

# Power Flow Formulation Based on a Mixed-Linear and Nonlinear System of Equations

Hamed Ahmadi  
School of Electrical and  
Computer Engineering  
University of British Columbia  
Vancouver, BC  
Email: hameda@ece.ubc.ca

José R. Martí  
School of Electrical and  
Computer Engineering  
University of British Columbia  
Vancouver, BC  
Email: jrms@ece.ubc.ca

**Abstract**—Power flow solutions are the key in many power system studies. In this paper, the power flow problem is formulated within rectangular coordinates. Using a voltage-dependent load model and linear approximation techniques, the problem is formulated as a system of mixed linear and nonlinear equations. An efficient matrix decomposition is then applied to facilitate establishing the Jacobian matrix in the linear subproblems of Newton's method. The accuracy of the proposed method is compared to the original nonlinear formulation and relative errors less than 0.1% are achieved. The average computation time of the proposed method is about 30% less than the conventional method.

**Index Terms**—Power flow, load voltage dependency, LU Decomposition.

## NOMENCLATURE

$a_p, b_p$	Parameters representing the voltage dependency of load's active power.
$a_q, b_q$	Parameters representing the voltage dependency of load's reactive power.
$G, B$	Network conductance and susceptance matrices, respectively.
$J$	Jacobian matrix.
$m$	Number of generator buses.
$n$	Number of total buses.
$P, Q$	Active and reactive power, respectively.
$V$	Bus voltage magnitude.
$\bar{V}$	Complex bus voltage.
$\delta$	Bus voltage angle.
$\vartheta$	Imaginary part of bus voltage.
$\nu$	Real part of bus voltage.

## I. INTRODUCTION

Power flow analysis is one of the fundamental tools in power systems. Conventionally, the loads are treated as constant P-Q, i.e. fixed amounts of active and reactive power are consumed by the load regardless of the voltage at the bus delivering electricity to that particular load. However, the actual active and reactive power consumed by the loads are functions of the voltage magnitude and the frequency at the load terminals [1]. As the power system analysis is deployed for real-time system operation and control, more accurate and faster converging models are required. The aim of this study is to represent the

loads according to their actual behavior and introduce a power flow solution algorithm which is both numerically stable and fast converging.

Power flow equations essentially form a system of nonlinear equations. In order to solve this system, different methods have been proposed in the literature [2]. The iterative Newton method (also known as Newton-Raphson method) has been widely accepted and used [3], [4]. The basic idea of this method is to replace the nonlinear functions by their first-order Taylor series expansion around a starting point, then iterate until the difference between two sequential results is less than some tolerance. This method involves solving a system of linear equations at each iteration. Different methods are available in the literature for solving systems of linear equations [2].

One of the widely used methods for solving systems of linear equations is the LU factorization [2]. In this method, the matrix of coefficients of linear equations is factorized into a product of two new matrices, one lower triangular (L) and the other upper-triangular (U). Although this method is fast and efficient, it may not lead to a solution in cases of ill-conditioned systems [5]. Modified versions of Newton's method have been applied to overcome this problem [6]. Other techniques have also been studied to increase the speed and/or robustness of linear subproblems. These include the Quasi-Newton method [7], the inexact Newton method [8], the Krylov method [9], and the Jacobian-free Newton-GMRES method [10].

In addition to the different methodologies applied to solve the power flow equations, there are also studies focused on the formulation of the problem in different ways. Traditionally, the power flow is formulated in polar coordinates, which are the angle and magnitude of the bus voltages [3]. Alternatively, it is possible to formulate the problem in rectangular coordinate, which are the real and imaginary parts of the bus voltages [11]. It is shown in [11] that this formulation can often lead to faster solutions because some parts of the Jacobian matrix are fixed and do not require recalculation at each iteration. However, the Jacobian matrix has no constant blocks because in the part that contains constant elements, the diagonal entries are still variable at each iteration. The method proposed in our

paper addresses this issue by providing constant blocks in the Jacobian matrix.

