A Scalable Network Resource Allocation Mechanism With Bounded Efficiency Loss

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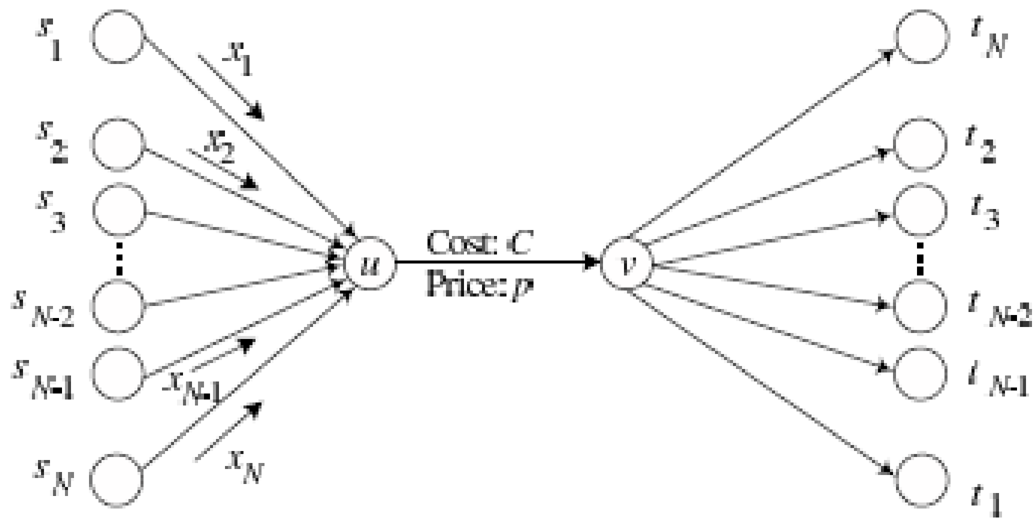

$C(q)$ is convex and nondecreasing.

Assumption 1: For each n, and over the domain $x_n \geq 0$, the utility function $U_n(x_n)$ is concave, nondecreasing, and continuously differentiable (where we interpret $U'_n(0)$ as the right directional derivative of U_n at 0).

We assume that both utility and cost are measured in monetary units, so that an efficient allocation is characterized as an optimal solution of the following optimization problem:

$$\text{maximize} \quad \sum_n U_n(x_n) - C\left(\sum_n x_n\right) \qquad (1)$$

$$\text{subject to} \quad x_n \geq 0, \quad n = 1, \ldots, N. \qquad (2)$$

We refer to the objective function (1) as the *aggregate surplus* [17]. Since $p(q) \to \infty$ as $q \to \infty$, while U_n only grows at most linearly, it follows that an optimal solution exists.

Notice that this problem is convex and easy to solve in a centralized fashion.

However, we are talking about a distributed network with selfish users!

1. Price Taker Users (Users are not really smart!)

Each user n chooses a desired rate x_n. Given the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, the link sets a single price $\mu(\boldsymbol{x}) = p(\sum_n x_n)$. User n then pays $x_n \mu(\boldsymbol{x})$. We first consider the case where, given a price $\mu > 0$, user n chooses x_n to maximize

$$P_n(x_n; \mu) = U_n(x_n) - \mu x_n. \tag{3}$$

Notice that in the previous expression, each user is acting as a *price taker*; that is, he does not anticipate the effect of a change in his strategy on the resulting price.

Proposition 1: There exists a *competitive equilibrium*
, that is, a vector $\boldsymbol{x}$ and a scalar
μ such that $\mu = p(\sum_n x_n)$, and

$$P_n(x_n; \mu) = \max_{\overline{x}_n \geq 0} P_n(\overline{x}_n; \mu), \quad n = 1, \ldots, N. \tag{4}$$

Any such vector $\boldsymbol{x}$ solves (1) and (2). If the functions U_n are strictly concave, such a vector $\boldsymbol{x}$ is unique.

2. Price Anticipator Users (Smart Users!)

When the price taking assumption is violated, however, the model changes into a game and the guarantee of Proposition 1 is no longer valid.

Q: What do we mean by "price anticipating?"

