Linear Power Flow Formulation Based on a Voltage-Dependent Load Model

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***Abstract*—The power flow (PF) solution is a fundamental tool in power system analysis. Standard PF formulations are based on the solution of a system of nonlinear equations which are computationally expensive due to the iterations needed. On the other hand, distribution system (DS) automation algorithms should be fast enough to meet real-time performance. The conventional load representation, as constant P-Q, becomes less accurate as we get closer to the actual load's level. In this paper, a new load model is proposed which represents the loads' voltage dependency. A simple curve-fitting technique is used to derive a voltage-dependent load model which splits the load as a combination of an impedance and a current source. With this representation and some numerical approximations on the imaginary part of the nodal voltages, it is possible to formulate the load flow problem as a linear power flow (LPF) solution which does not require iterations. The approximation has been tested in systems up to three thousand nodes with excellent results. The LPF formulation is particularly important in the context of optimization algorithms for automated smart distribution systems. The extension of the technique to unbalanced distribution systems will be presented in future work.**

***Index Terms*—Distribution system, linear power flow analysis, voltage-dependent load model.**

I. Introduction

THE present movement towards distribution systems automation involves the deployment of an automatic measuring infrastructure (AMI) which will make vast amount of information available for improved system operation. This information will make it possible to have a better understanding of the load's voltage dependency. An essential part of DS automation is to provide computationally efficient calculations for online real-time automated actions. Power flow solutions are an essential part of these calculations. There has been a large effort throughout the years towards enhancing power flow calculations in terms of computational speed and convergence characteristics. At the transmission system level, fast and reliable methods such as Gauss-Seidel [1], Newton-Raphson [2] and fast decoupled PF [3] have been used for many years. Distribution systems, however, are different from transmission systems in a number of aspects, such as the X/R ratio, the line's length, the use of underground cables, radial structures, and the unbalance of the phase currents. Due to these differences, PF algorithms developed for transmission systems often fail [4] or lose efficiency when applied to distribution systems. A number of PF solution methods have been developed to account for the specific nature of distribution systems, including the *backward/forward sweep* method [5] and the *ladder network theory* [6] method.

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The backward/forward sweep method is an iterative algorithm which is relatively time-demanding, especially for real-sized distribution systems. An improved version of this method is presented in [7] for radial DS in which the linear proportional principle is adopted to find the ratio of the real and imaginary components of the specified voltages with respect to the calculated voltage at the substation bus during the forward sweep. Using this method in a typical test system, the execution time has been reduced by 35.7% compared to the conventional backward/forward sweep. The PF problem has also been formulated as biquadratic equations in [8], which is still based on iterative computations of backward/forward sweeps. A direct method for PF solutions is proposed in [9] in which the loads are treated first as current source injections, and simple matrix calculations and iterative computations are used to find the bus voltages. A recursive formulation is proposed in [10] which includes three nonlinear equations for each branch, called *DistFlow* equations, that can be solved using Newton's method. By defining new variables in the PF formulation, the problem is converted in [11] into a convex optimization problem, or specifically, a conic quadratic problem; such problems can be solved by interior point methods. Some other methods for PF analysis in DS are also available in the literature, e.g. [12]-[15].

In all the mentioned references on PF calculations, the loads are modeled as constant P-Q. However, as the system gets closer to the loads voltage levels, the voltage dependency of the actual loads becomes more important in the representation. For example, in the framework of *Voltage VAR Optimization* (VVO), the load voltage dependency plays an inevitable role and the performance of the VVO algorithms are highly dependent on the accuracy of load modeling [16]. In addition, the introduction of distributed generation (DG) in distribution systems may change the unidirectional characteristics of the power flow, on which some of the previous methods have been built. Also, there might exist weakly-meshed DS in addition to the radial ones, which creates an additional limiting factor for some of the aforementioned approaches.

The load voltage dependency has an important impact on the load power consumption. It is shown in the B.C. Hydro system that decreasing the substation voltage by 1%, the active and reactive demand decrease by 1.5% and 3.4%, respectively [17]. The first author has studied different types of loads, both experimentally [18] and theoretically [19], to assess their voltage dependency. In the traditional approach, the voltage-dependent behavior of the load for voltages around the operating point has been approximated by exponential or polynomial functions [20]. From a modeling perspective, this is a problem of fitting an appropriate curve to a plot of active power versus voltage and a plot of reactive power versus voltage, with data collected from measurements along the time line. This paper proposes a synthesis of the voltage dependency of the loads using an impedance (Z) in parallel with a current source (I). In a real-time application, the values of the impedance and current source are recalculated as the load demand changes.