In conventional power flow algorithms, a constant P-Q model for the loads is assumed, while the real load behavior is basically different. It is shown in a real B.C. Hydro' system [12] that by reducing the voltage magnitude by 2.6% the active and reactive power consumption of the aggregated load connected to a feeder are reduced by 4% and 9%, respectively. This results indicate the importance of modeling the load voltage dependency in power system studies. Voltage behavior of a variety of loads is studied and reported in terms of classical representations in [13]-[15]. The IEEE recommendations for load modeling [16] presents a framework to divide the loads into three major groups, namely residential, commercial and industrial loads.

In this paper, a voltage-dependent load model is used which leads to a faster and more efficient power flow solution. It is shown in [17] that a voltage-dependent load model also leads to a linear power flow formulation. Some numerical techniques from linear algebra are also applied to facilitate the solution procedure. The new formulation takes advantage of the limits imposed by the system operation on the complex voltages and replaces the nonlinear functions with good linear approximations. This leads to linear equations for the load buses and bilinear equations for the generation buses. For this reason, we will refer to this method as mixed linear and nonlinear power flow formulation. With this method, a major part of the Jacobian matrix remains constant and only a small part needs to be updated per iteration. A special block LU factorization is adopted to form the LU factorization automatically, which further enhances the solution procedure.

The rest of the paper is organized as follows. In Section II, the load modeling and power flow formulation are discussed. Numerical results are reported in Section III. The main findings of the study are reviewed in Section IV.

## II. MIXED LINEAR-NONLINEAR POWER FLOW FORMULATION

### A. Load Modeling

The voltage dependency of loads has been described previously using two well-known models, namely the exponential model and the polynomial (ZIP) model [1]. Mathematically, the problem of modeling the load voltage dependency is essentially a curve-fitting problem. Point-wise measurements of voltage-active power ( $V, P$ ) and voltage-reactive power ( $V, Q$ ) data can be fitted with appropriate curves. The exponential model has only one parameter to be determined for  $P$  and one for  $Q$ . The polynomial model, on the other hand, has three parameters. The model employed in this paper has two parameters, as suggested in [17], as follows:

$$\frac{P(V)}{P_0} = a_p \left( \frac{V}{V_0} \right)^2 + b_p \left( \frac{V}{V_0} \right) \quad (1a)$$

$$\frac{Q(V)}{Q_0} = a_q \left( \frac{V}{V_0} \right)^2 + b_q \left( \frac{V}{V_0} \right) \quad (1b)$$

where the zero subscripts stand for the nominal values.

Although this model has less free variables to fit the curve, the performance is still reasonable. Also note that the sum of parameters in (1) should be unity (e.g.  $a_p + b_p = 1$ ). In order to find the values of the parameters in (1), a least-squares problem can be solved. This problem and its solution are discussed in Appendix A. The next section discusses how this load model can enhance the power flow solution.

### B. Power Flow Formulation

In power flow calculations, three types of buses are assumed: the generation buses, or P-V buses, for which the amount of active power generated and the voltage magnitude are fixed by the generator prime mover and excitation system, respectively; the load buses, or P-Q buses, for which the amount of active and reactive power demands are known; and the slack bus, for which the real and imaginary parts of the voltage are known. Instead of solving for voltage magnitudes and angles, we formulate the problem in a way that the variables are real and imaginary parts of the bus voltages. The sum of the currents being injected to Bus  $i$  from the network can be calculated as

$$I_i = \sum_{k=1}^n \bar{V}_k Y_{ik} = \sum_{k=1}^n (G_{ik}\nu_k + B_{ik}\vartheta_k) + j \sum_{k=1}^n (B_{ik}\nu_k - G_{ik}\vartheta_k) \quad (2)$$

The net apparent power injected to Bus  $i$  is calculated as

$$S_i^* = P_i - jQ_i = \bar{V}_i^* I_i \quad (3)$$

Substituting the value of  $I_i$  from (2) into (3), the net active and reactive power injections can be derived as

$$P_i = \nu_i \sum_{k=1}^n (G_{ik}\nu_k - B_{ik}\vartheta_k) + \vartheta_i \sum_{k=1}^n (B_{ik}\nu_k - G_{ik}\vartheta_k) \quad (4a)$$

$$Q_i = \vartheta_i \sum_{k=1}^n (G_{ik}\nu_k - B_{ik}\vartheta_k) - \nu_i \sum_{k=1}^n (B_{ik}\nu_k - G_{ik}\vartheta_k) \quad (4b)$$

For a P-V bus, the left-hand side of (4a) is a constant value. In addition, the fixed voltage magnitude at a P-V bus enforce the following equation:

$$\nu_i^2 + \vartheta_i^2 = V_i^2 \quad (5)$$

Note that since the reactive power generation, as a dependent variable, is not known, (5) is considered in the formulation of power flow for the P-V buses instead of (4b).