We use the notation $\boldsymbol{x}_{-n}$ to denote the vector of all rates chosen by users other than n; i.e., $\boldsymbol{x}_{-n} = (x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$. Then, given $\boldsymbol{x}_{-n}$, each user n chooses $x_n \geq 0$ to maximize

$$Q_n(x_n; x_{-n}) = U_n(x_n) - x_n p\left(\sum_m x_m\right). \quad (5)$$

The payoff function Q_n is similar to the payoff function P_n, except that the user now *anticipates* that the price will be set according to $p(\sum_m x_m)$. A *Nash equilibrium* of the game defined by $(Q_1, \ldots, Q_N)$ is a vector $\boldsymbol{x} \geq 0$ such that for all n

$$Q_n(x_n; \boldsymbol{x}_{-n}) \geq Q_n(\overline{x}_n; \boldsymbol{x}_{-n}), \quad \text{for all } \overline{x}_n \geq 0. \quad (6)$$

Minor Questions:

1. Does the Nash equilibrium always exist?

   Answer: Yes. → Rosen's Existence Theorem → Convex Game

2. Does the Nash equilibrium unique?

   Answer: Yes.

3. How can we obtain Nash equilibrium?

   Answer: By analyzing users' "**Best Reponses**".

Because the payoff Q_n is concave in x_n for fixed $\boldsymbol{x}_{-n}$, a vector $\boldsymbol{x}$ is a Nash equilibrium if and only if the following first-order conditions are satisfied for each n, where $q = \sum_m x_m$:

$$U_n'(x_n) = p(q) + x_n p'(q), \quad \text{if } x_n > 0; \tag{7}$$

$$U_n'(0) \leq p(q), \quad \text{if } x_n = 0. \tag{8}$$

Major Questions:

Before seeing the question, a few things you should keep in mind:

- Clearly, the network aggregate surplus (i.e., our objective function for our networking design problem) is not necessarily optimal at Nash equilibrium.
- Clearly, Nash equilibrium changes if we change the system parameters (e.g., utility functions, price functions, etc.)
  - This will also change the network efficiency

Here is our question:

What is the **worst-case efficiency** among all possible Nash equilibria points?

This is also called “Price of Anarchy”.

In fact, we want to know how bad the performance can be if the users are **smart** and **selfish** and DO NOT FOLLOW what the network admin wants.

Assumption: Prices are “affine”:

$$p(q) = a q + b$$

Lemma:

Worst-case efficiency occurs when the utility functions are also linear:

U_i(x) = \gamaa_i x_i

We start by computing the maximal aggregate surplus under these assumptions. Since the price function is $p(q) = aq + b$, the maximal aggregate surplus is achieved when $p(q^S) = 1$, i.e., when $q^S = (1 - b)/a$; this rate is entirely allocated to user 1. The maximal aggregate surplus is thus

$$\frac{1-b}{a} - \frac{(1-b)^2}{2a} - \frac{b(1-b)}{a} = \frac{(1-b)^2}{2a}.$$

Since the maximal aggregate surplus is fixed as $(1-b)^2/(2a)$, by (7) and (8) the worst case game is identified by solving the following optimization problem (with unknowns $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n, q$):

$$\text{minimize} \quad \sum_{n=1}^{N} \alpha_n x_n - C(q) \tag{12}$$

$$\text{subject to} \quad \alpha_n = p(q) + x_n p'(q), \text{ if } x_n > 0, \quad n = 1, \ldots, N \tag{13}$$

$$\alpha_n \leq p(q), \text{ if } x_n = 0, \quad n = 1, \ldots, N \tag{14}$$

$$\sum_{n=1}^{N} x_n = q > 0 \tag{15}$$

$$\alpha_1 = 1;\ 0 \leq \alpha_n \leq 1, \quad n = 2, \ldots, N \tag{16}$$

$$x_n \geq 0, \quad n = 1, \ldots, N. \tag{17}$$

We start by assuming that $q > 0$ is fixed, and optimize only over $\boldsymbol{x}$ and $\boldsymbol{\alpha}$. In this case, we start by noting that we may assume without loss of generality that $\alpha_n = p(q) + x_n p'(q)$ for all users $n = 2, \ldots, N$. Indeed, if $(\boldsymbol{\alpha}, \boldsymbol{x})$ is a feasible solution and $x_n > 0$ for some $n = 2, \ldots, N$, then (13) and (14) imply that $\alpha_n = p(q) + x_n p'(q)$. On the other hand, if $x_n = 0$ for some $n = 2, \ldots, N$, we can set $\alpha_n = p(q) = aq + b$; this preserves feasibility, but does not impact the term $\alpha_n x_n$ in the objective function (12). We can, therefore, restrict attention to feasible solutions for which

$$\alpha_n = p(q) + x_n p'(q) = aq + b + a x_n, \quad n = 2, \ldots, N. \quad (18)$$

Having done so, observe that the constraint (16), that $\alpha_n \leq 1$, may be written as

$$x_n \leq \frac{1 - aq - b}{a}, \quad n = 2, \ldots, N.$$

Finally, the constraint (16) that $\alpha_n \geq 0$ becomes redundant, as it is guaranteed by the fact that $a > 0$, $b \geq 0$, and $q > 0$.