Choosing the values of Z and I adequately results in a voltage-dependent characteristic that matches very closely (with a mean relative error less than 0.5% over a voltage range from 0.9-1.1 in per unit in the tests described in this paper) the actual load's behavior. Mathematically, this synthesis leads to a linear power flow (LPF) solution. In the past, current injection models have also been proposed, for example, in [21]-[23]. The superior performance of the current injection method against the backward/forward sweep method is shown in [24]. The model proposed in the present paper combines the current source in parallel with an impedance to better match the load's voltage dependency and leads to a linear formulation to solve the PF problem. As the load composition changes during the day, a new current and a new impedance need to be calculated by the curve-fitting routine. However, this calculation only adds an insignificant overload to the overall system solution. Distributed generation can be included in the proposed linear PF formulation using a standard constant P-Q model. In the most common contractual obligations of DGs [25] in North America, there are no reactive power requirements. In some specific contracts, a fixed amount of reactive power or a fixed power factor may be required. As an example, in Germany's new Grid Code [26], the system operator should be able to control a DG using one of the following schemes:

(a) A fixed power factor ($cos(\phi) =$ Constant)
(b) A fixed reactive power value ($Q =$ Constant)
(c) A voltage regulation characteristic ($Q = f(V)$)

Since the active power of the DG is known, for control schemes (a) and (b) the proposed method is able to model the DG as a negative constant P-Q load. For control scheme (c), if $f(V)$ can be expressed as a sum of a quadratic function and a linear function of V, we can still model the DG as a current source in parallel with an impedance as proposed in the LPF framework.

The rest of the paper is organized as follows. In Section II, the proposed load modeling and approximation techniques are explained. The LPF formulation is described in Section III. In Section IV, simulation results on a number of distribution systems of various sizes are presented. The paper concludes by summarizing the main findings of the study.

II. Load Modeling

A. Conventional Load Models

Power system loads present different behavior to grid voltage variations. For example, active and reactive power consumption by fluorescent lamps are highly affected by the voltage magnitude, while personal computers are less sensitive to voltage variations [18]. A common way of describing the dependency of active and reactive power consumption on the voltage magnitude is the exponential model [20], as follows:

$$\frac{P(V)}{P_0} = \left(\frac{V}{V_0}\right)^{\alpha} \tag{1}$$

$$\frac{Q(V)}{Q_0} = \left(\frac{V}{V_0}\right)^{\beta} \tag{2}$$

where P and Q are the load's active and reactive power consumption; V is the terminal voltage magnitude and the zero subscript means the nominal value; α and β are the active and reactive power exponents, respectively, which can be extracted from measurements. Some typical values for these exponents are given in [18], [27] and [28].

In addition to the exponential load model, the polynomial model (ZIP model) has also been widely used in power system studies. This model consists of three major parts: a constant-impedance (Z), a constant-current (I) and a constant-power (P). Mathematically, this model describes the load variation with voltage as [20]:

$$\frac{P(V)}{P_0} = F_Z\left(\frac{V}{V_0}\right)^2 + F_I\left(\frac{V}{V_0}\right) + F_P \tag{3}$$

$$\frac{Q(V)}{Q_0} = F'_Z\left(\frac{V}{V_0}\right)^2 + F'_I\left(\frac{V}{V_0}\right) + F'_P \tag{4}$$

where constants F and F' are fractions; the subscripts Z, I and P stand for constant-impedance, constant-current and constant-power contributions, respectively. Note that there are only two independent parameters in (3) and (4) because $F_Z + F_I + F_P = 1$ and $F'_Z + F'_I + F'_P = 1$.

B. Proposed Load Model

The constant P-Q load model, as well as the voltage-dependent load models of (1)-(4) above introduce nonlinearity in the solution of the PF equations. The load model proposed in this paper is an alternative to the voltage-dependent load models of (1)-(4) that allows for a linear formulation of the power flow equations (this modeling approach will be referred to as LPF load modeling from now on). The proposed model is a "fitted" ZI model, as follows:

$$\frac{P(V)}{P_0} = C_Z\left(\frac{V}{V_0}\right)^2 + C_I\left(\frac{V}{V_0}\right) \tag{5}$$

$$\frac{Q(V)}{Q_0} = C'_Z\left(\frac{V}{V_0}\right)^2 + C'_I\left(\frac{V}{V_0}\right) \tag{6}$$

in which constants C and C' are calculated by a curve-fitting procedure. Note that there is only one independent parameter in (5) and (6) because $C_Z + C_I = 1$ and $C'_Z + C'_I = 1$. From a mathematical synthesis point of view, exponents α and β in the model given by (1)-(2) can be calculated with a fitting procedure, while in the model described by (3)-(4) the polynomial coefficients are to be determined. The proposed synthesis of (5)-(6) is of the same type as (3)-(4) but without the zero-order term in the voltage; this eliminates the constant P-Q term in the synthesis and, as it is shown next, the nonlinearity in the power flow equations.

The proposed model approximates the load voltage dependency using an impedance Z, representing the quadratic voltage dependency, and a current source I, representing the linear voltage dependency. As shown in the paper, coefficients C_Z, C_I, C'_Z, C'_I can be found for a very good fit to the measured data. The fitting process can be formulated in terms of a simple convex quadratic optimization problem. The fitting objective for the active and reactive power is to minimize the difference between the fitted approximation and the measured data for a finite number of voltage points within a range of operating voltages (e.g. $\pm 10\%$). For example, for the active power in per unit values (writing in per unit allows for dropping V_0 and P_0),

$$\text{Minimize} \quad \sum_{i=1}^{N_v} \left[C_Z V_i^2 + C_I V_i - P(V_i) \right]^2 \tag{7a}$$

subject to

$$C_Z + C_I = 1 \tag{7b}$$

and similarly for the reactive power Q. In (7), N_v is the number of points selected within the voltage range; $\big(V_i, P(V_i)\big)$ is the i^{th} pair of measured (voltage, active power); C_Z and C_I are the unknown variables to be calculated. Note that at $V = V_0$, one has $P = P_0$. Hence, (7b) is implicitly considered in the problem. The constraint in (7b) reduces the number of variables to one. The solution of this problem is straightforward and is provided in Appendix.

