CTG Reading Group

An Introduction to Convex Optimization for Communications and Signal Processing

by Zhi-Quan Luo and Wei Yu (IEEE JSAC, Aug. 2006)

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Outline

- Part 1: Introduction
- Part 2: Background on Convex Optimization
- Part 3: Applications to Communications
- Summary

Part 1 – Introduction

- **Motivation:**
  Many communication problems → Convex optimization problems

- **Convex optimization:**
  → Minimize convex objective function
  → Optimal solutions
  → Efficient calculation

- **Applications:**
  → Optimization concepts used in engineering (Lagrangian duality, SDP relaxation method, etc.)

Part 2 – Background

Basic Concepts (1/2)

- **Convex sets:**

$$\theta x + (1-\theta) y \in S, \quad \forall \theta \in [0,1] \text{ and } x, y \in S$$

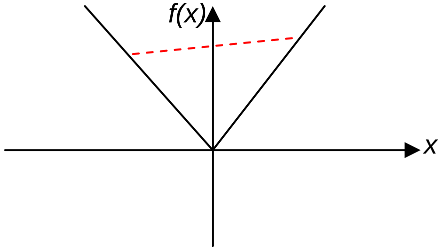
e.g.: The unit ball, $S = \left\{ x \,\middle|\, \|x\| \leq 1 \right\}$

- **Convex cones:**
  → Special convex set closed under positive scaling
  → E.g.: Nonnegative orthant, SOC, etc.

- **Convex functions:**

$$f\left(\theta x + (1-\theta) y\right) \leq \theta f(x) + (1-\theta) f(y), \quad \forall \theta \in [0,1]$$

e.g.: $f(x) = |x|$



Part 2 – Background

Basic Concepts (2/2)

- **Convex optimization problems:**

Requirements for convexity

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \le 0, \; i = 1, 2, \ldots, m, \\
& h_j(x) = 0, \; j = 1, 2, \ldots, r, \\
& x \in S
\end{aligned}
$$

Convex

Affine

Part 2 – Background

Properties (1/2)

- **Lagrangian Duality:**

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le 0,\ i = 1,2,\ldots,m, \\ & h_j(x) = 0,\ j = 1,2,\ldots,r, \\ & x \in S \end{aligned}$$

→ **Primal optimization problem**

$$\begin{aligned} \text{maximize} \quad & g(\lambda, \upsilon) \\ \text{subject to} \quad & \lambda \ge 0,\ \upsilon \in \Re^r \end{aligned}$$

→ **Dual optimization problem**

$$g(\lambda, \upsilon) := \min_{x \in S} L(x, \lambda, \upsilon)$$

$$L(x, \lambda, \upsilon) := f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{r} \upsilon_j h_j(x)$$

Part 2 – Background

Properties (2/2)

- **Karush-Kuhn-Tucker (KKT) Condition:**

Local optimality condition

$$f_i(x^*) \leq 0, \ \forall i = 1,2,\ldots,m$$

$$h_j(x^*) = 0, \ \forall j = 1,2,\ldots,r$$

→ **Primal feasibility of *x****

$$\lambda^* \geq 0,$$

→ **Dual feasibility**

$$\lambda_i^* f_i(x^*) = 0, \ \forall i = 1,2,\ldots,m$$

→ **Primal and dual feasibility**

$$\nabla_x L(x^*, \lambda^*, \upsilon^*) = 0$$

Part 3 – Applications

Conic Programming (1/3)

- **Downlink Beamforming Problem:**

$$\text{minimize} \quad \sum_{j=1}^{K} \|w_j\|^2$$
$$\text{subject to} \quad \frac{|h_i^H w_i|^2}{\sum_{j \neq i} |h_i^H w_i|^2 + \sigma^2} \geq \lambda_i, \quad \forall i$$

Not convex!

SDP relaxation →

$$\text{minimize} \quad \sum_{j=1}^{K} \mathrm{tr}(B_i)$$
$$\text{subject to} \quad tr(H_i B_i) - \gamma_i \sum_{j \neq i} tr(H_i B_j) \geq \gamma_i \sigma^2$$
$$B_i \succeq 0, \quad B_i \text{ is complex Hermitian}$$

SOCP →

$$\text{minimize} \quad \tau$$
$$\text{subject to} \quad \sqrt{1 + \frac{1}{\gamma_i}}\, h_i^H w_i \geq \left\| \begin{matrix} h_i^H W \\ \sigma \end{matrix} \right\|, \ \forall i$$
$$\sum_{j=1}^{K} \|w_j\| \leq \tau$$

Part 3 – Applications

Conic Programming (2/3)

- **Uplink-Downlink Duality:**

Downlink

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^{K} \|w_j\|^2 \\ &\text{subject to} && \frac{|h_i^H w_i|^2}{\sum_{j\neq i} |h_i^H w_i|^2 + \sigma^2} \geq \lambda_i, \quad \forall i \end{aligned}$$

Not convex!

Uplink

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{K} \lambda_i \sigma^2 \\ &\text{subject to} && \sum_{j=1}^{K} \lambda_j h_j h_j^H + I \succ \left(1 + \frac{1}{\gamma_i}\right) \lambda_i h_i h_i^H \end{aligned}$$

Part 3 – Applications

Conic Programming (3/3)

- **Capacity Region Duality:**

Broadcast channel

$$\text{maximize} \quad \sum_{k=1}^{K} \mu_k \log \frac{\left| H_k \left(\sum_{i=1}^{k} S_i \right) H_k^H + I \right|}{\left| H_k \left(\sum_{i=1}^{k-1} S_i \right) H_k^H + I \right|}$$

$$\text{subject to} \quad \sum_{i=1}^{K} \mathrm{tr}(S_i) \le \rho_T, \quad S_i \succeq 0$$

Not convex!

Multiple-access channel

$$\text{maximize} \quad \sum_{k=1}^{K} \mu_k \log \frac{\left| \sum_{i=1}^{k} H_i^H \hat{S}_i H_i + I \right|}{\left| \sum_{i=1}^{k-1} H_i^H \hat{S}_i H_i + I \right|}$$

$$\text{subject to} \quad \sum_{i=1}^{K} \mathrm{tr}(\hat{S}_i) \le \rho_T, \quad \hat{S}_i \succeq 0$$

Part 3 – Applications

SDP Relaxations

- **Two Examples:**
  → Eg-1: Multiuser Detection
  → Eg-2: Multicast Beamforming

  → Problem formulation: Nonconvex!
  → Use SDP relaxations (Refer to Slide-8)

Robust Optimization

Ideal case

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1,2,\ldots,m \end{array}$$

→

Corrupted by noise

$$\begin{array}{ll} \text{minimize} & \max_{\delta \in \Delta} f_0(x;\delta) \\ \text{subject to} & f_i(x;\delta) \le 0,\ \forall \delta \in \Delta,\ i = 1,2,\ldots,m \end{array}$$

Summary

- Convex optimization: Powerful tool in engineering
- Helpful concepts: Duality, SDP relaxation
- Open problems: Lots!

Further reading:

S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004. (also *freely* available online)