It follows from (16) together with (13) that a candidate solution satisfying (15) can only exist if $x_1 > 0$, in which case we have $1 = p(q) + x_1 p'(q) = aq + b + ax_1$, so that $x_1 = (1-aq-b)/a$. In particular, we conclude immediately that for a feasible solution to exist, we must have $0 < (1-aq-b)/a \leq q$. This yields the following reduced optimization problem:

$$\text{minimize} \quad \frac{1-aq-b}{a} + \sum_{n=2}^{N} (aq+b+ax_n)x_n - C(q) \tag{19}$$

$$\text{subject to} \sum_{n=2}^{N} x_n = q - \frac{1-aq-b}{a} \tag{20}$$

$$x_n \leq \frac{1-aq-b}{a}, \quad n = 2, \ldots, N \tag{21}$$

$$x_n \geq 0, \quad n = 2, \ldots, N. \tag{22}$$

The objective function (19) is equivalent to (12) upon substitution for α_n [from (13)] and x_1 [also from (13)]. The constraint (20) is equivalent to the allocation constraint (15); and the constraint (21) ensures $\alpha_n \leq 1$, as required in (16).

For fixed $q > 0$, the resulting problem is symmetric in the rates x_n for $n = 2, \ldots, N$. It is clear that a feasible solution exists if and only if

$$\frac{q}{N} \leq \frac{1 - aq - b}{a} \leq q. \tag{23}$$

In this case, the following symmetric solution is feasible:

$$x_n = \frac{q - \frac{(1-aq-b)}{a}}{N - 1}.$$

Furthermore, since the objective function (19) is strictly convex, this symmetric solution must in fact be optimal. If we substitute in the objective function (19), the resulting optimal value is strictly decreasing as N increases; the worst case occurs as $N \to \infty$, and the optimal objective value (19) becomes

$$\begin{aligned} &\frac{1 - aq - b}{a} + (aq + b)\left(q - \frac{1 - aq - b}{a}\right) - C(q) \\ &\quad = \frac{1 - b}{a} - q + (aq + b)\left(2q - \frac{1 - b}{a}\right) - \frac{aq^2}{2} - bq. \end{aligned} \tag{24}$$

Until now, we have kept the price function and the total rate q fixed, and found the worst case game. We now optimize over all possible choices of price function p (i.e., over $a > 0$ and $b \geq 0$), as well as over possible Nash equilibrium rates (i.e., over $q > 0$). Recall that the maximal aggregate surplus is $(1-b)^2/(2a)$. Thus, the worst case ratio is identified by the following optimization problem over q, a, and b:

$$\begin{aligned} \text{minimize} \quad & \frac{2a}{(1-b)^2}\left(\frac{1-b}{a} - q + (aq+b)\right. \\ & \left. \times\left(2q - \frac{1-b}{a}\right) - \frac{aq^2}{2} - bq\right) \\ \text{subject to} \quad & \frac{1-b}{2} \leq aq \leq 1-b, \quad a > 0,\ b \geq 0,\ q > 0. \end{aligned}$$

If we let $\overline{a} = aq$, then this problem becomes equivalent to the following problem:

$$\begin{aligned} \text{minimize} \quad & \frac{2\overline{a}}{(1-b)^2} \\ & \times\left(\frac{1-b}{\overline{a}} - 1 + (\overline{a}+b)\left(2 - \frac{1-b}{\overline{a}}\right) - \frac{\overline{a}}{2} - b\right) \\ \text{subject to} \quad & \frac{1-b}{2} \leq \overline{a} \leq 1-b, \quad \overline{a} > 0,\ b \geq 0. \end{aligned}$$

If we define $x = \overline{a}/(1-b)$ then the above problem becomes a single variable convex optimization problem. → Optimal = 2/3.

Key result:

Theorem 3: Suppose that Assumption 1 holds, and that $p(q) = aq + b$ for some $a > 0$, $b \geq 0$. Suppose also that $U_n(0) \geq 0$ for all n. If $\boldsymbol{x}^S$ is any solution to (1), (2), and $\boldsymbol{x}$ is any Nash equilibrium of the game defined by $(Q_1, \ldots, Q_n)$, then

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \frac{2}{3}\left(\sum_n U_n\left(x_n^S\right) - C\left(\sum_n x_n^S\right)\right). \tag{9}$$

Q: What does that mean?