The data for the synthesis in (7) can be obtained directly from load measurements or previous synthesis by (1)-(4). The ZIP model, exponential model and the LPF load model obtained by fitting a curve to experimental measurements for a 3-phase induction motor [18] are depicted in Fig.1. Figure 1 also shows the parameters for the fitted curves. As can be seen, the difference between the LPF and the exponential model is negligible. Also, the ZIP model provides slightly tighter approximation compared to the LPF and exponential models. For other types of loads that show stronger dependency on voltage, even tighter fits have been achieved using the LPF method. The parameters for some commonly-used types of loads are calculated through measurements [18] and are given in Table I. It should be noted that since the ZIP model has 2 independent variables (since $F_Z + F_I + F_P = 1$) for fitting while the proposed LPF load model and the exponential model have only one, more accurate fittings may be achieved by the ZIP model. However, the differences between the resulting curves are minor. Depending on the load type, it sometimes happens that negative values are obtained for F in (3)-(4), or C in (5)-(6). These negative values do not affect the mathematical solutions, although they do not have a physical meaning.

III. Proposed Power Flow Formulation

Conventionally, the power flow equations have been formulated based on a constant P-Q load model. This makes the equations nonlinear and iterative methods are required to find the solution. It is possible, however, with the load model proposed in this paper, to reformulate the PF problem and make the equations very close to linear for distribution systems. A similar attempt was made in [23] by replacing the loads with current source injections. In this method (*implicit Z-bus* method), however, an iterative procedure is still required to update the injected currents at every iteration. The updating mechanism for current injections at a generic node at iteration k is

$$\bar{I}^{(k)} = \left(\frac{\bar{S}^{(k)}}{\bar{V}^{(k)}} \right)^* \tag{8}$$

where values with a bar on top are complex numbers. The LPF load model suggested in the present paper introduces the modification that $\bar{S}$ is not constant but voltage-dependent. Considering in (8) that P and Q are voltage-dependent and separating the real and imaginary parts of $\bar{S}$ and $\bar{V}$, we can write

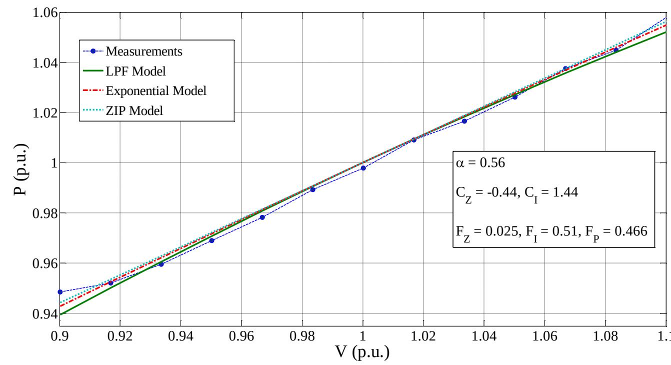
$$\bar{I}^{(k)} = \frac{P^{(k)}(V^{(k)}) - \mathrm{j}Q^{(k)}(V^{(k)})}{V_{\text{re}}^{(k)} - \mathrm{j}V_{\text{im}}^{(k)}} \tag{9}$$

where V_{re} and V_{im} are the real and imaginary parts of the voltage, respectively. Substituting the values of P and Q from

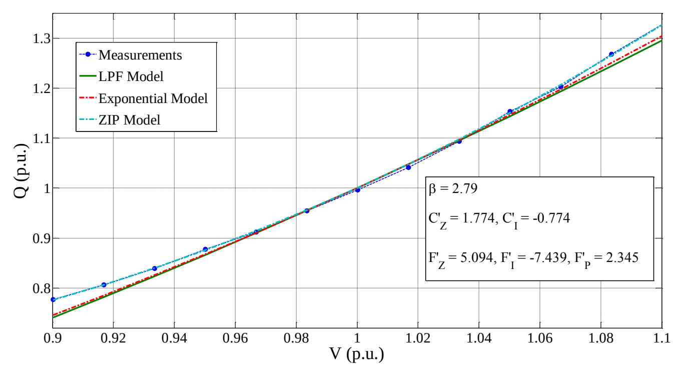
Table I

Parameters Representing the Voltage Dependency of Some Specific Loads Obtained Through Measurements [18]

Load Type	Compact Fluorescent Lamp	Incandescent Lamp	Electric Stove	Personal Computer
C_Z	-0.394	0.578	0.967	-0.984
C'_Z	-1.857	-5.012	0.960	-1.351



(a) Active Power



(b) Reactive Power

Figure 1. Comparison of the exponential, ZIP and proposed LPF load models for a 3-phase induction motor (Motor ratings: 460 V, 3-phase, 1.4 HP, 1725 RPM) [18].

