \mathcal{R}_1)\Big) \supseteq \mathcal{X}_\tau$$

# Reachability in Lower Dimensions

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \Pi z_2$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{B}_2 u_2$$

# Reachability in Lower Dimensions

$$\dot{z}_1 = \tilde{A}_{11} z_1 + \Pi\, z_2$$
$$\dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{B}_2 u_2$$

# Reachability in Lower Dimensions

Formulating a bound on conservatism of $\mathcal{Z}_\tau^1$ :

$$\text{Let } z_{1,0} \in \mathcal{Z}_0^1 \text{ and } z_1 \in \mathcal{Z}_\tau^1.$$

$$\left\| z_1 - e^{-\tau \tilde{A}_{11}} z_{1,0} \right\| \leq \int_0^\tau e^{(\tau - r)\left\| \tilde{A}_{11} \right\|} \left\| \Pi \right\| \zeta \, dr$$

$$\leq \left\| \Pi \right\| \zeta \lim_{N \to \infty} \sum_{i=1}^{N} \frac{\tau^i (\overline{\sigma}(\tilde{A}_{11}) \sqrt{k})^{i-1}}{i!} \quad =: \mu$$

effect of input

# Reachability in Lower Dimensions

Formulating a bound on conservatism of $\mathcal{Z}_\tau^1$ :

$$\text{Let } z_{1,0} \in \mathcal{Z}_0^1 \text{ and } z_1 \in \mathcal{Z}_\tau^1.$$

$$\left\| z_1 - e^{-\tau \tilde{A}_{11}} z_{1,0} \right\| \leq \int_0^\tau e^{(\tau - r)\left\|\tilde{A}_{11}\right\|} \left\|\Pi\right\| \zeta \, dr$$

$$\leq \left\|\Pi\right\| \zeta \lim_{N\to\infty} \sum_{i=1}^{N} \frac{\tau^i (\overline{\sigma}(\tilde{A}_{11})\sqrt{k})^{i-1}}{i!} =: \mu$$

$$\therefore \quad \boxed{\mathcal{Z}_\tau^1 \subseteq \left( \bigcup_{s\in[-\tau,0]} e^{\tilde{A}_{11} s} \mathcal{Z}_0^1 \right) \oplus \mathcal{B}(0, \mu)}$$

# Reachability in Lower Dimensions

Formulating a bound on conservatism of $\mathcal{Z}_\tau^1$ :

$$\text{Let } z_{1,0} \in \mathcal{Z}_0^1 \text{ and } z_1 \in \mathcal{Z}_\tau^1.$$

$$\left\| z_1 - e^{-\tau \tilde{A}_{11}} z_{1,0} \right\| \leq \int_0^\tau e^{(\tau - r)\|\tilde{A}_{11}\|} \|\Pi\| \zeta \, dr$$

$$\leq \|\Pi\| \zeta \lim_{N\to\infty} \sum_{i=1}^{N} \frac{\tau^i (\overline{\sigma}(\tilde{A}_{11})\sqrt{k})^{i-1}}{i!} \quad =: \mu$$

$$\therefore \quad \boxed{\mathcal{Z}_\tau^1 \subseteq \left( \bigcup_{s\in[-\tau,0]} e^{\tilde{A}_{11} s} \mathcal{Z}_0^1 \right) \oplus \mathcal{B}(0,\mu)}$$

# Outline

- Introduction and motivation
- Structure decomposition
- Reachability in lower dimensions
- Example

# Example

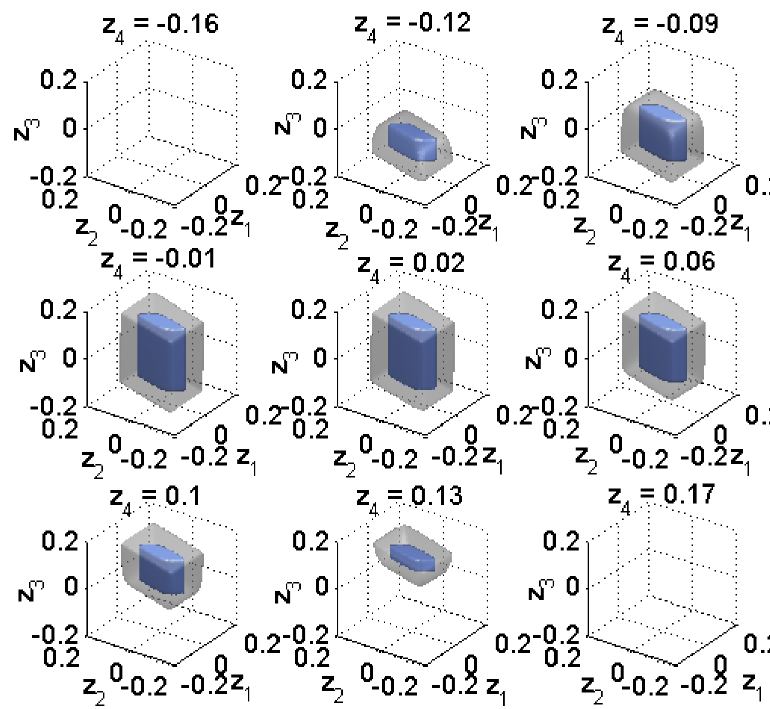
## Longitudinal aircraft dynamics (4D)

$$A = \begin{bmatrix} -0.0030 & +0.0390 & 0 & -0.3220 \\ -0.0650 & -0.3190 & +7.7400 & 0 \\ +0.0200 & -0.1010 & -0.4290 & 0 \\ 0 & 0 & +1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} +0.0100 \\ -0.1800 \\ -1.1600 \\ 0 \end{bmatrix}$$