For a P-Q bus, the voltage-dependent load model introduced in (1) is used to model the load behavior. Using the current injection method, the total current flowing to a bus has to be equal to the current drawing at that bus. The drawn current at Bus  $i$  with a voltage-dependent load described by (1) is

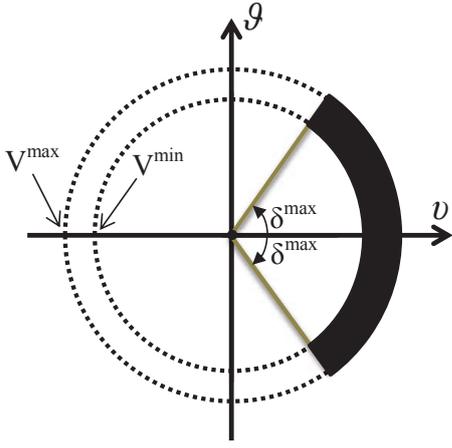


Figure 1. Limits on the voltage real and imaginary parts. The shadowed area shows the allowable operating range.

Table I  
PARAMETERS FOR THE LINEAR APPROXIMATION.

$u_0$	$u_1$	$u_2$	$\hat{u}_0$	$\hat{u}_1$	$\hat{u}_2$
0.0146	0.9820	0	0	0	0.9976

calculated, in per unit, as

$$I_i = \frac{P_i - jQ_i}{\bar{V}_i^*} = \left[ (a_{p,i} + b_{p,i} \frac{1}{\bar{V}_i}) \nu_i + (a_{q,i} + b_{q,i} \frac{1}{\bar{V}_i}) \vartheta_i \right] + j \left[ (a_{p,i} + b_{p,i} \frac{1}{\bar{V}_i}) \vartheta_i - (a_{q,i} + b_{q,i} \frac{1}{\bar{V}_i}) \nu_i \right] \quad (6)$$

In normal operation of power systems, the voltage magnitudes are usually kept within  $\pm 5\%$  (i.e.  $V^{\min} = 0.95\text{p.u.}$  and  $V^{\max} = 1.05\text{p.u.}$ ). Besides, the bus voltage angles are limited due to angle stability constraints (i.e.  $\delta_i \leq \delta^{\max}$ ). This is visualized in Fig. 1. With these considerations, the operating range for real and imaginary parts of the voltage are limited to the narrow area shown in Fig. 1. Based on this fact, the nonlinear parts of (6), i.e.  $\frac{\nu_i}{\bar{V}_i}$  and  $\frac{\vartheta_i}{\bar{V}_i}$ , can be replaced by linear approximations. Mathematically, one can write the following for a general bus:

$$\frac{\nu}{V} = \frac{\nu}{\sqrt{\nu^2 + \vartheta^2}} \approx u_0 + u_1 \nu + u_2 \vartheta \quad (7a)$$

$$\frac{\vartheta}{V} = \frac{\vartheta}{\sqrt{\nu^2 + \vartheta^2}} \approx \hat{u}_0 + \hat{u}_1 \nu + \hat{u}_2 \vartheta \quad (7b)$$

In order to find the parameters in (7), a least-squares problem needs to be solved. The details are explained in Appendix B. The results of these linearization are given in Table I. Note that these calculations are all done off-line only once and the results are used afterwards in power flow solutions.