(5)-(6) into (9), and temporarily dropping the iteration index yields:

$$\bar{I} = \underbrace{\frac{P_0 C_Z V_{\text{re}} + Q_0 C'_Z V_{\text{im}}}{V_0^2}}_{\text{Impedance}} + \underbrace{\frac{P_0 C_I V_{\text{re}} + Q_0 C'_I V_{\text{im}}}{V_0 V}}_{\text{Current}}$$
$$+ \mathrm{j}\Big[\underbrace{\frac{P_0 C_Z V_{\text{im}} - Q_0 C'_Z V_{\text{re}}}{V_0^2}}_{\text{Impedance}} + \underbrace{\frac{P_0 C_I V_{\text{im}} - Q_0 C'_I V_{\text{re}}}{V_0 V}}_{\text{Current}}\Big] \tag{10}$$

In distribution systems, taking the voltage angle of the feeding substation as reference (zero value), the imaginary part of the voltage V_{im} is smaller than the real part V_{re} by several orders of magnitude, as is also assumed in [8]. This feature allows us to eliminate the imaginary part of the voltage in the current parts of (10). There are two common nonlinear terms in the current parts of (10), which are reproduced here for simplicity:

$$\frac{V_{\text{re}}}{V} = \frac{V_{\text{re}}}{\sqrt{V_{\text{re}}^2 + V_{\text{im}}^2}} \tag{11a}$$

$$\frac{V_{\text{im}}}{V} = \frac{V_{\text{im}}}{\sqrt{V_{\text{re}}^2 + V_{\text{im}}^2}} \tag{11b}$$

Applying the assumption $V_{\text{im}} \approx 0$ to (11), we can write

$$\frac{V_{\text{re}}}{V} \approx 1 \tag{12a}$$

$$\frac{V_{\text{im}}}{V} \approx 0 \tag{12b}$$

With these approximations, (10) can be simplified to its real and imaginary parts as

$$I_{\text{re}} = \Re\{\bar{I}\} \approx \frac{Q_0 C'_Z}{V_0^2} V_{\text{im}} + \frac{P_0 C_Z}{V_0^2} V_{\text{re}} + \frac{P_0 C_I}{V_0} \tag{13a}$$

$$I_{\text{im}} = \Im\{\bar{I}\} \approx \frac{P_0 C_Z}{V_0^2} V_{\text{im}} - \frac{Q_0 C'_Z}{V_0^2} V_{\text{re}} - \frac{Q_0 C'_I}{V_0} \tag{13b}$$

As it is shown next, these approximations eliminate the need for iterations in the PF solutions.

A. Power Flow Formulation in Rectangular Coordinates

Many software packages for system optimization are available that work with real variables. To use these packages, it is required to keep the real and imaginary parts of the complex voltages and currents separate. Writing the current drawn by a constant-impedance load as $\bar{I} = \bar{Y}\bar{V}$ and separating real and imaginary parts of $\bar{V}$ and $\bar{Y}$ yields

$$\begin{aligned} \bar{I} = \bar{Y}\bar{V} &= (G + \mathrm{j}B)(V_{\text{re}} + \mathrm{j}V_{\text{im}}) \\ &= (GV_{\text{re}} - BV_{\text{im}}) + \mathrm{j}(BV_{\text{re}} + GV_{\text{im}}) \end{aligned} \tag{14}$$

It can be seen from (14) that the voltage-dependent part of the drawn current given by (13a)-(13b) can be synthesized by an equivalent admittance with the following parameters:

$$G = \frac{P_0 C_Z}{V_0^2} \tag{15a}$$

$$B = -\frac{Q_0 C'_Z}{V_0^2} \tag{15b}$$

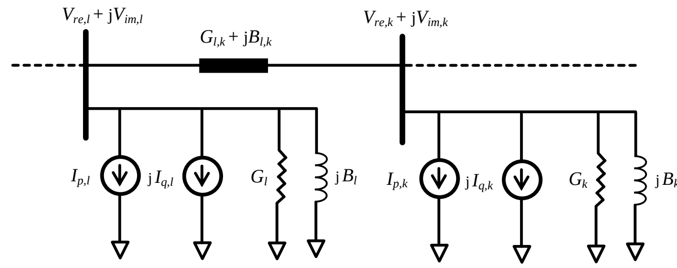


Figure 2. Generic part of a DS obtained based on the LPF load synthesis.

while the rest of the terms in (13a)-(13b) can be represented as a constant-current source. Eventually, the proposed load model for LPF analysis is a mixture of the constant-impedance and constant-current parts introduced. The LPF load model can then be represented by circuit elements as shown in Fig. 2. The quantities for the constant-current sources I_p and I_q are given by the last terms in (13a)-(13b), respectively, i.e.:

$$I_p = \frac{P_0 C_I}{V_0} \tag{16a}$$

$$I_q = -\frac{Q_0 C'_I}{V_0} \tag{16b}$$

Applying Kirchhoff's Current Law to a generic distribution system, considering the substation(s) as voltage source(s) and representing the loads using the LPF load model yields, in complex numbers form

$$\begin{bmatrix} \bar{Y}_{\text{AA}} & \bar{Y}_{\text{AB}} \\ \bar{Y}_{\text{BA}} & \bar{Y}_{\text{BB}} \end{bmatrix} \begin{bmatrix} \bar{V}_{\text{A}} \\ \bar{V}_{\text{B}} \end{bmatrix} = \begin{bmatrix} \bar{I}_{\text{A}} \\ \bar{I}_{\text{B}} \end{bmatrix} \tag{17}$$

in which $\bar{Y}$, partitioned into four sub-matrices, is the system admittance matrix which includes the equivalent admittances of the loads given by (15) in the diagonal elements; $\bar{V}_{\text{A}}$ is the vector of known voltages and $\bar{I}_{\text{A}}$ is the unknown vector of corresponding nodal current injections at the substation(s). $\bar{I}_B$ is the constant-current part of the load given by (16). $\bar{V}_{\text{B}}$ is the vector of unknown voltages, which can be computed as:

$$[\bar{V}_{\text{B}}] = [\bar{Y}_{\text{BB}}]^{-1}[\bar{I}_{\text{B}}] - [\bar{Y}_{\text{BB}}]^{-1}[\bar{Y}_{\text{BA}}][\bar{V}_{\text{A}}] \tag{18}$$