*Source: A. Bryson, Control of Spacecraft and Aircraft. Princeton Univ. Press, 1994.*

$$\mathcal{Z}_0 = \left\{ z \in \mathbb{R}^4 \mid \|z\| > 0.15,\ z = T^{-1}x,\ x \in \mathcal{X}_0 \right\}$$

$$\mathcal{U} = \left\{ u \in \mathbb{R} \mid u \in [-13.3°, 13.3°] \right\}$$




Full-order: 17352.0 s
Transformation-based: 33.5 s

Produced using the Level-Set Toolbox

# Implications

- Formal verification:

$$\mathcal{Z}_\tau \cap T^{-1}\mathcal{I} = \emptyset \Leftrightarrow \widehat{\mathcal{X}}_\tau \cap \mathcal{I} = \emptyset \Rightarrow \mathcal{X}_\tau \cap \mathcal{I} = \emptyset$$

$$\left\{\mathcal{Z}_\tau^i \cap \mathrm{Proj}(T^{-1}\mathcal{I}, i)\right\}_{i=1}^{i=2} = \emptyset \Rightarrow \mathcal{X}_\tau \cap \mathcal{I} = \emptyset$$

- Safety-preserving control synthesis:

$$u(t) = \begin{cases} u_p(x,t), & x(t) \in \mathrm{int}\, \widehat{\mathcal{X}}_\tau^{\complement} \\ u^*(x,t), & x(t) \in \partial \widehat{\mathcal{X}}_\tau^{\complement} \end{cases}$$

# Summary

- Computations scale poorly with dimension
- Safety control, severe non-convexity
- Appropriate coordinate transformation
- Significant computational advantage
- Easily extendible to hybrid systems
- Motivated by safety-based control of anesthesia

THE UNIVERSITY OF BRITISH COLUMBIA

# Overapproximating the Reachable Sets of LTI Systems Through a Similarity Transformation

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# HJI PDE
## (non-disjoint control input)

$$\mathcal{S}(A,B): A = \mathrm{diag}(A_{11}, A_{12}),\ \ B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},\ \ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{U}$$

$$\mathcal{X}_0 = \{x \mid g(x) \leq 0\}$$

— — — — — — — — — — — — — — — — — — — — — — — — — — —

$$\frac{\partial \phi(x,t)}{\partial t} + \min\left\{0, H(x, \nabla_x \phi(x,t))\right\} = 0,$$

$$t \in [-\tau, 0], \quad \phi(x,0) = g(x)$$

$$H(x,p) = \sup_{u \in \mathcal{U}} \left(p^T A x + p^T B u\right)$$

# HJI PDE
## (non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \sup_{u \in \mathcal{U}} \Big( p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)$$

# HJI PDE
## (non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \sup_{u \in \mathcal{U}} \Big( p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)$$

# HJI PDE
## (non-disjoint control input)

$$H_{\mathcal{S}}(x,p) = \sup_{u\in\mathcal{U}} \Big( p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + p_1^T B_{11} u_1 + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + p_1^T B_{12} u_2 \Big)$$

$$H_{\mathcal{S}}(x,p) \neq \sum_{k=1}^{2} H_{\mathcal{S}_k}(x_k, p_k).$$

# HJI PDE
## (disjoint control input)

$$H_{\mathcal{S}}(x,p) = \sup_{u \in \mathcal{U}} \Big( p_1^T A_{11} x_1 + p_2^T A_{22} x_2 + \cancel{p_1^T B_{11} u_1} + p_2^T B_{22} u_2 + p_2^T B_{21} u_1 + \cancel{p_1^T B_{12} u_2} \Big)$$

$$H_{\mathcal{S}}(x,p) = \sum_{k=1}^{2} H_{\mathcal{S}_k}(x_k, p_k).$$

# Attainability vs. Reachability (HJI PDE)

$$\mathcal{X}_0 = \{x \mid g(x) \le 0\},\ t \in [-\tau, 0]$$

$$\nabla_t \phi(x,t) + \min\left\{0, H(x, \nabla_x \phi(x,t))\right\} = 0, \quad \phi(x,0) = g(x),$$

| Reachability | Attainability |
| --- | --- |
| $H(x,p) = \sup_{u \in \mathcal{U}} \inf_{d \in \mathcal{D}} p^T f(x,u,d)$ | $H(x,p) = \inf_{u \in \mathcal{U}} \sup_{d \in \mathcal{D}} p^T f(x,u,d)$ |