Now, we are ready to present the power flow formulation. For the P-V buses, it suffices to have (4a) and (5), which are  $2(m-1)$  bilinear equations in  $\nu$  and  $\vartheta$  (by dropping the

$$\begin{bmatrix} \boxed{2(n-m) \times 2(n-1)} \\ \boxed{2(m-1) \times 2(n-1)} \end{bmatrix} = \begin{bmatrix} \boxed{J_{11}} & \boxed{J_{12}} \\ \boxed{J_{21}} & \boxed{J_{22}} \end{bmatrix}$$

Figure 2. Jacobian matrix pattern for the mixed linear and nonlinear system of equations. The first row block contains constant values.

equations for the slack bus). Kirchhoff's Current Law imposes that (6) and (2) have to be equal. Separating the real and imaginary parts leads to:

$$a_{p,i} \nu_i + b_{p,i} (u_0 + u_1 \nu_i + u_2 \vartheta_i) + a_{q,i} \vartheta_i + b_{q,i} (\hat{u}_0 + \hat{u}_1 \nu_i + \hat{u}_2 \vartheta_i) = \sum_{j=1}^n (G_{ij} \nu_j + B_{ij} \vartheta_j) \quad (8a)$$

$$a_{p,i} \vartheta_i + b_{p,i} (\hat{u}_0 + \hat{u}_1 \nu_i + \hat{u}_2 \vartheta_i) - a_{q,i} \nu_i - b_{q,i} (u_0 + u_1 \nu_i + u_2 \vartheta_i) = \sum_{j=1}^n (B_{ij} \nu_j - G_{ij} \vartheta_j) \quad (8b)$$

which yields a system of  $2(n-m)$  linear equations on  $\nu$  and  $\vartheta$ .

### C. Solution of the Power Flow Problem

As discussed above, we have now a system of mixed linear and nonlinear equations to solve. The Newton method suggests the utilization of the Jacobian matrix and iterative solution of linear subproblems. Ordering the equations appropriately, i.e. the P-Q buses first and then the P-V buses, yields a special structure for the Jacobian matrix. The first-order partial derivatives of the linear equations are constant and do not require recalculation at each iteration. It is worthwhile to mention that this constant part of the matrix is indeed comparatively larger than the variable part based on the fact that there are more load buses in the network than generation buses. This fact is pictorially represented in Fig. 2 by the size of the blocks.

It is of interest to take advantage of the special structure of the Jacobian matrix. Let the upper block in the Jacobian matrix in Fig. 2 be horizontally divided into two block columns, namely  $J_{11} \in \mathbb{R}^{2(n-m) \times 2(n-m)}$  and  $J_{12} \in \mathbb{R}^{2(n-m) \times 2(m-1)}$ . Similarly, divide the second block into two block columns, namely  $J_{21} \in \mathbb{R}^{2(m-1) \times 2(n-m)}$  and  $J_{22} \in \mathbb{R}^{2(m-1) \times 2(m-1)}$ . Then, linear algebra allows us to write

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ J_{21} J_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} J_{11} & 0 \\ 0 & Q_S \end{bmatrix} \begin{bmatrix} I & J_{11}^{-1} J_{12} \\ 0 & I \end{bmatrix} \quad (9)$$

where  $I$  and  $0$  are the identity and zero matrices of appropriate size;  $Q_S$  is the Schur complement of  $J$  defined by:

$$Q_S = J_{22} - J_{21}J_{11}^{-1}J_{12} \quad (10)$$

The following remarks follow from (9):

- This is a factorization of the Jacobian matrix into lower triangular, block diagonal, and upper triangular matrices (LDU decomposition).
- $J_{11}$  is a constant square matrix which is the only matrix that appears in inverse form. Thus, just one matrix inverse calculation before the iterative process suffices.
- The LDU decomposition facilitates the solution of the linear subproblems at each iteration.

Preconditioning techniques are also applicable to this process to take care of the ill-conditioned matrices. This is beyond the scope of this study and is well-established in the literature, e.g. [18].

### III. NUMERICAL RESULTS

In this section, the power flow problem is solved using the IEEE 300-bus test system of [19]. This test system consists of 69 generator buses and 411 branches. For illustrative purposes, the buses are reordered so that the P-V buses are written first. In a real system, each bus (which is usually a substation) may show different voltage behavior according to its load composition. The voltage dependency of the loads can be approximately considered as constant during some time interval (e.g. one hour). With this assumption, a few field tests can provide a general idea for choosing the values of the parameters in (1). For instance, a step change in voltage, which may occur naturally by the operation of under load tap-changing transformers, gives a good approximation of load voltage dependency [12]. In order to have an accurate enough model of load voltage dependency, it is inevitable that a through study is required throughout the system. In this paper, we assume that the parameters in (1) are known beforehand. For simplicity and without loss of generality, we assume that the parameters for active and reactive powers are equal. Furthermore, we assume that all the buses have the same parameters.