Separating the real and imaginary parts of (17), the proposed PF equations in rectangular coordinates can be represented by the following partitioned matrix equation:

$$\begin{bmatrix} \begin{bmatrix} G_{1,1:n} & -B_{1,1:n} \\ B_{1,1:n} & G_{1,1:n} \end{bmatrix} \\ \begin{bmatrix} G_{2,1:n} & -B_{2,1:n} \\ B_{2,1:n} & G_{2,1:n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} G_{n,1:n} & -B_{n,1:n} \\ B_{n,1:n} & G_{n,1:n} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} V_{\text{re},1} \\ V_{\text{re},2} \\ \vdots \\ V_{\text{re},n} \end{bmatrix} \\ \begin{bmatrix} V_{\text{im},1} \\ V_{\text{im},2} \\ \vdots \\ V_{\text{im},n} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_{p,1} \\ I_{q,1} \\ I_{p,2} \\ I_{q,2} \\ \vdots \\ I_{p,n} \\ I_{q,n} \end{bmatrix} \tag{19}$$

in which G and B are the real and imaginary parts of $\bar{Y}$, respectively. The same matrix reduction technique used in (17) and (18) can be used here to eliminate the known variables.

By rearranging the rows in (19), a more compact form can be obtained as

$$\begin{bmatrix} \hat{G} & -\hat{B} \\ \hat{B} & \hat{G} \end{bmatrix} \begin{bmatrix} \hat{V}_{\text{re}} \\ \hat{V}_{\text{im}} \end{bmatrix} = \begin{bmatrix} \hat{I}_p \\ \hat{I}_q \end{bmatrix} \tag{20}$$

in which the hat sign ($\hat{.}$) indicates a matrix/vector form of the corresponding variables. The system of linear equations in (20) yields the power flow solutions.

In order to include distributed generation (DG) in the proposed formulation, we assume in the present work that the active power generation of the DG unit is given and that the DG has the contractual obligation of being reactive power neutral (or to generate only a predefined amount of reactive power). A similar assumption was made in [23]. Systems with large DG penetration under different contractual obligations will need additional considerations. With the indicated assumption, DGs can be modeled as a constant negative P-Q load. Since each nodal load is an aggregation of different types of loads, even if a constant P-Q load exists at that node, there are usually other loads which can be modeled as voltage-dependent and the coefficients in (5) and (6) can still be derived.

IV. Simulation Results

In this section, the proposed method is applied to sample test systems with 14, 70, and 135 nodes, described in [29], [30], and [31], respectively. A larger system consisting of 3249 nodes was also tested by combining the 135-node and 70-node systems, 16 times each. These test systems are indicated in Table II. For simplicity, the voltage dependency of all the loads is assumed to be identical. Also, both active and reactive power are considered to have the same voltage-dependency.

A comparison between the results obtained using the implicit Z-bus method of [23], assuming a voltage-dependent load model, and the proposed method was carried out. The performance of the proposed method was evaluated based on the average error between the voltage magnitudes calculated by our LPF method and the implicit Z-bus method as:

$$\eta = \frac{1}{n} \sum_{i=1}^{n} \frac{|V_i' - V_i|}{V_i'} \tag{21}$$

where the primed voltages are obtained from the implicit Z-bus method and the unprimed voltages are obtained from the proposed method. The histogram of errors for the 70-node and the 3249-node test systems are shown in Fig. 3. For the 70-node system, the maximum relative error is about 0.01% which happens when $C_Z = 1$. For other values of C_Z the maximum error is even less than 0.01%. For the 3249-node system, the relative error does not exceed 0.15% for considered values of C_Z. Figure 4 shows the variation of η as a function of C_Z for different test systems (C_Z' is assumed equal to C_Z for the test). Note that each value of C_Z leads to corresponding values for C_I using (7b). As can be seen in Fig. 4, the errors are very small (less that 0.11%) over the range of values considered for C_Z. In the especial case of $C_Z = 1$, i.e. constant-impedance (no current source in the model), the error is zero. It is worthwhile mentioning that, depending on the value of C_Z, the Z-bus method needs at least 4 iterations to reach a tolerance of 1e-4 p.u. in the magnitude of all nodal voltages. Increasing the value of C_Z, the number of required iterations increases. This result is consistent with the results found in [32]. In terms of iterations, the proposed LPF model requires only one solution and it is, therefore, at least four times faster. A comparison between CPU times and number of iterations is given in Table III.

Table II
Dimensions of the Employed Test Systems

Test Case	Nodes	Branches	Feeders	Load(MVA)
14-node	14	13	3	28.70 + i17.30
70-node	70	69	4	4.47 + i3.06
135-node	135	134	8	18.31 + i7.93
3249-node	3249	3248	192	364.50 + i175.86

Table III
CPU Times for 20 Times Power Flow Calculations Using the LPF and the Implicit Z-Bus Methods

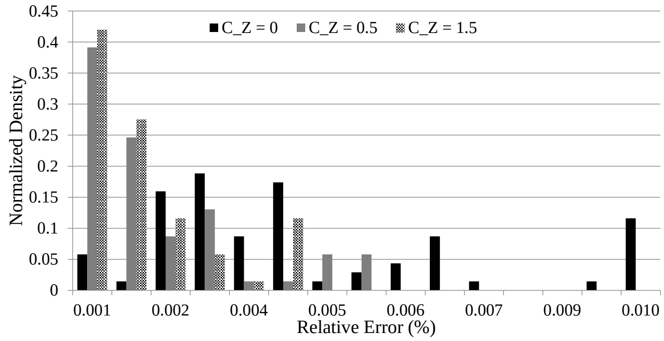
Test System	Method	Iterations	CPU Time (s)
3249-node	Implicit Z-Bus	4	1.142
	LPF	1	0.244
70-node	Implicit Z-Bus	5	0.0511
	LPF	1	0.0075

Figure 5 shows the voltage profile of the distribution feeders in the 70-node system, as one moves away from the substation (Node 1). The figure compares the results obtained using the Z-bus method and the LPF method. Part (a) is for a load model with $C_Z = C_Z' = -0.5$ and part (b) is for $C_Z = C_Z' = 0.5$. Both the Z-bus solution and the LPF solution are basically identical while the LPF method as about six times faster (see Table III). The value of C_Z affects the voltage profile. But this effect is ignored in the traditional constant P-Q model. The significance of the effect of C_Z on the voltage profile depends on the system under study. In Fig. 6, a linear function is fitted to each average voltage curve along the feeders (the actual curve is, for the 70-node system as an example, as in Fig. 5). Average voltage curves are plotted in Fig. 6 as a function of C_Z for the various test systems. In general, the system average voltage increases as the value of C_Z increases. It can be seen that the 70-node system is more sensitive to the value of C_Z compared to the other systems.

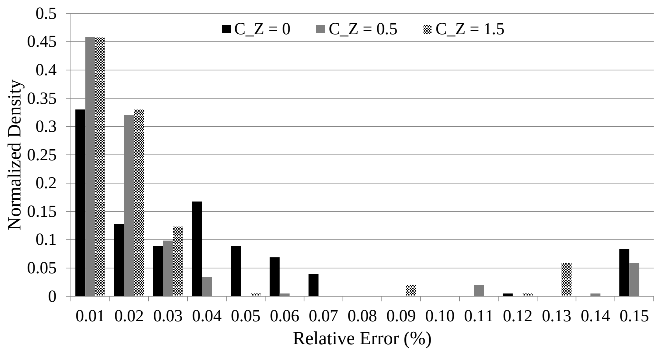
A. Parameter Selection of Voltage-Dependent Load Models

In existing distribution systems, the load's voltage dependency characteristics are usually not directly known. However, the conventional classification of loads into residential, commercial and industrial, recommended by the IEEE [33], is a good guide to obtain an estimate of voltage-dependent load models. An example is the study conducted by Ontario Hydro to obtain the parameters for a ZIP load model for different load types [34]. In our paper, we will refer to this approach as "composite" load model.

It is the objective of the advanced measuring infrastructure (AMI) deployment in the so-called smart grids to collect extensive measurements of the load demand and the loads' behavior.



(a) The 70-node test system



(b) The 3249-node test system

Figure 3. Histograms of relative errors between nodal voltages obtained by the LPF method and the implicit Z-bus method for different load types.

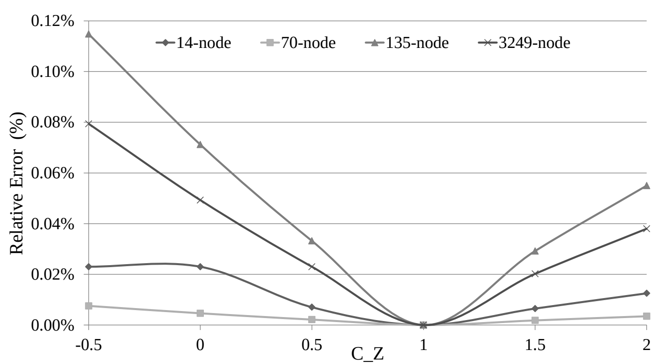
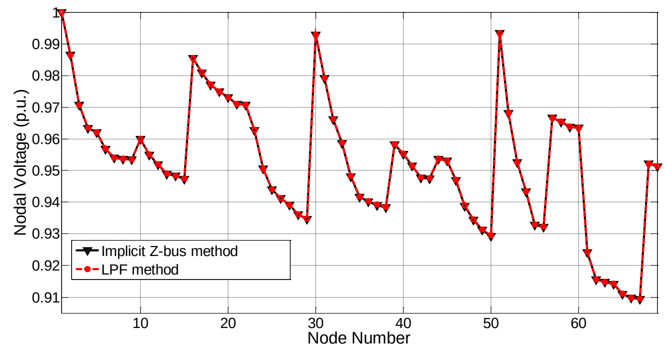
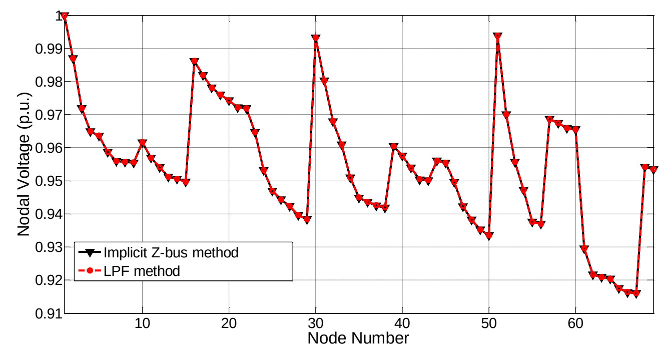