As claimed earlier, the power flow solution obtained using the linear approximation and the original nonlinear formulation are reasonably close. This is shown here by the numerical results for the IEEE 300-bus test system. Figure 3 shows the density of the relative error between the voltage magnitudes obtained using the mixed linear and nonlinear method and the original nonlinear formulation. The original power flow is solved using MATPOWER [20]. The error is calculated using the following formula:

$$\eta_i = 100 \frac{|V_i - V_i'|}{V_i} \quad (11)$$

where  $V_i$  is obtained using the original formulation and  $V_i'$  is obtained using the mixed linear and nonlinear formulation. As can be seen, the error is quite negligible and, in this case, is

always less than 0.1%.

In order to show the impact of the voltage dependency in the load modeling on the power flow solution, two well-known cases are simulated. In the first case, the loads are modeled as constant-impedance ( $a_p = 1, b_p = 0$ ). In the second case, the loads are modeled as constant-current ( $a_p = 0, b_p = 1$ ). In general, a mixture of the constant-current and constant-impedance load models can be used to represent the voltage dependency of actual loads. The differences in terms of voltage magnitudes for the two cases are calculated by (11) and the

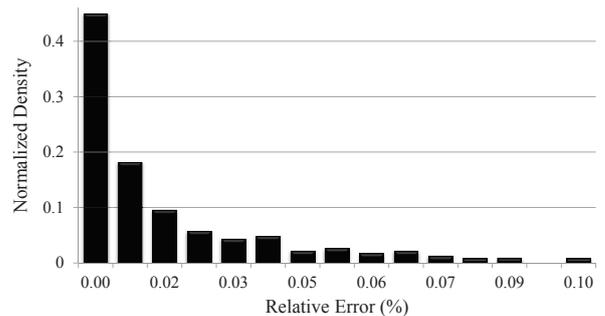


Figure 3. Normalized density of the relative voltage magnitude error ( $\eta$ ) between the results obtained using the linear approximation formulation and the original nonlinear formulation using the IEEE 300-bus test system. ( $a_p = 0.5, b_p = 0.5$ )

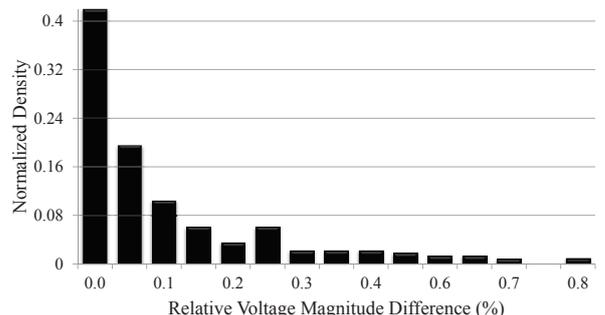


Figure 4. Normalized density of the relative voltage difference between two load models: constant-impedance  $a_p = 1, b_p = 0$  and constant-current  $a_p = 0, b_p = 1$  using the IEEE 300-bus test system.

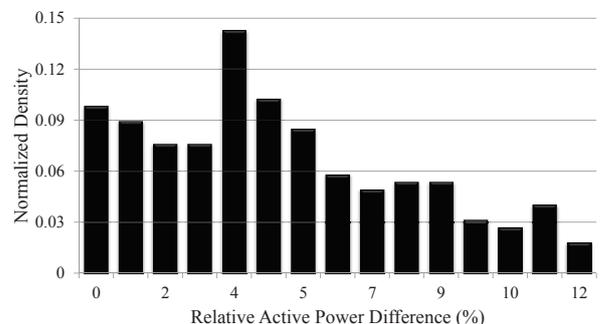


Figure 5. Normalized density of the relative difference between the load active power demand for two load models: constant-impedance  $a_p = 1, b_p = 0$  and constant-current  $a_p = 0, b_p = 1$  using the IEEE 300-bus test system.

corresponding error density is shown in Fig. 4. As can be seen, the voltage differences are always less than 0.8%. However, by comparing the load active power consumption, the impact of load modeling is more clear. Figure 5 shows the density of the active power differences for the two cases. As can be seen, the different load models yield variations of up to 12% in active power consumption. Similar values are also obtained for the reactive power.