Figure 4. Average relative errors between nodal voltages obtained by the LPF method and the implicit Z-bus method (obtained by (21)) as a function of C_Z.



(a) $C_Z = C'_Z = -0.5$



(b) $C_Z = C'_Z = 0.5$

Figure 5. Nodal voltages for the 70-node test system obtained by the proposed LPF method and the implicit Z-bus method.

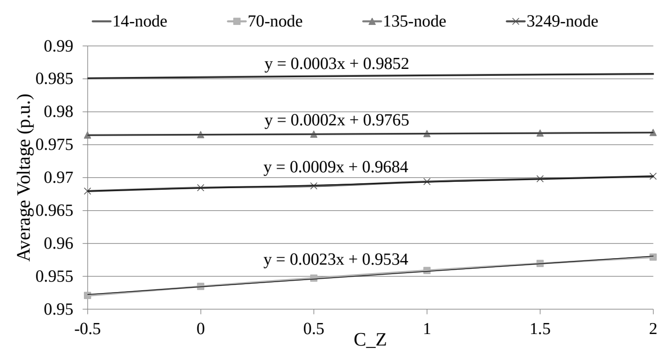


Figure 6. System average voltage as a function of C_Z for various test systems.

In the future smart distribution systems, it will be possible with adequate measuring devices to obtain load behavior functions for specific time spans (for example, every 10 minutes). Even in the existing systems with no smart devices connected, the main contributors to the load composition along a feeder are usually known.

As an example of the effect of the load composition, the 14-node test system shown in Fig. 7 is studied next. In this system, a hypothetical load type is assumed for each node. Reference [34] provides the parameters for the ZIP load model for various types of loads in real systems. It is simple to compute the parameters for the LPF load model (i.e. C_Z, C_I, C'_Z and C'_I) using the ZIP model parameters through (7). Table IV shows the parameters for the ZIP and LPF load models for different load compositions.

It is expected for the test system that because the nodal voltages are below 1 p.u., the power consumption will be lower than the nominal value. Figure 8 shows the active and reactive power consumptions of loads represented by the constant P-Q model and the composite load model. As expected, using a composite load model results in a reduced power demand in all nodes. The voltage profile of the system is shown in Fig. 9. The composite load model results in an increase in the nodal voltage magnitudes as compared to the constant P-Q model.

V. Discussion and Future Work

The advantages of linearizing the power flow solution are not limited to the indicated fourfold performance increase in

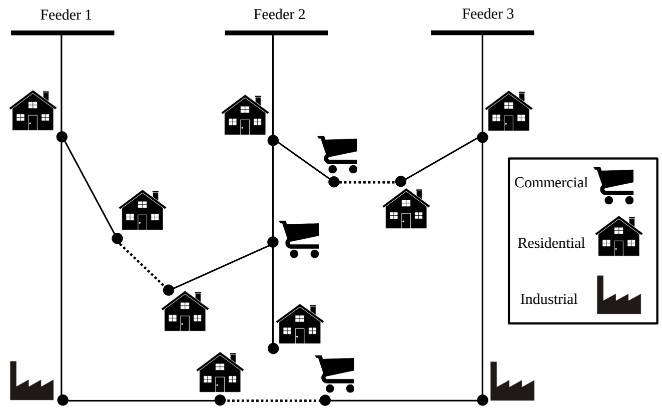


Figure 7. The 14-node test system with hypothetical load composition.

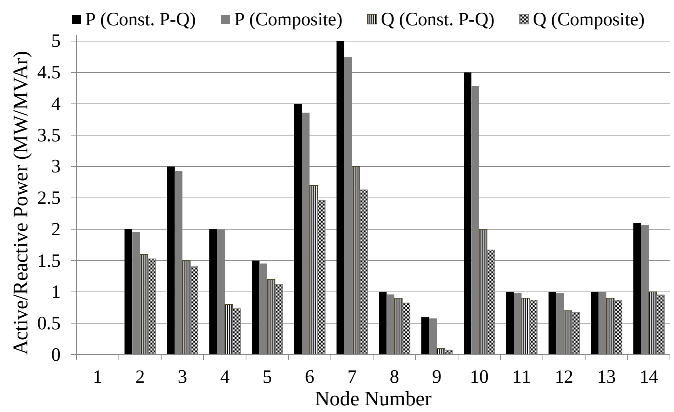


Figure 8. Nodal active and reactive power demands obtained using the constant P-Q and composition-based load models for the 14-node system.

Table IV
PARAMETERS ASSUMED FOR DIFFERENT LOAD COMPOSITIONS [34]

	Composition	F_Z	F_I	F_P	C_Z	C_I
Active Power	Residential	0.24	0.62	0.13	-0.10	1.1
	Commercial	0.16	0.80	0.04	0.12	0.88
	Industrial	-0.07	0.24	0.83	-0.90	1.90
Reactive Power	Residential	2.44	-1.94	0.50	1.93	-0.93
	Commercial	3.26	-3.10	0.84	2.42	-1.42
	Industrial	1.00	0	0	1.00	0

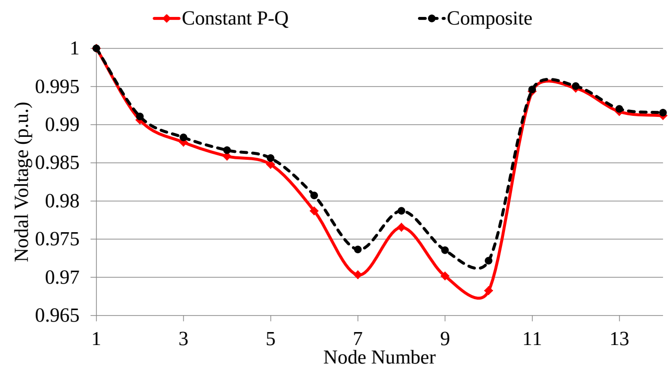


Figure 9. Nodal voltages obtained using the constant P-Q and composite load models for the 14-node system.

power flow analysis of distribution systems. A significant advantage is also achieved in power system optimization studies. For example, distribution systems reconfiguration has been a challenging problem for many years [29]-[31]. The nonlinear set of PF equations makes the problem of DS reconfiguration a mixed-integer nonlinear programming problem. This family of combinatorial optimization is known to be NP-hard and, generally speaking, there is no solver that can guarantee reaching a global optima within polynomial time. As a result, researchers have been focusing on heuristic methods to solve this problem. However, heuristics are not fully reliable and there is no guarantee of the globality of the solution.

With a linear set of equations for PF analysis, the DS reconfiguration problem can be formulated and solved more efficiently than with conventional nonlinear formulations. The speed and robustness of convergence of linear optimization methods can lead to DS automation for more economic and reliable operation of future smart grids. The authors are currently developing a mixed-integer linear programming (MILP) formulation for DS reconfiguration, for which there are well-developed software capable of solving large-scale problems efficiently in polynomial time. It is expected that the linear PF formulation will be able to reduce the computation burdens of this problem and to allow for online applications. As indicated, in addition to speed, a major advantage of linear optimization is that it guarantees a global optima.

The VVO algorithms are also in need of fast and linear power flow solutions to be used in the optimization problems embedded within those routines. These algorithms have to be solved in near-realtime or even realtime and, therefore, the proposed LPF has a great potential of being used in those algorithms. Moreover, the voltage dependency of the loads significantly affects the accuracy of the VVO algorithms [16], which the LPF is also capable of representing the loads using their voltage dependencies.

VI. CONCLUSION

The property of load voltage dependency is utilized in this study to find a load synthesis that leads to a linear power flow formulation for electrical distribution systems. The idea behind the proposed method is a simple curve-fitting algorithm for the load measurements data. The conventional load-voltage relationship in the form of exponential or polynomial functions can be converted into the proposed load synthesis with negligible error over the normal operating range of voltages. Using this load synthesis and the property of small imaginary parts in the nodal voltages of distribution systems, the nonlinear power flow equations are converted into linear equations. Compared to a traditional constant P-Q load representation, the advantages of the proposed model can be summarized as follows:

- A voltage-dependent load model provides more realistic power flow results than a constant P-Q model by more

accurately representing the actual behavior of the loads in response to voltage variations.
- The proposed voltage-dependent load model leads to a linear power flow formulation for distribution systems.
- A linear power flow formulation results in an improvement of at least four times in execution speed compared to the implicit Z-bus algorithm.
- A linear power flow formulation allows the development of more efficient optimization algorithms with guaranteed global optima.
- The method is applicable to both radial and meshed configurations.

The power flow method implemented in this paper assures balanced conditions. The extension of the power flow solutions to three-phase unbalanced systems is well-developed in the literature and will be incorporated in our algorithm in future work. Also, the range of validity of the proposed model depends on the range of validity of the load models used. Therefore, the LPF in its current presentation may not be suitable for voltage stability studies.

Appendix

In order to find the optimum solution of (7), the affine constraint (7b) is used to drop one dependent variable, say C_I. Assume that the voltage and active power measurements are in per unit with respect to their nominal values, i.e. V_0 and P_0. This allows for dropping the constants V_0 and P_0 in the equations. The reduced problem can be written as

$$\text{Minimize} \quad f(C_Z) = \sum_{i=1}^{N_v} \left[C_Z \left(V_i^2 - V_i \right) - P(V_i) + V_i \right]^2 \quad (22)$$

Taking the derivative of f with respect to C_Z and putting it equal to zero (Karush–Kuhn–Tucker conditions for optimality), one has:

$$C_Z = \frac{\sum_{i=1}^{N_v} (V_i^2 - V_i)(P(V_i) - V_i)}{\sum_{i=1}^{N_v} (V_i^2 - V_i)^2} \quad (23)$$

Based on the fact that the objective function is convex quadratic, this is the global optima. The solution for C_I can be retrieved using (7b).

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