A comparison in the solution of the power flow problem showed that an average time saving (for 100 times of running the solution routine) of about 30% can be achieved by using the mixed-linear and nonlinear formulation, as compared to the nonlinear formulation solved by MATPOWER [20]. In MATPOWER, a standard Newton's method is used to solve the system of nonlinear equations.

#### IV. CONCLUSION

The loads' voltage dependency characteristics are used in this paper along with a linear approximation technique to reformulate the general power flow problem. With the majority of the equations being linear, advantage can be taken in the numerical process to solve the system of mixed linear and nonlinear equations. Using the technical limits imposed on the voltage in a system, a narrow operating range for the real and imaginary parts of the voltage is derived which allows for efficient linearization of the nonlinear functions where these variables appear in the formulation. The ideas described in this paper have also the potential of being applied to speed up the optimal power flow calculations. This application is still under investigations by the authors and will be addressed in future work.

#### APPENDIX A

##### LEAST-SQUARES FOR CURVE FITTING

Assume a data set of  $N$  couples  $(V, P)$  is provided from measurements in which the voltages are in per unit of the nominal voltage and the powers are in per unit of the nominal power. A least-squares problem would find the best fit to this set of data for the model given in (1). Note that in (1), the following always hold:

$$a_p + b_p = 1 \quad (12)$$

Therefore, the least-squares problem is formulated as follows:

$$\min . f = \sum_{i=1}^N (a_p v_i^2 + (1 - a_p)v_i - p_i)^2 \quad (13)$$

The same procedure can be applied for the reactive power. To solve the least-squares problem, partial derivatives with respect to  $a_p$  have to be derived:

$$\frac{\partial f}{\partial a_p} = 2 \sum_{i=1}^N [(v_i^2 - v_i)(a_p(v_i^2 - v_i) - p_i + v_i)] \quad (14)$$

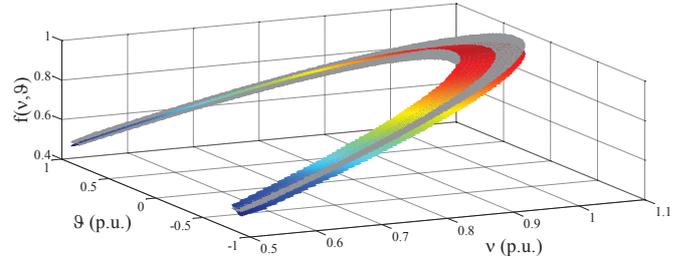


Figure 6. Linear approximation ( $\hat{f}$ ) and original nonlinear function ( $f$ ) for  $V^{\max} = 1.05$ ,  $V^{\min} = 0.95$ ,  $\delta_{\max} = 60^\circ$ .

The extreme point of  $f$  occurs at the root of above equation, which is

$$a_p = \frac{\sum_{i=1}^N (v_i^2 - v_i)(p_i - v_i)}{\sum_{i=1}^N (v_i^2 - v_i)^2} \quad (15)$$

To retrieve the value of  $b_p$ , substitute the value of  $a_p$  into (12).

#### APPENDIX B

##### LEAST-SQUARES FOR LINEAR APPROXIMATION OF NONLINEAR FUNCTIONS

The same approach pursued before can be used to find linear approximations for the nonlinear functions in (7). Assume  $N$  points of function evaluation of  $f(v, \vartheta) = v/\sqrt{v^2 + \vartheta^2}$  and the approximation to be  $\hat{f}(v, \vartheta) = u_0 + u_1 v + u_2 \vartheta$ . The least-squares solution yields (all the sums are from 1 to  $N$ ):

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sum v_i^2 & \sum v_i \vartheta_i & \sum v_i \\ \sum v_i \vartheta_i & \sum \vartheta_i^2 & \sum \vartheta_i \\ \sum v_i & \sum \vartheta_i & N \end{bmatrix}^{-1} \begin{bmatrix} \sum v_i f_i \\ \sum \vartheta_i f_i \\ \sum f_i \end{bmatrix} \quad (16)$$

Figure 6 shows the approximation ( $\hat{f}$ ) and the original function ( $f$ ).

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